Multidimensional Heat Transfer

Heat Diffusion Equation

$$r c_p \frac{\partial T}{\partial t} = k(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}) + \dot{q} = k\nabla^2 T + \dot{q}$$

- This equation governs the Cartesian, temperature distribution for a three-dimensional unsteady, heat transfer problem involving heat generation.
- For steady state $\partial / \partial t = 0$
- No generation $\dot{q} = 0$
- To solve for the full equation, it requires a total of six boundary conditions: two for each direction. Only one initial condition is needed to account for the transient behavior.

Two-D, Steady State Case

For a 2 - D, steady state situation, the heat equation is simplified to

$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$, it needs two boundary conditions in each direction.

There are three approaches to solve this equation:

- **Numerical Method**: Finite difference or finite element schemes, usually will be solved using computers.
- **Graphical Method**: Limited use. However, the conduction shape factor concept derived under this concept can be useful for specific configurations. (see 8-10 and table 8.7 for selected configurations)
- **Analytical Method**: The mathematical equation can be solved using techniques like the method of separation of variables. (review Engr. Math II)

Conduction Shape Factor

This approach applied to 2-D conduction involving two isothermal surfaces, with all other surfaces being adiabatic. The heat transfer from one surface (at a temperature T_1) to the other surface (at T_2) can be expressed as: $q=Sk(T_1-T_2)$ where k is the thermal conductivity of the solid and S is the conduction shape factor.

- The shape factor can be related to the thermal resistance:

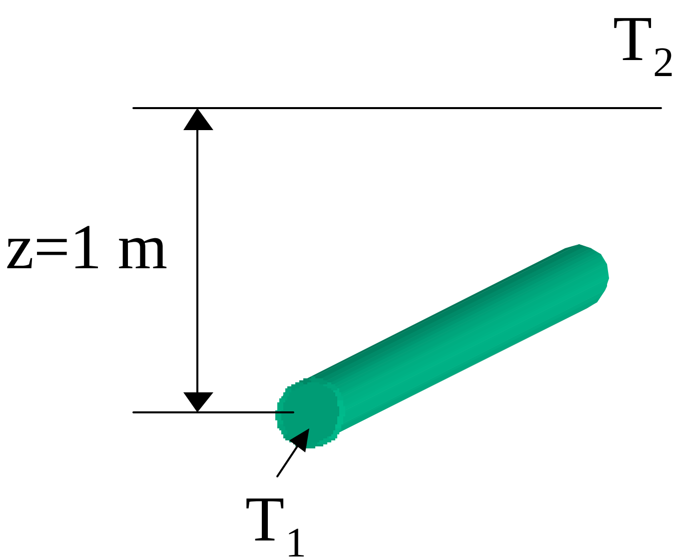
$q=Sk(T_1-T_2)=(T_1-T_2)/(1/kS)= (T_1-T_2)/R_t$

where $R_t = 1/(kS)$

- 1-D heat transfer can use shape factor also. Ex: heat transfer inside a plane wall of thickness L is $q=kA(\Delta T/L)$, $S=A/L$
- Common shape factors for selected configurations can be found in Table 8.7

Example

An Alaska oil pipe line is buried in the earth at a depth of 1 m. The horizontal pipe is a thin-walled of outside diameter of 50 cm. The pipe is very long and the averaged temperature of the oil is 100°C and the ground soil temperature is at -20 °C (k_{soil}=0.5W/m.K), estimate the heat loss per unit length of pipe.



