Thermal Conduction

Objectives

The objectives of this experiment are:

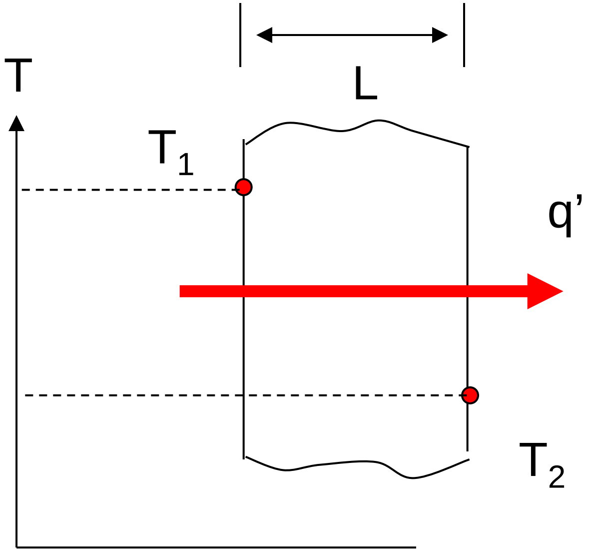
1. **Use *Fourier's Law* of thermal conduction to find out the heat transfer through a metal rod.**

2. **To estimate the *thermal contact resistance* between two metallic surfaces.**

3. **Measure and plot the temperature distribution in a 2D surface with prescribed temperature boundary conditions and compare it with analytical and numerical solutions.**

Fourier’s Law of heat conduction

- **Fourier’s law is an empirical relation based on experimental observation.**
- **It relates the heat transfer through a surface to its temperature gradient and the surface area**



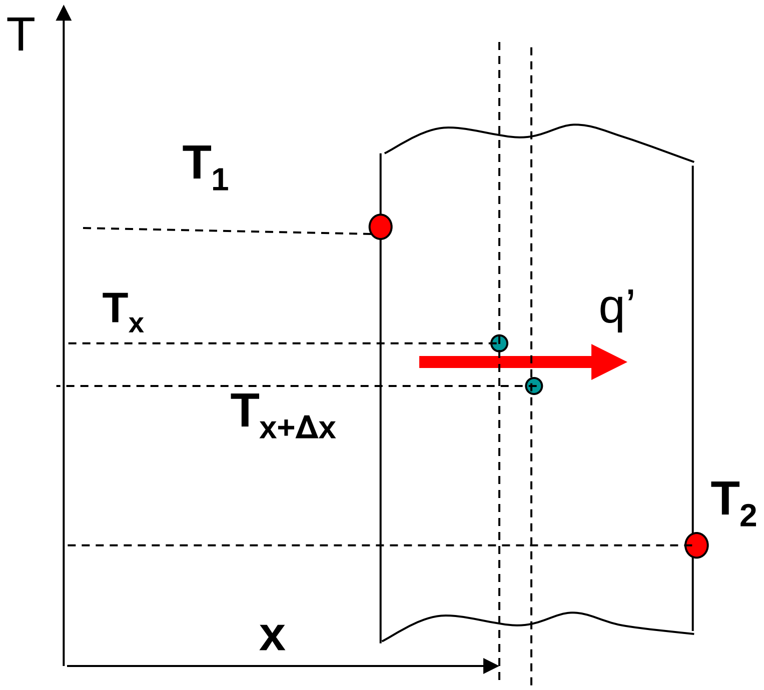
$$q' \alpha \quad A\left(\frac{T_1 - T_2}{L}\right)$$

$$\boxed{q' = kA\left(\frac{T_1 - T_2}{L}\right)}$$

Fourier law of heat conduction

- **where *k* is the proportionality constant known as the coefficient of thermal conductivity, a thermo-physical property of the material (*W/m K*)**

