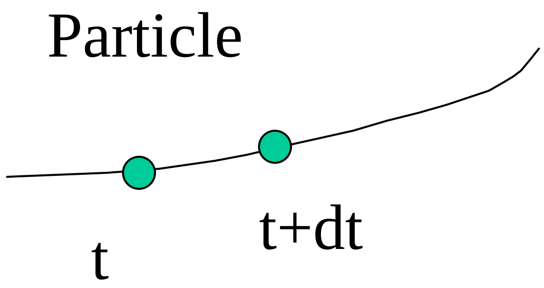
Particle Acceleration



Tracking the particle as we follow it path:

$$\vec{V}_{P@\text{time } t} = \vec{V}(x, y, z, t)$$

Note: V(x,y,z,t) is the velocity field of the entire flow, not the velocity of a particle.

As the particle moves, its velocity changes to

$$\vec{V}_{P@\text{time } t+dt} = \vec{V}(x+dx, y+dy, z+dz, t+dt)$$

The acceleration of a particle (substantial acceleration) is given by

$$\vec{a}_P = \frac{d\vec{V}_P}{dt} = \frac{\partial \vec{V}}{\partial t} + \frac{\partial \vec{V}}{\partial x}\frac{dx_P}{dt} + \frac{\partial \vec{V}}{\partial y}\frac{dy_P}{dt} + \frac{\partial \vec{V}}{\partial z}\frac{dz_P}{dt}$$

$$= \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}. \quad \text{where } u = \frac{dx_P}{dt}, \quad v = \frac{dy_P}{dt}, \quad w = \frac{dz_P}{dt}$$

