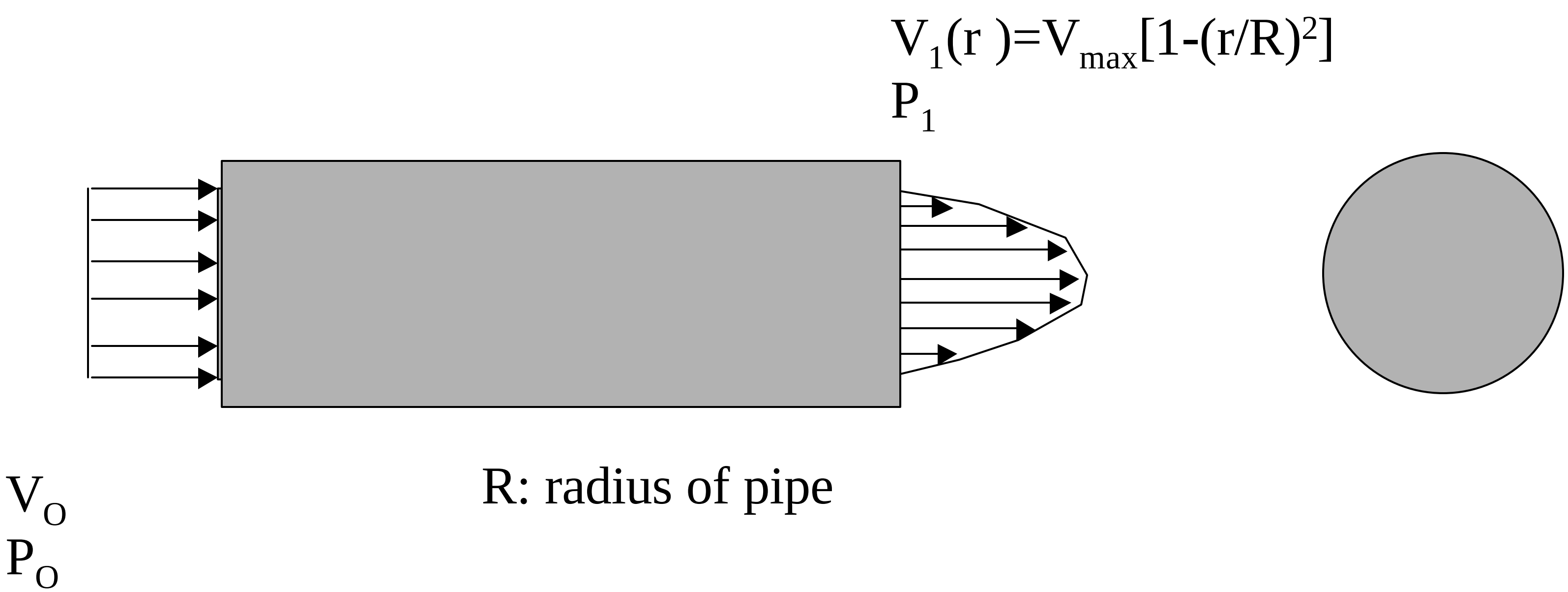
Pipe Flow Example

Water flows steadily into the circular pipe with a uniform inlet velocity profile as shown. Due to the presence of viscosity, the velocity immediately adjacent to the inner pipe wall becomes zero and this phenomenon is called the no-slip boundary condition. It is found out that the velocity distribution reaches a parabolic profile at a distance downstream of the entrance and can be represented as $V_1(r)=V_{max}[1-(r/R)^2]$, where V_{max} is the velocity at the center of the pipe and r is the radial distance measured away from the center axis. Use the mass conservation equation, determine $V_{max.}$



Mass Conseration

Mass conservation: $\frac{\partial}{\partial t}\int_{CV} r\, d\forall + \int_{CS} r\vec{V}\cdot d\vec{A} = 0$

0, steady state

$$\int_{IN} r\vec{V}\cdot d\vec{A} + \int_{OUT} r\vec{V}\cdot d\vec{A} = 0$$

$$-rV_O A_O + \int_{OUT} r V_1 dA_1 = 0, \text{ since } \vec{V} \,\&\, d\vec{A} \text{ are in the same direction}$$

$$V_O A_O = \int_{r=0}^{r=R} V_1(r)(2p\, rdr), \text{ since } dA_1 = d(p\, r^2) = 2p\, rdr$$

$$V_O(p R^2) = \int_{r=0}^{r=R} V_{max}\left(1 - \frac{r^2}{R^2}\right)(2p\, rdr) = 2p\, V_{max}\int_0^R \left(r - \frac{r^3}{R^2}\right)dr$$

$$= 2p\, V_{max}\left(\frac{R^2}{4}\right) = \frac{1}{2}p R^2 V_{max}$$

Therefore, $V_{max} = 2V_O$

Momentum Conservation

Assume there is no significant forces acting between the pipe and the fluid except the pressure forces normal to the pipe inlet (P_O) and the section 1 (P_1). Use linear momentum conservation equation, estimate the pressure difference between these two sections in order to accelerate the velocity profile inside the pipe from the inlet to a parabolic profile at section 1. Assume the pressure is uniform both at the inlet and section 1.

$$\text{Momentum conservation: } \vec{\mathrm{F}}_{\mathrm{S}} + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} r\vec{V}d\forall + \int_{\mathrm{CS}} r\vec{V}(\vec{\mathrm{V}}\cdot \mathrm{d}\vec{\mathrm{A}})$$

$$P_O A_O - P_1 A_1 + R_x = \int_{\text{inlet}} r\vec{V}(\vec{\mathrm{V}}\cdot \mathrm{d}\vec{\mathrm{A}}) + \int_{\text{section 1}} r\vec{V}(\vec{\mathrm{V}}\cdot \mathrm{d}\vec{\mathrm{A}})$$

$$(P_O - P_1)A_O = -rV_O^2 A_O + \int_0^R rV_1^2 d(p r^2)$$

$$= -rV_O^2 A_O + \int_0^R rV_{\max}^2\left(1-\frac{r^2}{R^2}\right)(2p\,rdr)$$

Momentum Conservation (cont.)

$$(P_O - P_1)A_O = -rV_O^2 A_O + \int_0^R rV_{\max}^2 \left(1 - \frac{r^2}{R^2}\right)^2 (2p\, rdr)$$

$$= -rV_O^2 A_O + 2p rV_{\max}^2 \int_0^R \left(1 - \frac{2r^2}{R^2} + \frac{r^4}{R^4}\right)(rdr)$$

$$= -rV_O^2 A_O + 2p rV_{\max}^2 \int_0^R \left(r - \frac{2r^3}{R^2} + \frac{r^5}{R^4}\right)dr$$

$$= -rV_O^2 A_O + 2p rV_{\max}^2 \left[\frac{r^2}{2} - \frac{2r^4}{4R^2} + \frac{r^6}{6R^4}\right]_{r=0}^{r=R}$$

$$= -rV_O^2 A_O + 2p rV_{\max}^2 \left(\frac{R^2}{6}\right) = -rV_O^2 A_O + \left(\frac{4}{3}\right) rV_O^2 A_O$$

$$P_O - P_1 = \left(\frac{1}{3}\right) rV_O^2$$