Fourier law of heat conduction in differential form



$$q' = kA\left(\frac{T_1 - T_2}{L}\right)$$

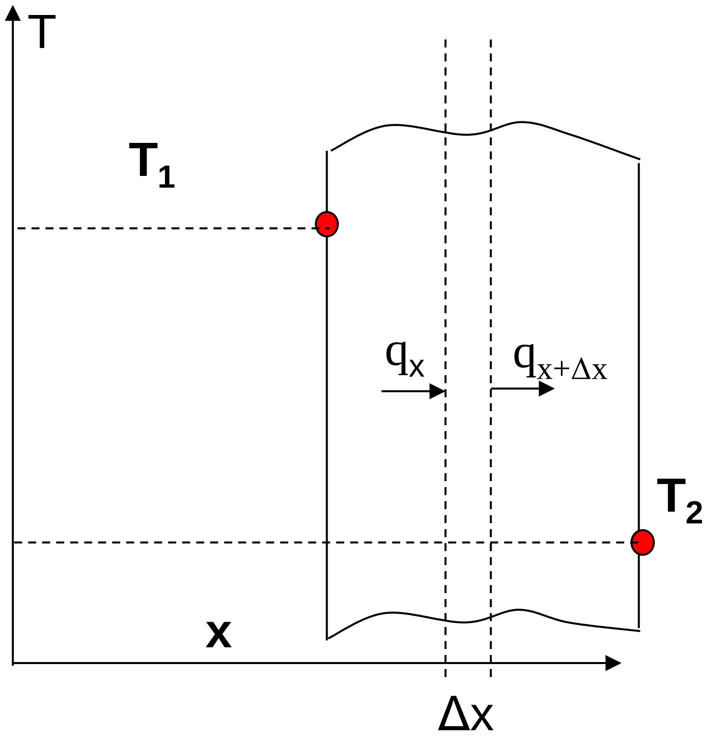
$$q' = kA\left(\frac{T_x - T_{x+\Delta x}}{\Delta x}\right)$$

The material layer Δx may be assumed to be homogenous when Δx →0

$$q' = -kA \lim_{\Delta x \to 0}\left(\frac{T_{x+\Delta x} - T_x}{\Delta x}\right)$$

$$\boxed{q' = -kA\frac{dT}{dx}}$$

Steady, 1D heat equation (without considering heat generation)



$$q' = -KA\frac{dT}{dx}$$

Goal is to find an expression for temperature distribution inside the plate

Conservation of energy in a differential element *Adx* can be written as

$$q'_x - q'_{x+\Delta x} = 0$$

$$q'_x - \left(q'_x + \frac{d\left(q'_x\right)}{dx}\Delta x \right) = 0$$

$$\cancel{q'_x} - \cancel{q'_x} + \frac{d\left(q'_x\right)}{dx}\Delta x = 0$$

$$\frac{d\left(q'_x\right)}{dx} = 0 \qquad q'_x = -kA\frac{dT}{dx} \qquad \frac{d\left(-kA\frac{dT}{dx}\right)}{dx} = 0$$

If *K* and *A* are constants

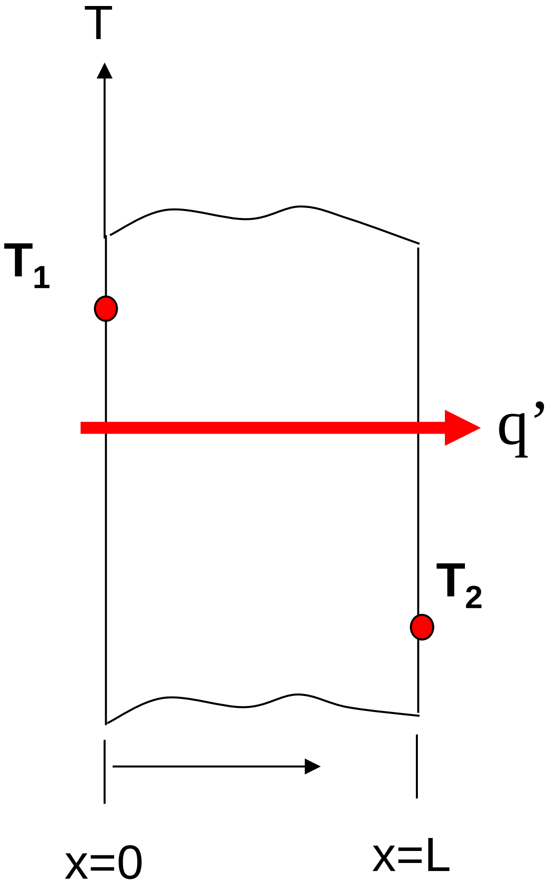
$$\frac{d\left(-kA\frac{dT}{dx}\right)}{dx} = 0 \Rightarrow \frac{d\left(\frac{dT}{dx}\right)}{dx} = 0 \Rightarrow \boxed{\frac{d^2T}{dx^2} = 0}$$

