# Governing Equations in Differential Form

Very often, we would like to examine the detailed variation of a fluid flow field instead of just evaluating the integral effects. For example, local flow behavior near the solid surface determines both the convective heat transfer and the skin friction between the surface and the fluid.

In these situations, governing equations in a differentil form are needed:

The integral formula such as the mass conservation equation:

$\frac{\partial}{\partial t}\int_{CV} r\, d\forall + \int_{CS} r\vec{V}\cdot d\vec{A} = 0$ is still valid. However, it has to be evaluated inside an infinitesimal element in order to be able to predict the local flow behavior.

- Take the limit such that the control volume approach to infinitesimal small: consequently, all fluid properties within the volume can be considered constant.
- Use the divergence theorem to convert the surface integration term into a volume integration term: $\int_{CS} \vec{B}\cdot d\vec{A} = \int_{CV} \nabla\cdot\vec{B}d\forall$

# Continuity Equation

$$\frac{\partial}{\partial t}\int_{CV} r\, d\forall + \int_{\mathrm{CS}} r\,\vec{\mathrm{V}}\cdot \mathrm{d}\vec{\mathrm{A}} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\int_{CV} r\, d\forall + \int_{CV} \nabla\cdot(r\vec{V})d\forall = 0 \text{ using divergence theorem}$$

Also, taking the CV to the limit of infinitesimally samll

$$\frac{\partial r}{\partial t} + \nabla\cdot(r\vec{V}) = 0 \quad \text{this is the continuity equation (mass conservation)}$$

In Cartesian coordiante: $\frac{\partial r}{\partial t} + \frac{\partial(r u)}{\partial x} + \frac{\partial(r v)}{\partial y} + \frac{\partial(r w)}{\partial z} = 0$

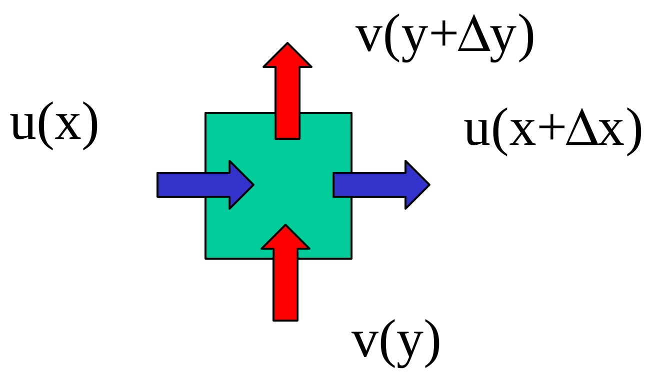
Special cases: $\Rightarrow$ steady state $\frac{\partial}{\partial \mathrm{t}} = 0,\ \nabla\cdot(r\vec{V}) = 0,\ \frac{\partial(r u)}{\partial x} + \frac{\partial(r v)}{\partial y} + \frac{\partial(r w)}{\partial z}$

$\Rightarrow$ Incompressible $r$ =constant

$$r\nabla\cdot\vec{\mathrm{V}}=0,\ \nabla\cdot\vec{\mathrm{V}}=0,\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

# Physical Interpretation

Consider two dimensional flow: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$ on a small fluid CV $\Delta x \Delta y$




Situation 1: $\frac{\partial u}{\partial x} > 0, \quad u(x+\Delta x) > u(x)$

More fluid leaving CV than entering along the x -direction, therefore, there should be more fluid entering than leaving along the y-direction.

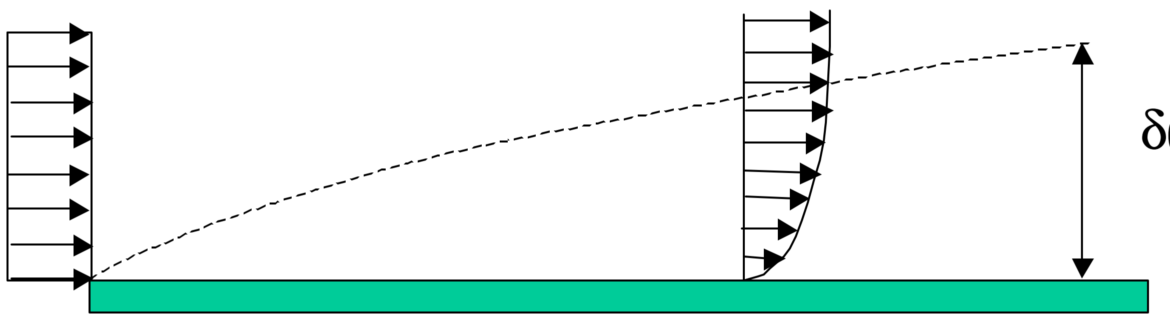
$v(y) > v(y+\Delta y)\text{:}\ \frac{\partial v}{\partial y} < 0$

Situation 2: $\frac{\partial u}{\partial x} < 0, \quad u(x+\Delta x) < u(x).$ More fluid entering CV than leaving along the x -direction, therefore, there should be more fluid leaving than entering along the y-direction. $v(y) < v(y+\Delta y)\text{:}\ \frac{\partial v}{\partial y} > 0$

Both situations should satisfy $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$ mass conservation

# Example (suction boundary layer control)

A laminar boundary layer can be approximated as having a velocity profile $u(x)=Uy/\delta$, where $\delta=cx^{1/2}$, c is a constant, U is the freestream velocity, and $\delta$ is the boundary layer thickness. Determine the v(vertical component) of the velocity inside the boundary layer.




As the boundary layer grows downstream, the u-velocity is slowed down by the presence of viscous effect and the no-slip condition at the solid surface. In order to satisfy the mass conservation equation, the v-velocity should be positive and removing the fluid away from the boundary layer.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad \frac{\partial}{\partial x}\left(\frac{Uy}{cx^{1/2}}\right)=\frac{-Uy}{2cx^{3/2}}=-\frac{\partial v}{\partial y}$$

# Example (cont.)

Integrate the v with respect to y: $v = \dfrac{Uy^2}{4cx^{3/2}} = \left(\dfrac{Uy}{cx^{1/2}}\right)\left(\dfrac{y}{4x}\right) = \dfrac{uy}{4x}$

The velocity ratio: v/u = y/4x increases away from the surface at a fixed x position;it decreases  further downstream at a fixed y location.

At the edge of the boundary layer $y = \boldsymbol{d} = cx^{1/2}$, $\quad v/u = \dfrac{c}{4x^{1/2}}$

It also decreases further downstream.