From Table 8.7, case 1.
L>>D, z>3D/2

$$S = \frac{2\pi L}{\ln(4z/D)} = \frac{2\pi(1)}{\ln(4/0.5)} = 3.02$$

$$q = kS(T_1 - T_2) = (0.5)(3.02)(100+20)$$

$$= 181.2(W) \text{ heat loss for every meter of pipe}$$

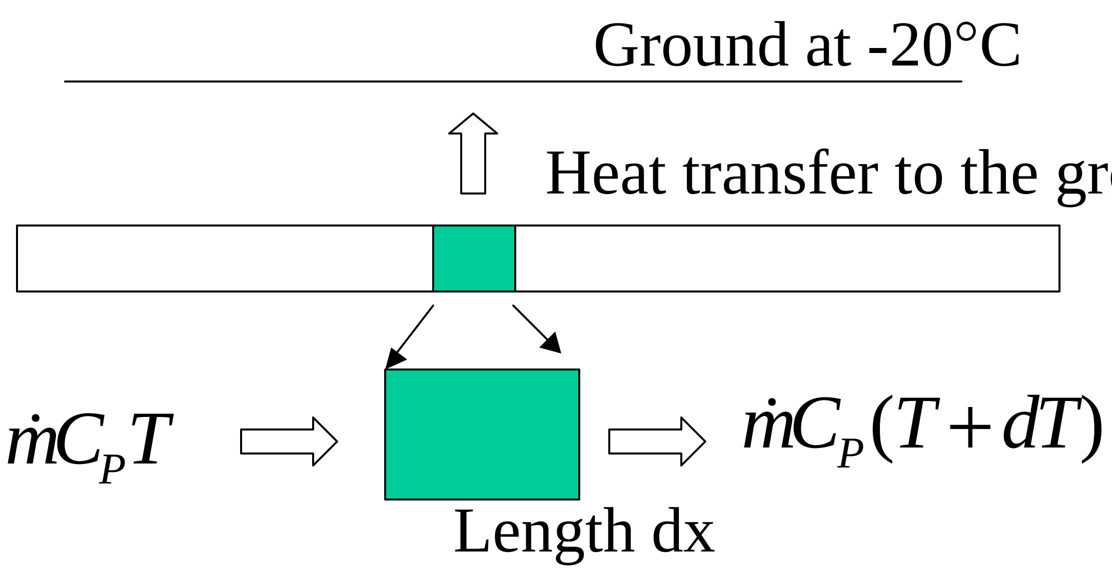
Example (cont.)

If the mass flow rate of the oil is 2 kg/s and the specific heat of the oil is 2 kJ/kg.K, determine the temperature change in 1 m of pipe length.

$$q = \dot{m}C_P\Delta T,\ \Delta T = \frac{q}{\dot{m}C_P} = \frac{181.2}{2000*2} = 0.045(^\circ C)$$

Therefore, the total temperature variation can be significant if the pipe is very long. For example, 45°C for every 1 km of pipe length.

- Heating might be needed to prevent the oil from freezing up.
- The heat transfer can not be considered constant for a long pipe



Example (cont.)

Heat Transfer at section with a temperature T(x)

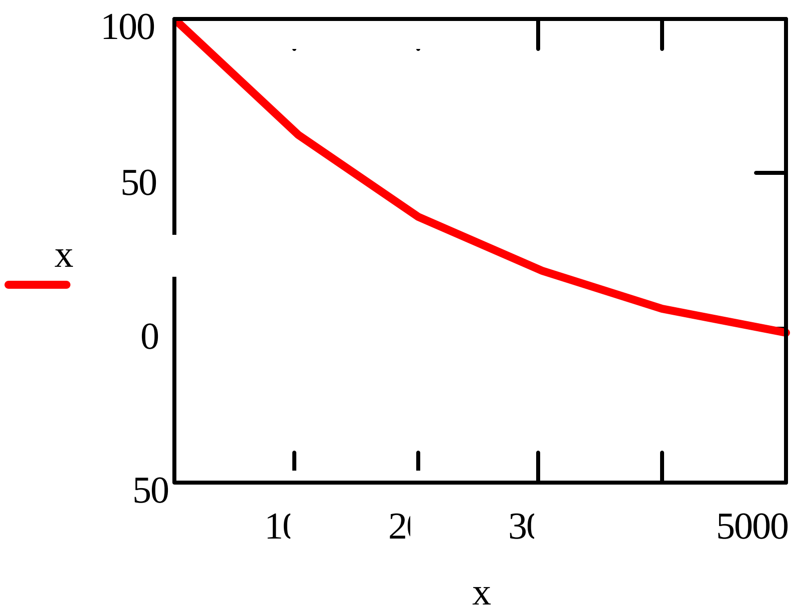
$$q = \frac{2\pi k(dx)}{\ln(4z/D)}(T+20) = 1.51(T+20)(dx)$$

Energy balance: $\dot{m}C_P T - q = \dot{m}C_P(T+dT)$

$$\dot{m}C_P\frac{dT}{dx} + 1.51(T+20) = 0,\ \frac{dT}{T+20} = -0.000378dx, \text{integrate}$$

$$T(x) = -20 + Ce^{-0.000378x},\ \text{at inlet x = 0, T(0) = 100°C, C = 120}$$

$$T(x) = -20 + 120e^{-0.000378x}$$



- Temperature drops exponentially from the initial temp. of 100°C
- It reaches 0°C at x=4740 m, therefore, reheating is required every 4.7 km.