This is 1 D, steady state heat equation (without considering heat generation)

Solving this equation we get the temperature distribution as

$$T_x = C_1 x + C_2$$

Two BCs needed for finding C_1 and C_2



$$T_x = C_1 x + C_2$$

$$x = 0, \quad T = T_1 \qquad x = L, \quad T = T_2$$

$$C_2 = T_1 \qquad T_2 = C_1 L + T_1$$

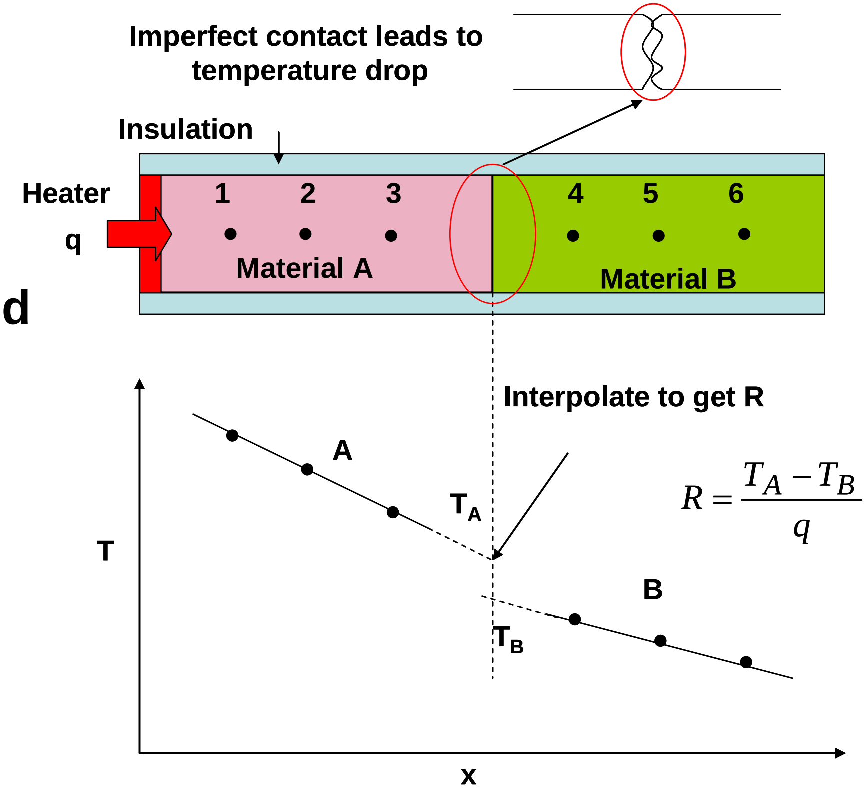
$$C_1 = \frac{T_2 - T_1}{L}$$

$$T_x = \frac{T_2 - T_1}{L} x + T_1$$

$$\boxed{T_x = T_1 - (T_1 - T_2)\frac{x}{L}}$$

Thermal contact resistance (R)

