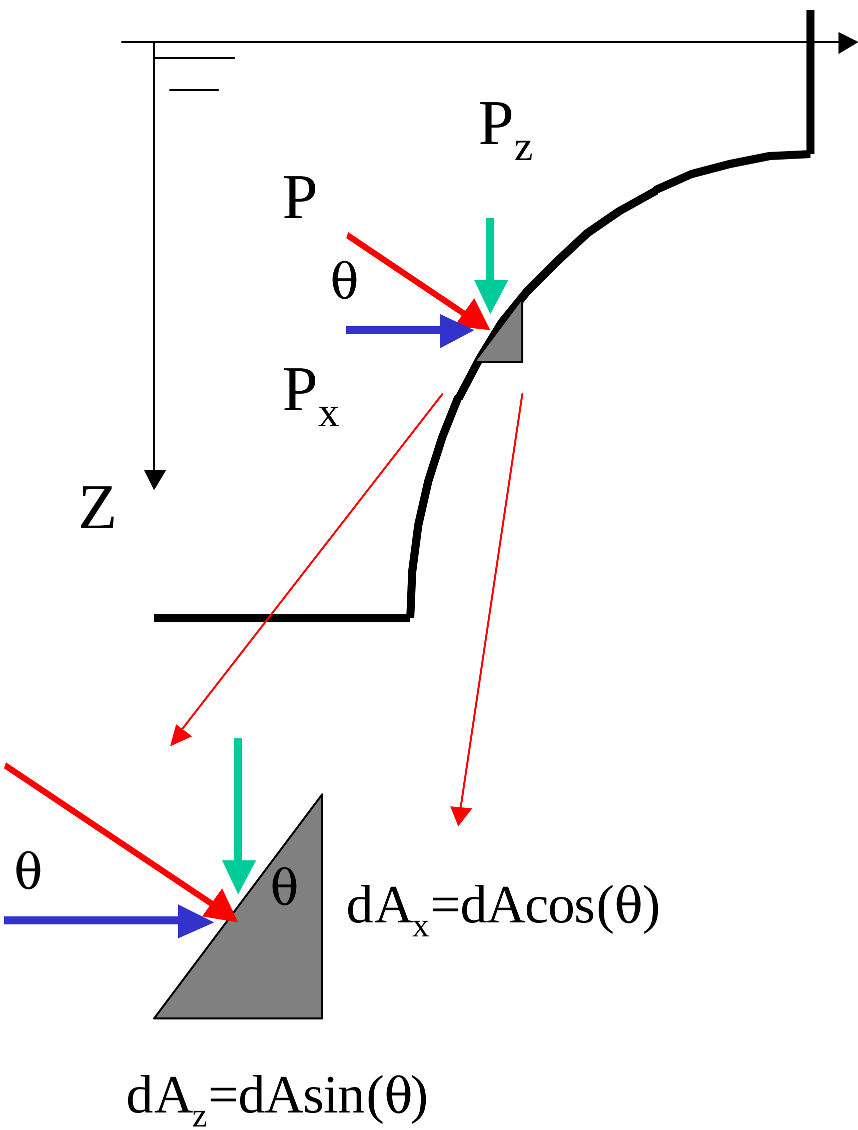
Hydrostatic Forces on Curved, Submerged Surfaces



Pressure is always acting perpendicular to the solid surface since there is no shear motion in static condition.

