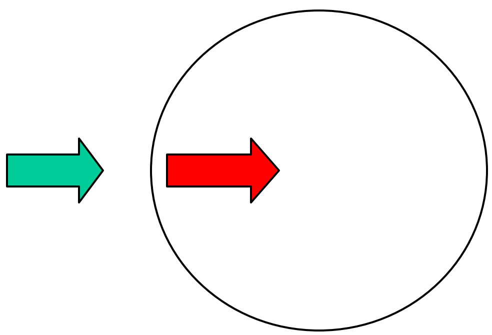
Spatial Effects

QUESTION: When can we neglect the temperature variation inside a solid? We can do that if the heat transfer inside the solid is much more effective than that occurs between the solid and the ambient fluid. If can be demonstrated as the following:



- Heat transfer from the fluid to the solid through convection: $q=hA(T_S-T_\infty)$ and the thermal resistance of this process is $R_{conv}=1/(hA)$.
- Heat transfer from the exterior of the solid to the interior of the solid is through the conductive heat transfer:

$$q=\frac{4p\,k(T_{S,1}-T_{S,2})}{(1/r_1)-(1/r_2)}$$

The thermal resistance is

$$R_{sphere}=\frac{1}{4p\,k}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

Biot Number

In order to be able to use the lumped capacitance assumption, the temperature variation inside the solid should be much smaller than the temperature difference between the surface and the fluid: $T_S - T_\infty \ll T_{S,1} - T_{S,2}$. In other words, the thermal resistance between the solid and the fluid should be much greater compared to the thermal resistance inside the solid. $R_{conv} \gg R_{solid}$.

$$\frac{1}{hA} = \frac{1}{h(4\pi r_2^2)} \text{(of the order } \frac{1}{\mathrm{hL_C}^2}\text{)} \gg \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \text{(of the order } \frac{1}{\mathrm{kL_C}}\text{)}$$

$$1 \gg \frac{h r_2^2}{k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \text{ (of the order } \frac{\mathrm{hL_C}}{\mathrm{k}} = \mathbf{Bi}, \text{ defined as the } \mathbf{Biot} \text{ number)}$$

where L_C is a characteristic length scale: relate to the size of the solid involved in the problem.

For example, $L_C = r_o/2$ (half-radius) when the solid is a cylinder.

$L_C = r_o/3$ (one-third radius) when the solid is a sphere.

$L_C = L$ (half thickness) when the solid is a plane wall with a 2L thickness.

Biot and Fourier Numbers

Define Biot number Bi=hL_C/k

In general, Bi<0.1 for the lumped capacitance assumption to be valid.

Use temperature variation of the Alumina particles in a plasma jet process as an example:

$$\frac{T(t)-T_\infty}{T_i-T_\infty}=\exp\left[-\left(\frac{hA_S}{\rho V c_P}\right)t\right]=\exp\left[-\left(\frac{hA_S}{Vk}\right)\left(\frac{kt}{\rho c_P}\right)\right]$$

For a sphere: $\frac{A_S}{V}=\frac{4\pi r_O^2}{\frac{4}{3}\pi r_O^3}=\frac{3}{r_O}=\frac{1}{L_C}$

$$\frac{T(t)-T_\infty}{T_i-T_\infty}=\exp\left[-\left(\frac{h}{kL_C}\right)\left(\frac{kt}{\rho c_P}\right)\right]=\exp\left[-\left(\frac{hL_C}{k}\right)\left(\frac{kt}{\rho c_P L_C^2}\right)\right]$$

$$=\exp\left[-(Bi)\left(\frac{\alpha t}{L_C^2}\right)\right]=\exp(-Bi\cdot\tau)$$

Define Fourier number (Fo): $\tau=\frac{\alpha t}{L_C^2}$ as dimensionless time

Spatial Effects

Re-examine the plasma jet example: h=30,000 W/m^2.K, k=10.5 W/m.K, and a diameter of 50 μm. Bi=(h/k)(r_O/3)=0.0238<0.1. Therefore, the use of the LCM is valid in the previous example. However, if the diameter of the particle is increased to 1 mm, then we have to consider the spatial effect since Bi=0.476>0.1. The surface temperature can not be considered as the same as the temperature inside the particle. To determine the temperature distribution inside the spherical particle, we need to solve the unsteady heat equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)=\frac{1}{a}\frac{\partial T}{\partial t}$$

with initial condition of $T(r,t=0)=T_i$ and boundary conditions of the form:

$$\frac{\partial T}{\partial r}=0 \text{ at r = 0: symmetric, temperature is either a max. or a min. at the center.}$$

$$-k\frac{\partial T}{\partial r}=h(T(r_O,t)-T_\infty) \text{ at r = } r_O\text{: conduction = convection at the surface.}$$