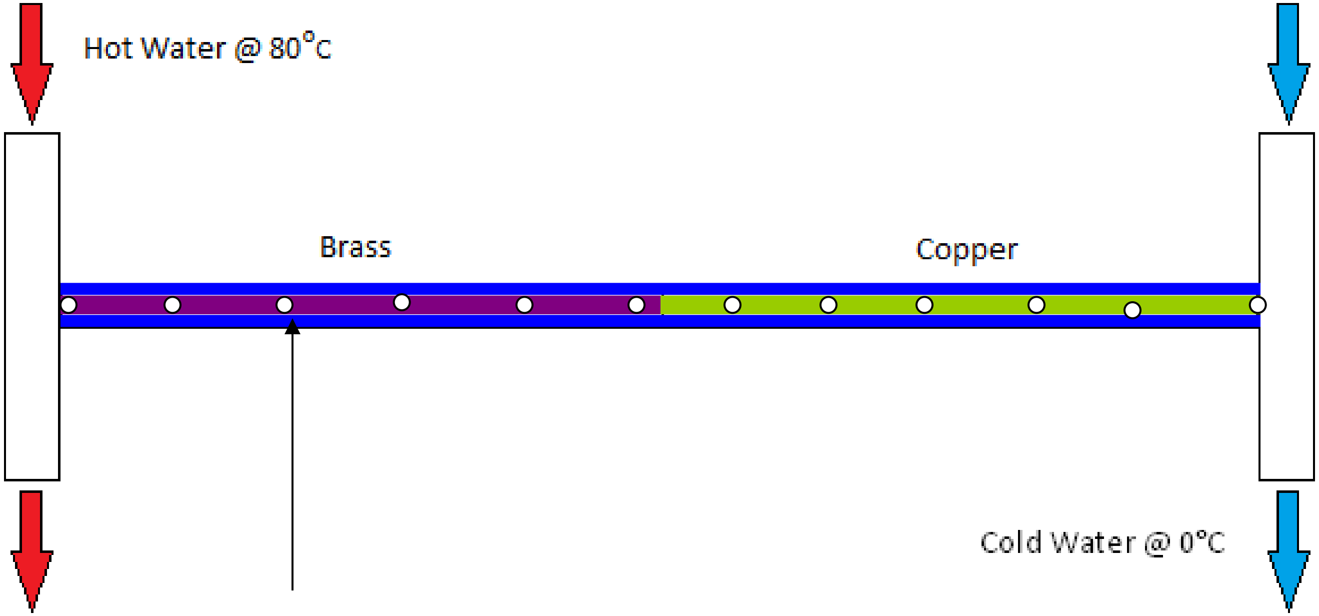
- **When heat flows between two surfaces in contact, the imperfections in the contact may lead to a temperature drop.**



- **The *thermal contact resistance, R,* is defined as the ratio of this temperature drop to the mean heat flow through the surface**

$$R = \frac{\Delta T}{q}$$

where Δ*T* the temperature drop across the surface, and q is the mean heat transfer rate through the surface.



Thermo couple locations

Schematic of 1D metal bars used for contact resistance experiment

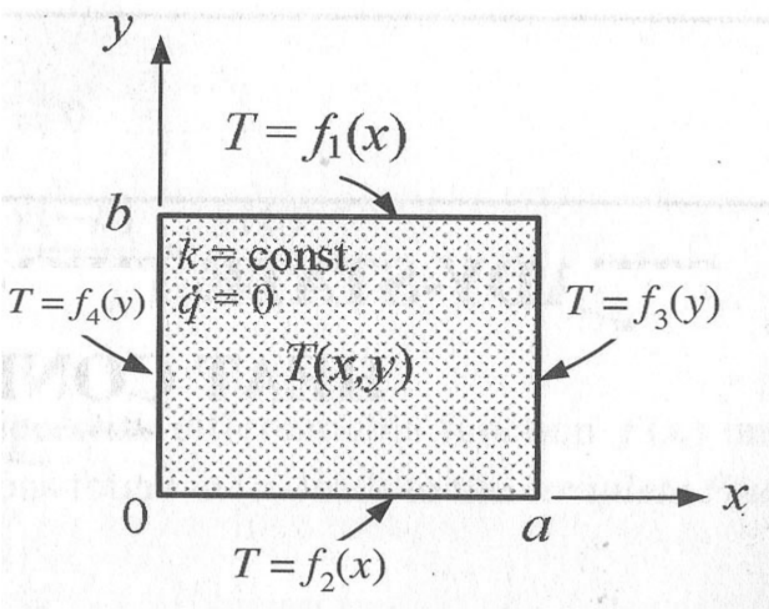
2 D heat conduction equation (steady, without heat generation)