$$\vec{P}=P\cos(\theta)\vec{i}+P\sin(\theta)\vec{j}$$

Integrate over the entire surface

$$d\vec{F}=dF_x\vec{i}+dF_z\vec{j}=\vec{P}dA$$

$$=(P\cos(\theta)\vec{i}+P\sin(\theta)\vec{j})dA$$

$$dF_x=PdA\cos(\theta)=PdA_x$$

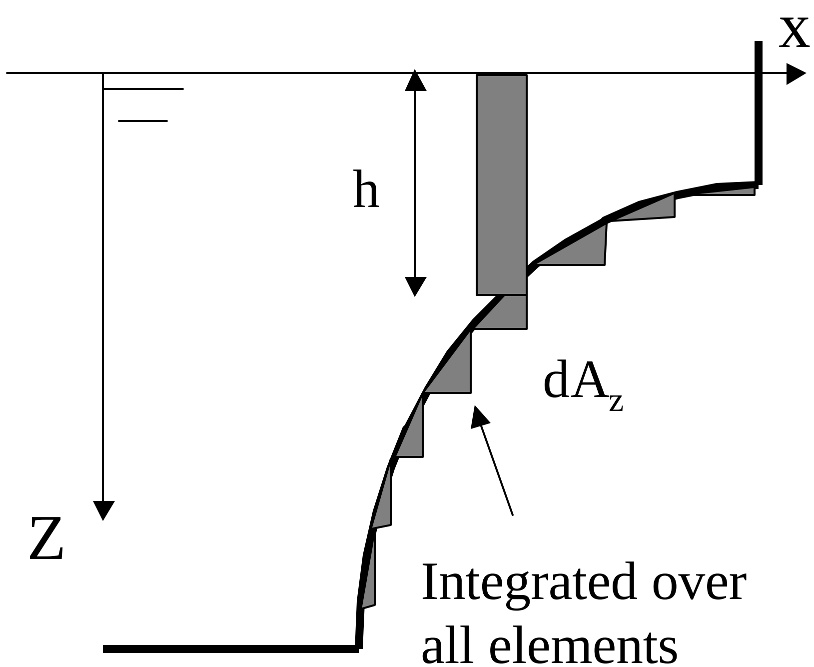
$$dF_z=PdA\sin(\theta)=PdA_z$$

Projected Forces

$$\mathrm{dF_x} = PdA\cos(q) = PdA_x, \quad dF_Z = PdA\sin(q) = PdA_Z$$

Integrate over the entire surface:

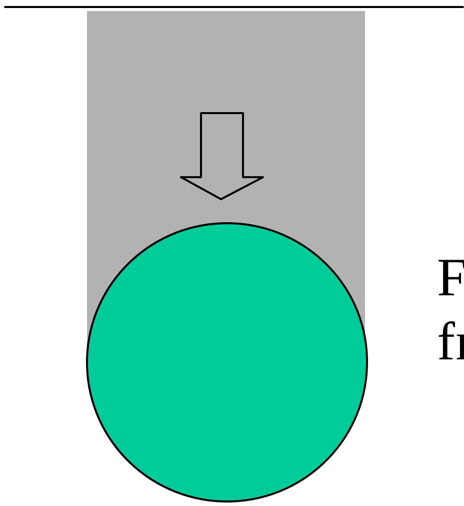
$$\mathrm{F_x} = \int dF_x = \int PdA_x = \int r\,ghdA_x, \quad \mathrm{F_z} = \int dF_z = \int PdA_z = \int r\,ghdA_z$$



$$F_z = r\,g\int hdA_z = r\,g\forall$$

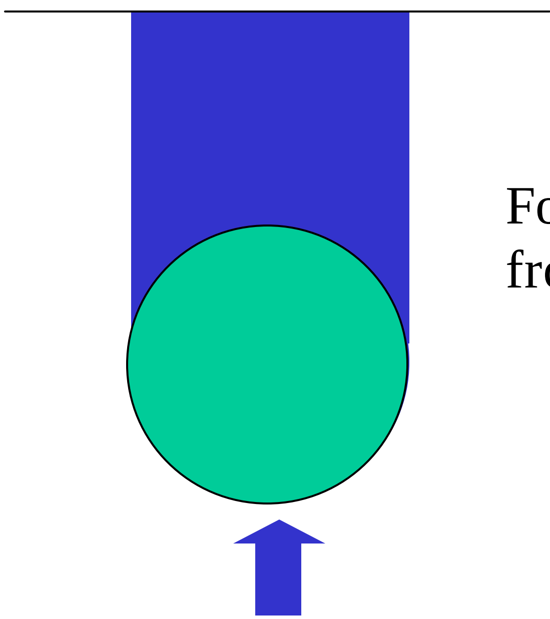
where $\forall$ is the fluid volume enclosed between the curved surface and the free surface. The force is equal to the weight of the total colume of fluid directly above the curved surface. The line action passes through the center of gravity of the volume of liquid being displaced.

