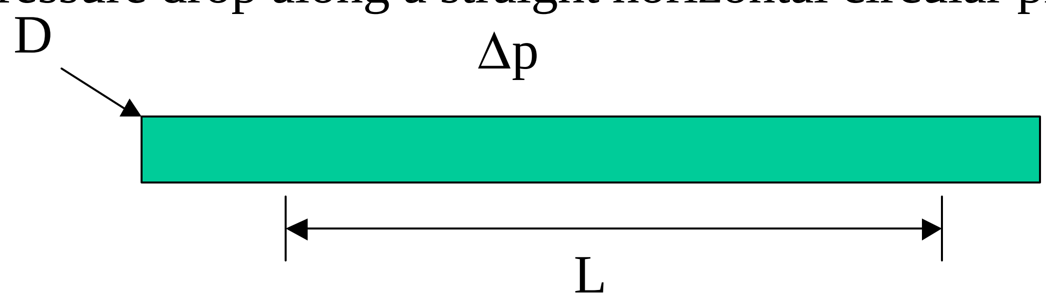
Buckingham Pi Theorem

This example is the same as example 7.2 in the textbook except that we assume the pipe is a smooth pipe. Using Buckingham Pi theorem, determine the dimensionless Π parameters involved in the problem of determining pressure drop along a straight horizontal circular pipe.



- Relevant flow parameters: Δp pressure drop, ρ density, V averaged velocity, μ viscosity, L pipe length, D pipe diameter. Therefore the pressure drop is a function of five variables. $\Delta p = f_1(\rho, V, \mu, L, D)$ **See step 1 p. 300**

Buckingham theorem states that the total number of these relevant dimensional parameters (n) can be grouped into n-m independent dimensionless groups. The number m is usually equal to the minimum of independent dimensions required to specify the dimensions of all relevant parameters.

Dimensional Analysis

➢ Primary dimensions M(mass), L(length), t(time), and T(temperature).
Example: to describe the dimension of density ρ, we need M and L
$[\rho]=[M/L^3]$, $[\Delta p]=[F/A]=[ma/A]=[ML/t^2/L^2]=[M/(Lt^2)]$
Similarly, $[\mu]=[M/(Lt)]$, $[V]=[L/t]$, $[L]=[L]$, $D=[L]$ **See steps 2 & 3 in p. 301**

Therefore, there are a total of three (3) primary dimensions involved: M, L, and t. We should be able to reduce the total number of the dimensional parameters to (6-3)=3.

➢ Now, we need to select a set of dimensional parameters that collectively they includes all the primary dimensions. We will select three since we have three primary dimensions involved in the problem. **See step 4 in p. 301**
Special notes: do not include μ into this set since it is usually less important compared to other parameters such as ρ (density), V(velocity) and a length scale.

We will select ρ, V and D for this example

Π Groups

➢ Set up dimensionless Π groups by combining the parameters selected previously with the other parameters (such as Δp, μ and L in the present example), one at a time. Identify a total of n-m dimensionless Π groups. You have to solve the dimensional equations to make sure all Π groups are dimensionless.

The first group: $\Pi_1 = \rho^a V^b D^c \Delta p$, a, b & c exponents are needed to non-dimensionalize the group. In order to be dimensionless:

$$[\frac{M}{L^3}]^a[\frac{L}{t}]^b[L]^c[\frac{ML}{t^2}] = M^0 L^0 t^0$$

So that a + 1 = 0, -3a + b + c + 1 = 0 & -b-2 = 0

Solved a = -1, b = -2, c = 0.

Therefore, the first Π group is $\Pi_1 = \frac{\Delta p}{r V^2}$

Π Groups

Use similar strategy, we can find the other two Π groups:

$$\Pi_2 = \frac{m}{rVD}, \qquad \Pi_3 = \frac{L}{D}$$

The functional relationship can be written as

$$\Pi_1 = f_2(\Pi_2, \Pi_3) \text{ or } \frac{\Delta \mathrm{p}}{r\mathrm{V}^2} = f_2\left(\frac{m}{rVD}, \frac{L}{D}\right)$$

Therefore, the pressure drop in the pipe is a function of only two parameters: the Reynolds number and the ratio between its length and diameter.

It can be understood that the pressure drop is linearly proportional to the length of the pipe. This has also been confirmed experimentally. Therefore:

$$\frac{\Delta p}{rV^2} = f_2\left(\mathrm{Re}, \frac{L}{D}\right) = \frac{L}{D} f_3(\mathrm{Re}), \text{ where } \mathrm{Re} = \frac{r\mathrm{VD}}{m}$$