Example

The solution can be determined using the method of seapration of variables.
Equations 9-10,11,12 show the one-term approximate solutions of the exact solutions for a plane wall, a cylinder and a sphere, respectively.
This approximation can also be represented in a graphical form as shown in figures 9-13, 14 & 15. (Sometimes they were called the Heisler charts)

Terminology :

$q_O^* = \frac{q_O}{q_i} = \frac{T_O - T_\infty}{T_i - T_\infty}$ where T_O is the center temperature,

T_∞ is the ambient temperature, T_i is the initial temperature.

$\frac{Q}{Q_{max}}$ where Q is the amount of heat transfer into(or out of) the solid,

Q_{max} is the maximum amount of energy transfer when $t \to \infty$

$Q_{max} = r c_P V (T_i - T_\infty)$ or the energy to increase (or decrease) the overall temperature of the solid from T_i to T_∞.

Example (cont.)

Recalculate the plasma jet example by assuming the diameter of the particles is 1 mm. (a) Determine the time for the center temperature to reach the melting point, (b) Total time to melt the entire particle. First, use the lumped capacitance method and then compare the results to that determined using Heisler charts. ($h=30000\ W/m^2$, $k=10.5$ W/m.K, $\rho=3970\ kg/m^3$, $r_O=0.5$ mm, $c_P=1560$ J/kg.K, $T_\infty=10000$ K, $T_i=300$ K, $T_{melt}=2318$, $h_{sf}=3577$ kJ/kg, $\alpha=k/(\rho c_P)=1.695\times10^{-6}(m^2/s)$.

First, $Bi=(h/k)(r_O/3)=0.476>0.1$ should not use LCM. However, we will use it anyway to compare the difference.

$$\frac{T(t)-T_\infty}{T_i-T_\infty}=\exp(-Bi\cdot\tau),\ \text{or}\ \frac{\text{T-10000}}{\text{300-10000}}=\exp(-0.476\cdot\tau)$$

To reach T=2318 K, $\tau=\frac{\alpha t}{L_C^2}$ =0.49, or t=0.008(s)

since $L_C=\frac{r_O}{3}$ for a spherical particle.

Example (cont.)

After the particle reaches the melting point, the heat transfer will be used to supply to the solid for phase transition

$$\int_{t_1}^{t_2} q_{conv}\,dt == \Delta E_{sf} = mh_{sf}\,,\ hA_S(T_\infty - T_{melt})(t_2 - t_1) = rVh_{sf}$$

$$t_2 - t_1 = \frac{rVh_{sf}}{hA_S(T_\infty - T_{melt})} = \frac{3970(0.001)(3577000)}{6(30000)(10000 - 2318)} = 0.01(s)$$

It will take an additional 0.01 s. to melt the particle

The total time to completely melt the particle will be 0.018 s.

To consider the spatial effects, we need to use the Heisler chart:

First, $\text{Bi}^{-1} = \dfrac{k}{hr_O} = 0.7$ (Note : this definition is different from the previous Biot number used to validate the lumped capacitance method)