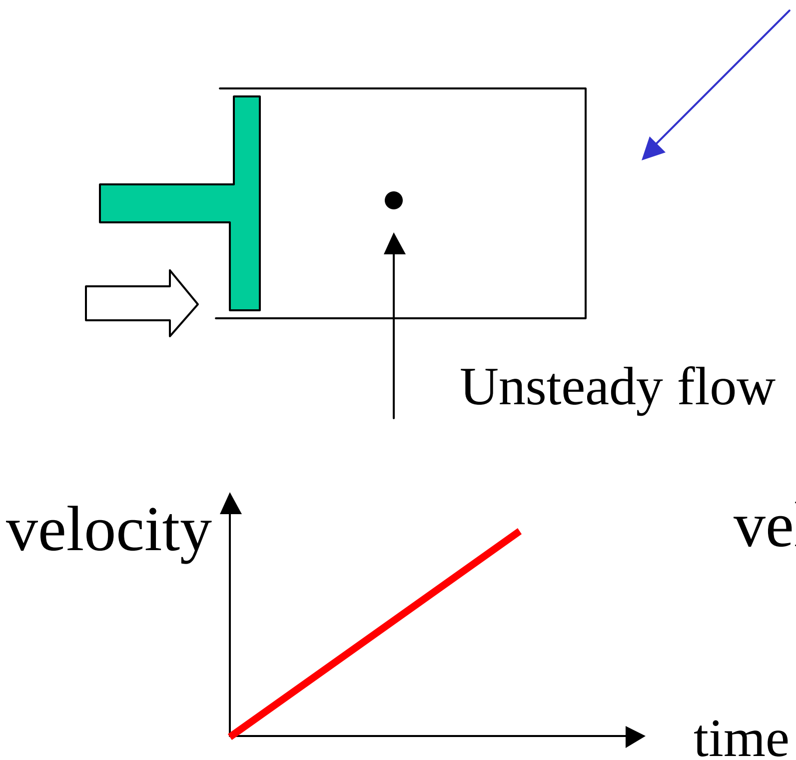
Physical Interpretation

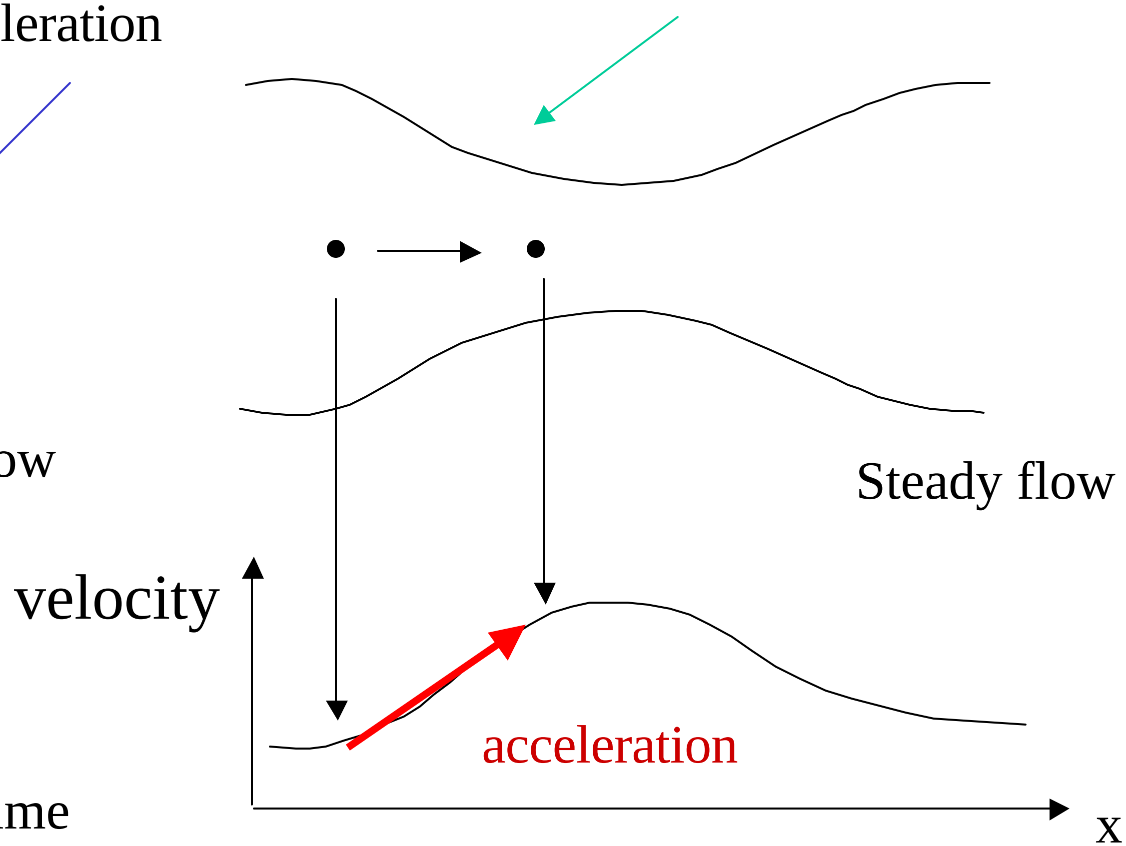
$$\vec{a}_P = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

Total acceleration of a particle ($\frac{D\vec{V}}{Dt}$)

Local acceleration ($\frac{\partial \vec{V}}{\partial t}$)

Convective acceleration ($u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$)