Buoyancy



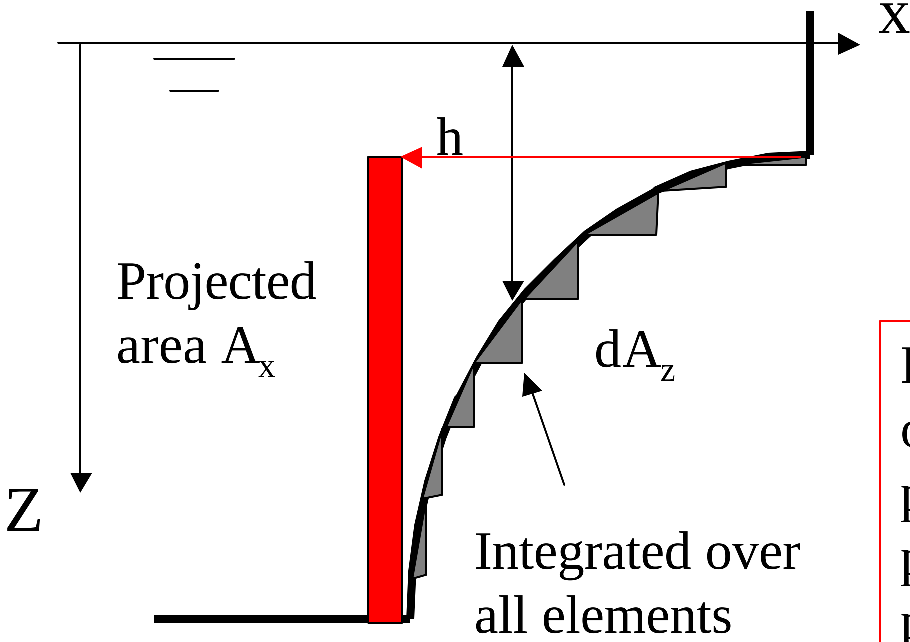
Force acting down $F_D = \rho g V_1$
from

Buoyancy = $F_U - F_D$
$= \rho g (V_2 - V_1) = \rho g V$
V: volume occupied by the object



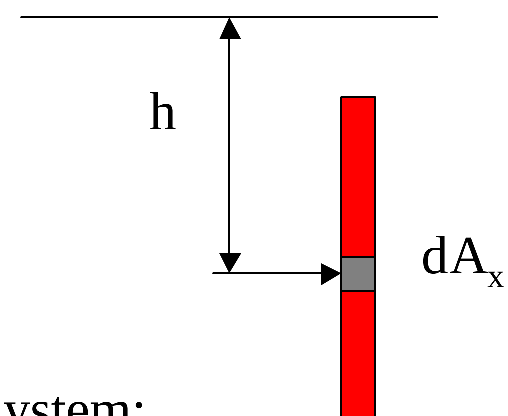
Force acting up $F_U = \rho g V_2$
from 

Horizontal Forces



$$F_x = \int dF_x = \int P dA_x = \int r\, g h dA_x$$

Finding F_x is to determine the force acting on a plane submerged surface oriented perpendicular to the surface. A_x is the projection of the curved surface on the yz plane. Similar conclusion can be made to the force in the y direction F_y.