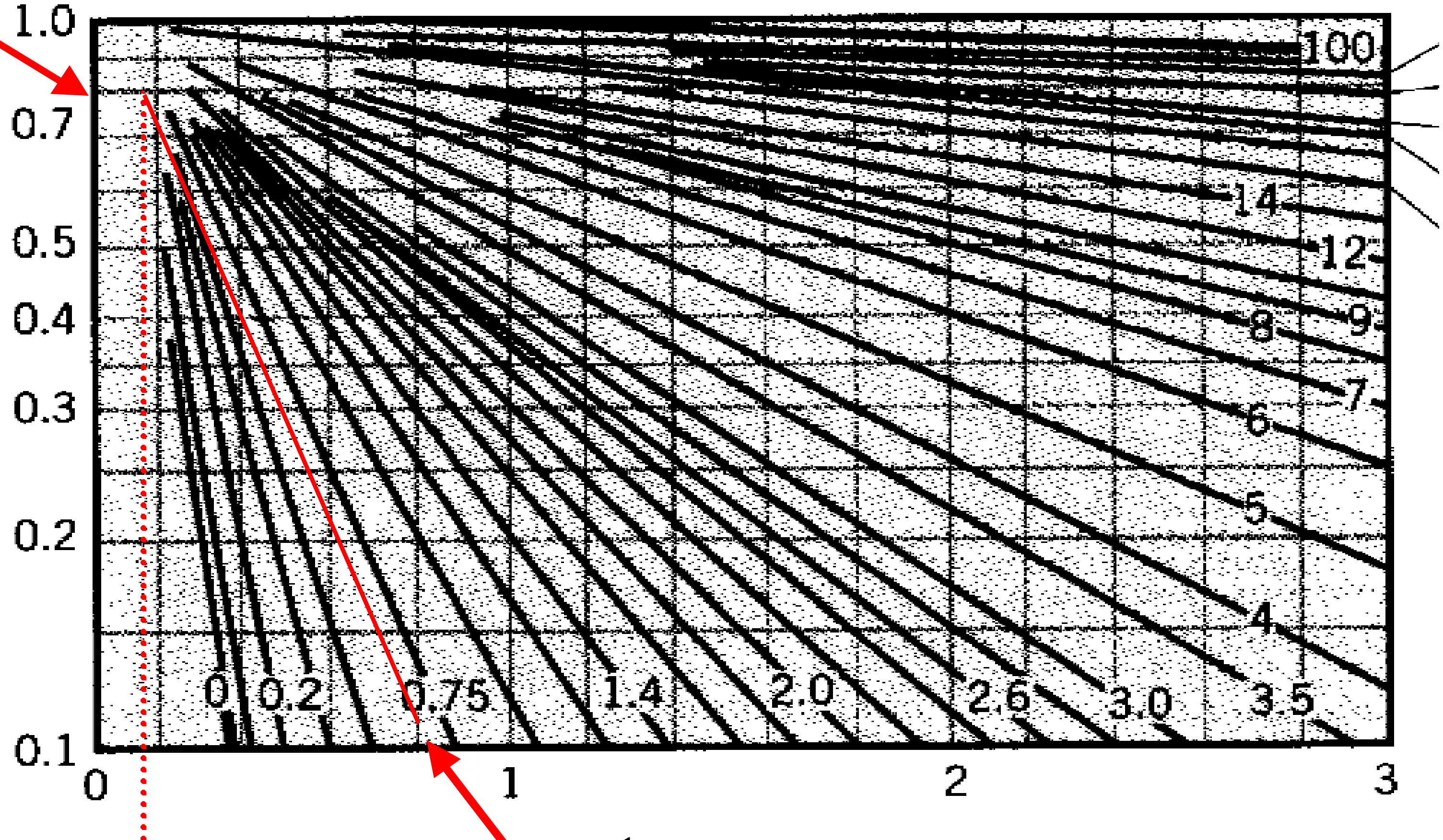
$$q_O^* = \frac{T_O - T_\infty}{T_i - T_\infty} = \frac{2318 - 10000}{300 - 10000} = 0.792.$$ Now look it up in figure 9-15

Heisler Chart

Note: this is the magnified upper left corner of Fig. 9-15

$\theta_O^*=0.792$

$$\frac{T_O - T_\infty}{T_i - T_\infty}$$



$Bi^{-1}=0.7$

$t^*=\tau$

$t^*=0.16=\alpha t/r_O^2$

$t=(r_O^2/\alpha)(0.16)=0.023$(s) much longer than the value calculated using LCM (t=0.008 s)

Approximate Solution

We can also use the one-term approximation solution in equation 9-12 to calculate.

Determine the constants in Table 9.1: for Bi=1.43 through interpolation, $A_1 = 1.362$, $l_1 = 1.768$.

$$q^* = A_1 \exp(-l_1^2 t)\frac{1}{l_1 r^*}\sin(l_1 r^*),\ r^* = (r/r_O)$$

$$q_O^* = A_1 \exp(-l_1^2 t) \text{ at the center}$$

$$t = \frac{-1}{l_1^2}\ln\left(\frac{q_O^*}{A_1}\right) = \frac{-1}{(1.768)^2}\ln\left(\frac{0.792}{1.362}\right) = 0.173$$