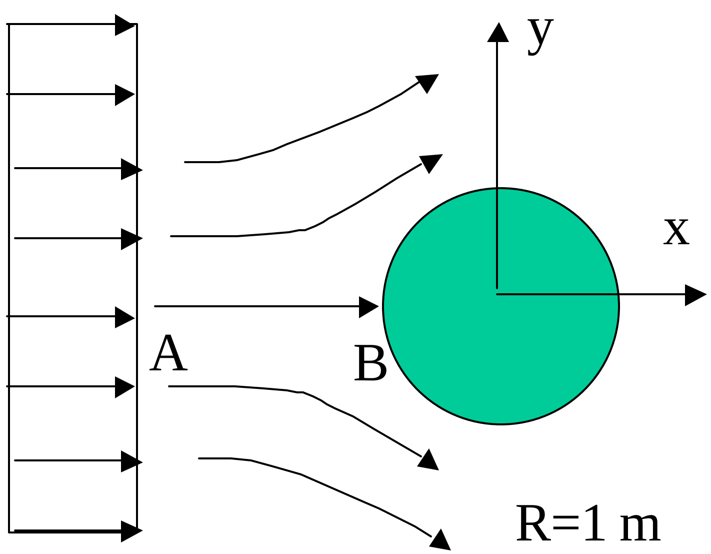


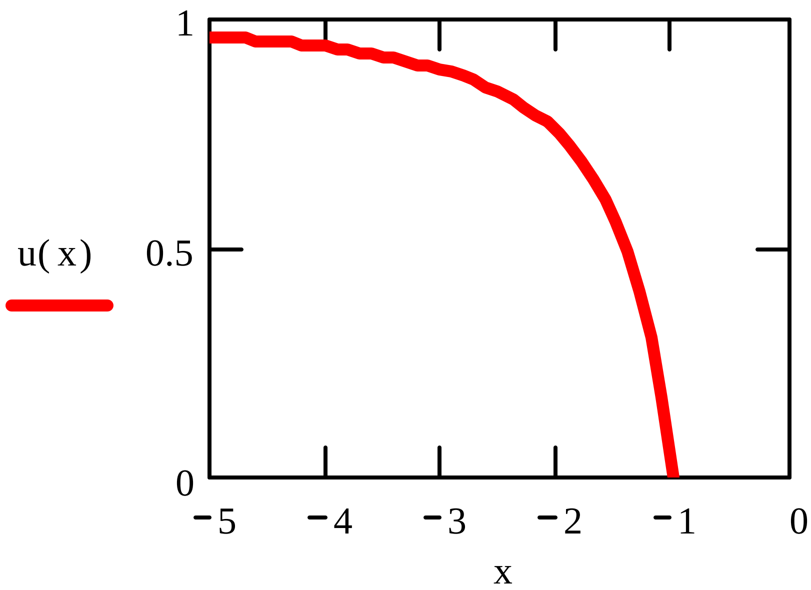


Example

An incompressible, inviscid flow past a circular cylinder of diameter d is shown below. The flow variation along the approaching stagnation streamline (A-B) can be expressed as:

$$\vec{V}(x, y=0) = u(x)\vec{i}, \qquad \text{where } u(x) = U_O\left(1 - \frac{R^2}{x^2}\right) = 1 - \frac{1}{x^2}$$





U_O=1 m/s

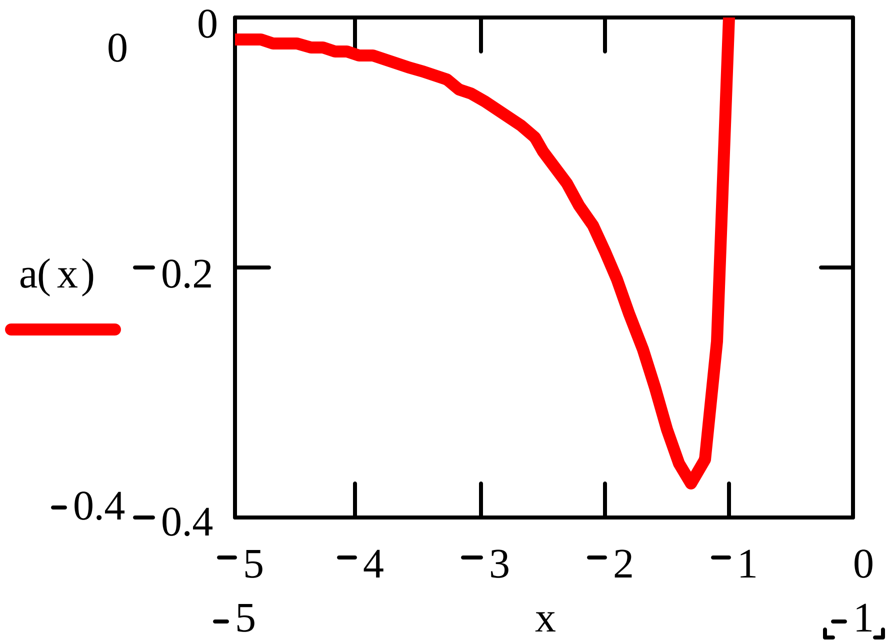
Along A-B streamline, the velocity drops very fast as the particle approaches the cylinder. At the surface of the cylinder, the velocity is zero (stagnation point) and the surface pressure is a maximum.