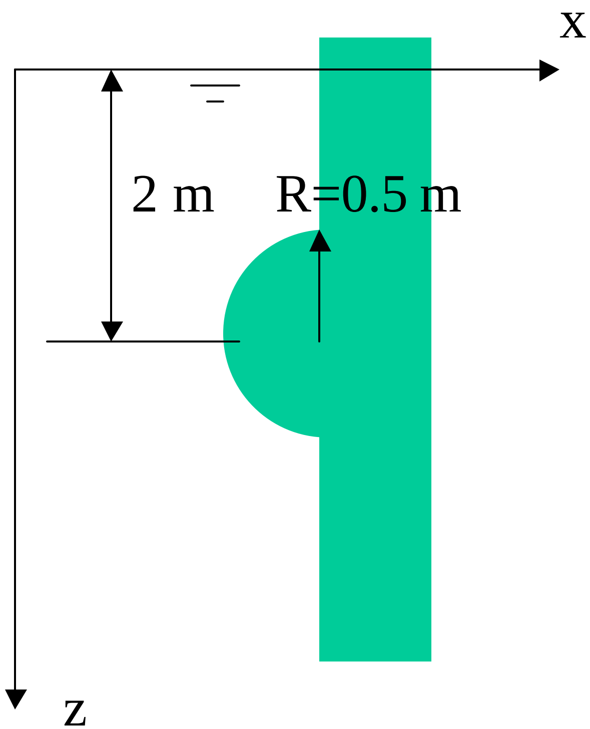


$$F_x = \int dF_x = \int P dA_x = \int r\, g h dA_x$$

Equivalent system:
A plane surface perpendicular to the free surface

Examples

Determine the magnitude of the resultant force acting on the hemispherical surface.



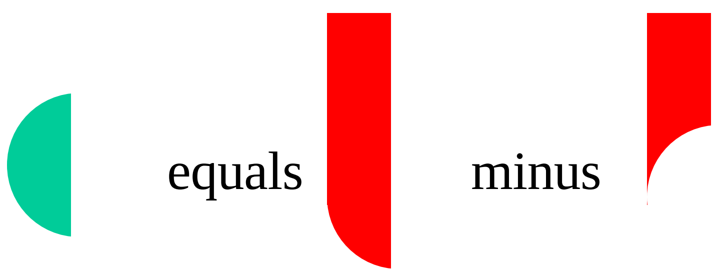
$$F_x = \int P dA_x = r g \int y dA_x = r g h_C A_x$$

A_x is the projection of the sphere along the x direction and it has a shape of a circle $A_x = p(0.5)^2 = 0.785(m^2)$

$$F_x = (1000)(9.8)(2)(0.785) = 15386(N)$$

$$F_z = \int P dA_z = r g \int z dA_z = r g \forall = r g(\forall^+ - \forall^-)$$

$$= (1000)(9.8)\left(\tfrac{1}{2}\right)\left(\tfrac{4}{3} p R^3\right) = 2564(N)$$

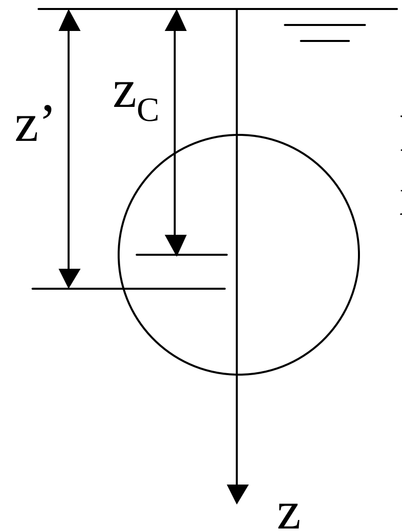


$$\vec{F} = F_x \vec{i} + F_z \vec{k} = 15386\vec{i} - 2564\vec{k}$$

The line of action of the force must go through the center of the hemisphere, O (why?)

Line of Action

Horizontal direction: line of action goes through z'

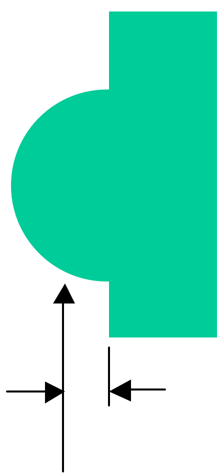


Projection
in x-direction

$$z' = z_C + \frac{I_{\hat{x}\hat{x}}}{A z_C} = (2) + \frac{\pi R^4 / 4}{\pi R^2 (2)}$$

$$= 2.03125(m)$$

Vertical direction: the line of action is 3R/8 away from the center of the hemisphere



The resultant moment of both forces with respect to the center of the hemisphere should be zero:
$F_x(2.03125-2)-F_z(0.1875)$
$=15386(0.03125)-2564(0.1875)=0$

location of the centroid for a hemisphere
is 3R/8=0.1875(m) away from the equator plane