It is close to the value (0.16) estimated from using the chart

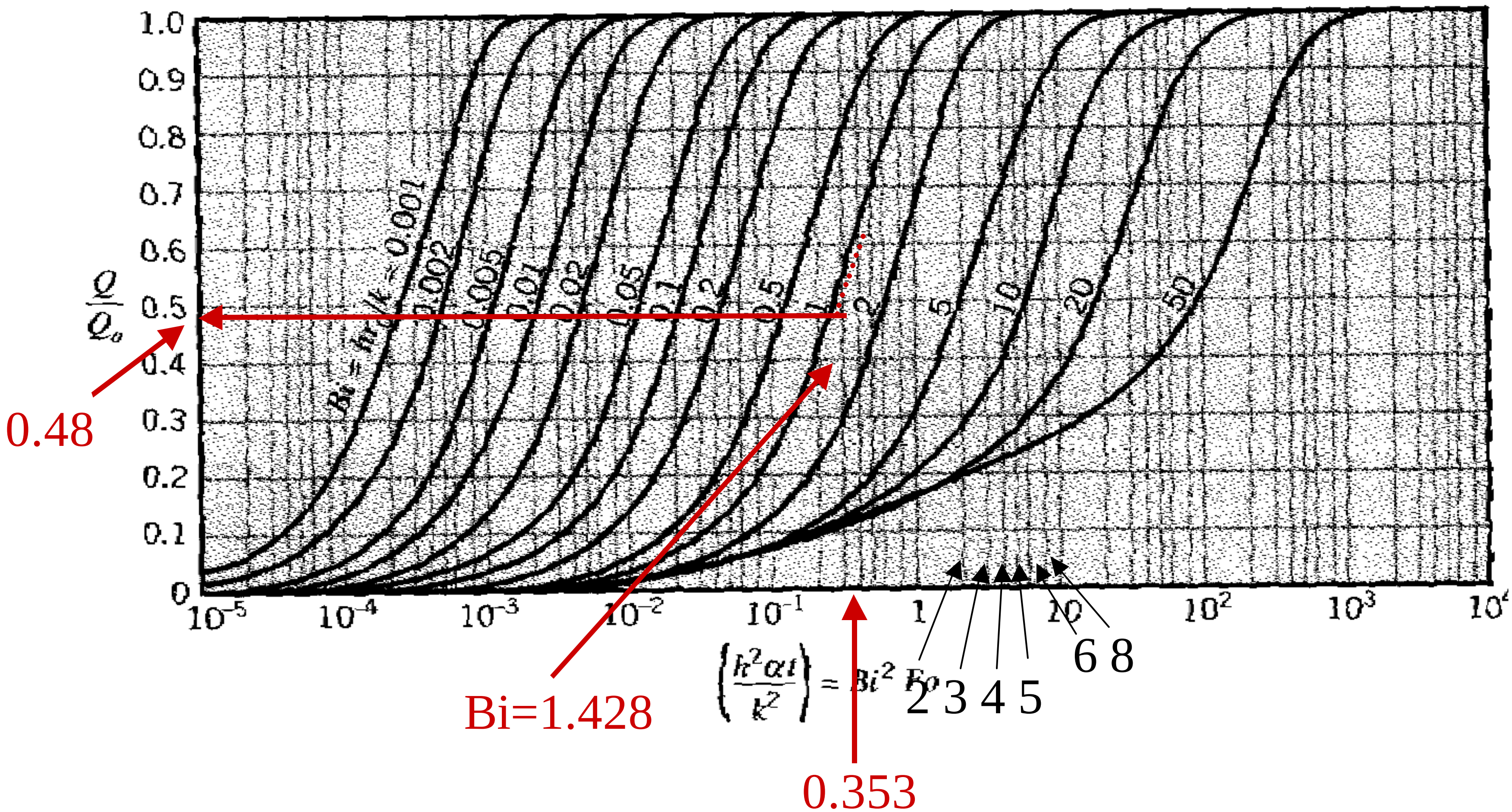
Heat Transfer

How much heat has been transfered into the particle during this period of time?

Determine $Q_{max} = r c_P V (T_i - T_\infty)$

$= (3970)(1560)(1/6)p(0.001)^2(300 - 10000) = 31{,}438(J)$

$Bi = 1.428,\ Bi^2 t = (1.428)^2(0.173) = 0.353$



Heat Transfer (cont.)

Total heat transfer during this period:
$Q=Q_{max}(Q/Q_{max})=31438(0.48)=15090(J)$

• The actual process is more complicated than our analysis. First, the temperature will not increase once the solid reaches the melting temperature unless the solid melts into liquid form. Therefore, the actual heat transfer process will probably slower than the estimation.

• If the outer layer melts then we have double convective conditions: convection from the plasma gas to the liquid alumina and then from there to the inner solid.

• To make matter even more complicated, the interface between the melt alumina and the solid is continuously moving inward.

• No analytical solutions or numerical analysis is necessary.