Example (cont.)

Determine the acceleration experienced by a particle as it flows along the stagnation streamline.

$$\vec{a} = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + 0 + 0, \quad \text{since v = w = 0 along the stagnation streamline.}$$

$$\text{Therefore, } a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}, \quad a_y = a_z = 0, \quad a_x = (1 - \frac{1}{x^2})(\frac{2}{x^3}) \quad \text{for steady state flow}$$



- The particle slows down due to the strong deceleration as it approaches the cylinder.
- The maximum deceleration occurs at x=-1.29R=-1.29 m with a magnitude of a(max)=-0.372(m/s^2)

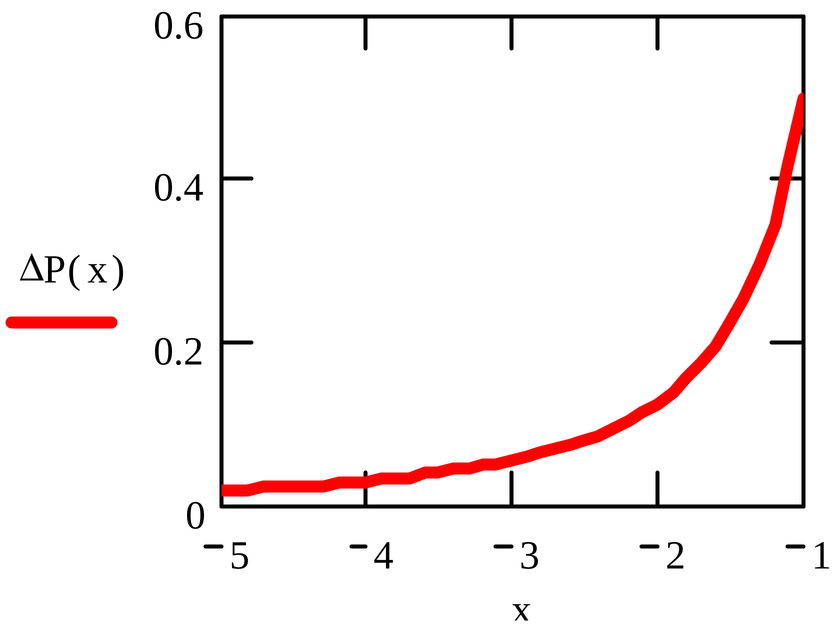
Example (cont.)

Determine the pressure distribution along the streamline using Bernoulli's equation. Also determine the stagnation pressure at the stagnation point.

Bernoulli's equation: $$\frac{\mathrm{P(x)}}{r} + \frac{\mathrm{u}^2(x)}{2} = \frac{P_\infty}{r} + \frac{U_O^2}{2}$$

$$P(x) - P_{atm} = \frac{r}{2}(U_O^2 - u^2(x)) = \frac{r}{2}\left(1 - \left(1 - \frac{1}{x^2}\right)\right) = \frac{r}{2}\left(\frac{1}{x^2}\right)$$

$$\Delta P(x) = \frac{P(x) - P_{atm}}{r} = \frac{1}{2x^2}$$

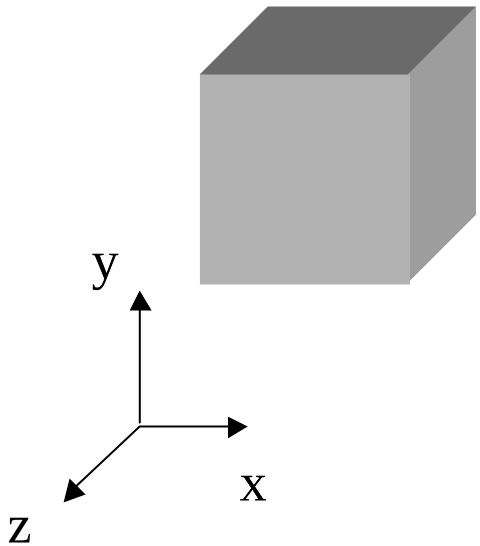


- The pressure increases as the particle approaches the stagnation point.
- It reaches the maximum value of 0.5, that is P_{stag}-P_∞=(1/2)ρU_O^2 as u(x)→0 near the stagnation point.

Momentum Conservation

From Newton's second law : Force = (mass)(acceleration)

Consider a small element $dxdydz$ as shown below.

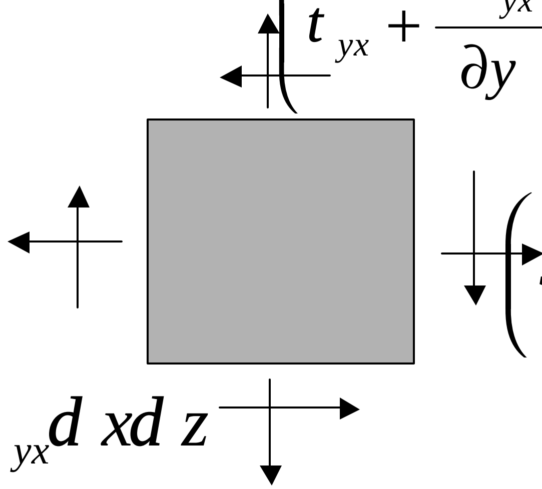


The element experiences an acceleration

$$\mathrm{m}\frac{\mathrm{D}\vec{\mathrm{V}}}{\mathrm{Dt}} = r(d\,xd\,yd\,z)\left(\frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}\right)$$

as it is under the action of various forces:

normal stresses, shear stresses, and gravitational force.



$$\left(t_{yx} + \frac{\partial t_{yx}}{\partial y}d\,y\right)d\,xd\,z$$

$$s_{xx}d\,yd\,z$$

$$\left(s_{xx} + \frac{\partial s_{xx}}{\partial x}d\,x\right)d\,yd\,z$$

$$t_{yx}d\,xd\,z$$

Momentum Balance (cont.)

Net force acting along the x-direction:

$$\frac{\partial s_{\text{xx}}}{\partial \text{x}} dxdydz + \frac{\partial t_{\text{yx}}}{\partial \text{x}} dxdydz + \frac{\partial t_{\text{zx}}}{\partial \text{x}} dxdydz + r g_x dxdydz$$

Normal stress

Shear stresses (note: τ_{zx}: shear stress acting on surfaces perpendicular to the z-axis, not shown in previous slide)

Body force

The differential momentum equation along the x-direction is

$$\frac{\partial s_{\text{xx}}}{\partial \text{x}} + \frac{\partial t_{\text{yx}}}{\partial \text{x}} + \frac{\partial t_{\text{zx}}}{\partial \text{x}} + r g_x = r\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

similar equations can be derived along the y & z directions

Euler's Equations

For an inviscid flow, the shear stresses are zero and the normal stresses are simply the pressure: $t = 0$ for all shear stresses, $s_{xx} = s_{yy} = s_{zz} = -P$

$$-\frac{\partial P}{\partial \mathrm{x}} + r\,g_x = r\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

Similar equations for y & z directions can be derived

$$-\frac{\partial P}{\partial \mathrm{y}} + r\,g_y = r\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$-\frac{\partial P}{\partial \mathrm{z}} + r\,g_z = r\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Note: Integration of the Euler's equations along a streamline will give rise to the Bernoulli's equation.

Navier and Stokes Equations

For a viscous flow, the relationships between the normal/shear stresses and the rate of deformation (velocity field variation) can be determined by making a simple assumption. That is, the stresses are linearly related to the rate of deformation (Newtonian fluid). (see chapter 5-4.3) The proportional constant for the relation is the dynamic viscosity of the fluid (μ). Based on this, Navier and Stokes derived the famous Navier-Stokes equations:

$$r\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial \mathrm{x}}+r g_x+m\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$r\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial P}{\partial \mathrm{y}}+r g_y+m\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$r\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial \mathrm{z}}+r g_z+m\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$