We have derived 1D, steady heat equation as

$$\frac{d^2T}{dx^2} = 0$$

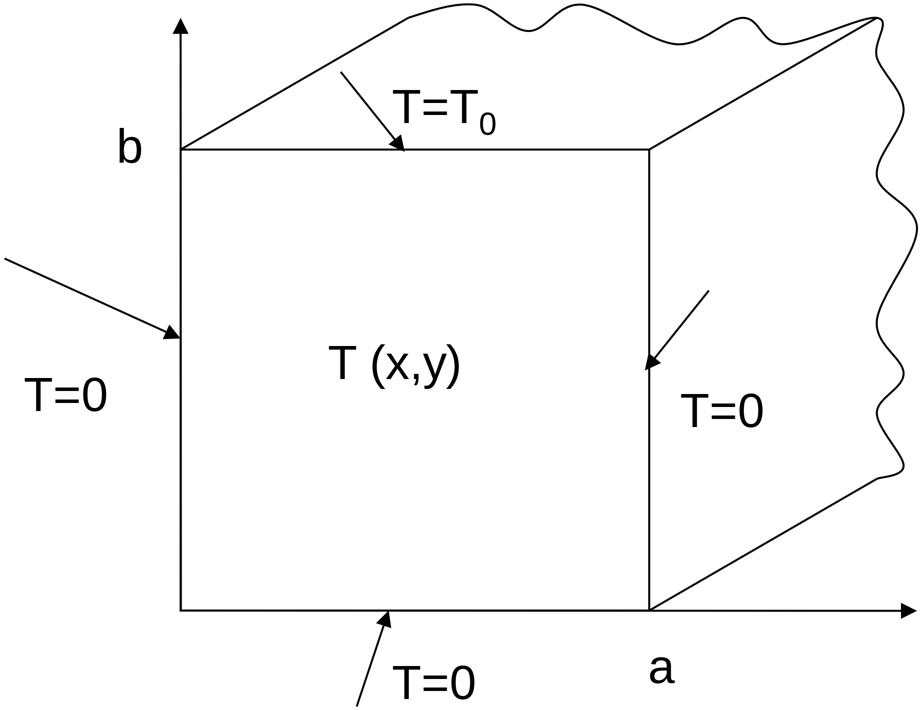
In 2D, this equation will be of the form

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0}$$



This is also called a Laplace's equation

Q.1 How many boundary conditions needed for solving this equation
Q.2 How can we solve this 2D equation and find out the temperature distribution



$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$

$T(0,y)=0, \quad 0<y<b$

$T(a,y)=0, \quad 0<y<b$

$T(x,0)=0, \quad 0<x<a$

$T(x,b)=T_0 \quad 0<x<a$

$$T(x,y)=?$$

Method of separation of variables

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$T(x, y) = ?$$

$T(0,y)=0, \quad 0<y<b$

$T(a,y)=0, \quad 0<y<b$

$T(x,0)=0, \quad 0<x<a$

$T(x,b)=T_0 \quad 0<x<a$

This method requires the assumption of the existence of a product solution of the form

$$T(x, y) = X(x)Y(y)$$

Where *X* is a function of *x* alone
***Y* is a function of y alone**

Separating the variables

$$Y\frac{d^2 X}{dx^2} + X\frac{d^2 Y}{dy^2} = 0$$

$$-\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{Y}\frac{d^2 Y}{dy^2}$$

Since each term is function of only one variable both terms must be a constant

$$-\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{Y}\frac{d^2Y}{dy^2}=\lambda$$

$$\frac{d^2X}{dx^2}+\lambda X(x)=0 \qquad \frac{d^2Y}{dy^2}-\lambda Y(y)=0$$

For $\lambda=0,\quad \lambda<0$ ***there is no solution***

For $\lambda>0$ ***the general solution is of the form***

$$X(x)=A\sin\lambda x+B\cos\lambda x$$

$$Y(y)=C\sinh\lambda y+D\cosh\lambda y$$

$$T(x,y)=\left(A\sin\ \lambda x+B\cos\ \lambda x\right)\left(C\sinh\ \lambda y+D\cosh\ \lambda y\right)$$

$$T(x,y) = \left(A\sin\lambda x + B\cos\lambda x\right)\left(C\sinh\lambda y + D\cosh\lambda y\right)$$

$T(0,y)=0, \quad 0<y<b$
$T(a,y)=0, \quad 0<y<b$
$T(x,0)=0, \quad 0<x<a$
$T(x,b)=T_0 \quad 0<x<a$

Put the BC and solve for constants A,B,C, D

The final solution can be written in the form

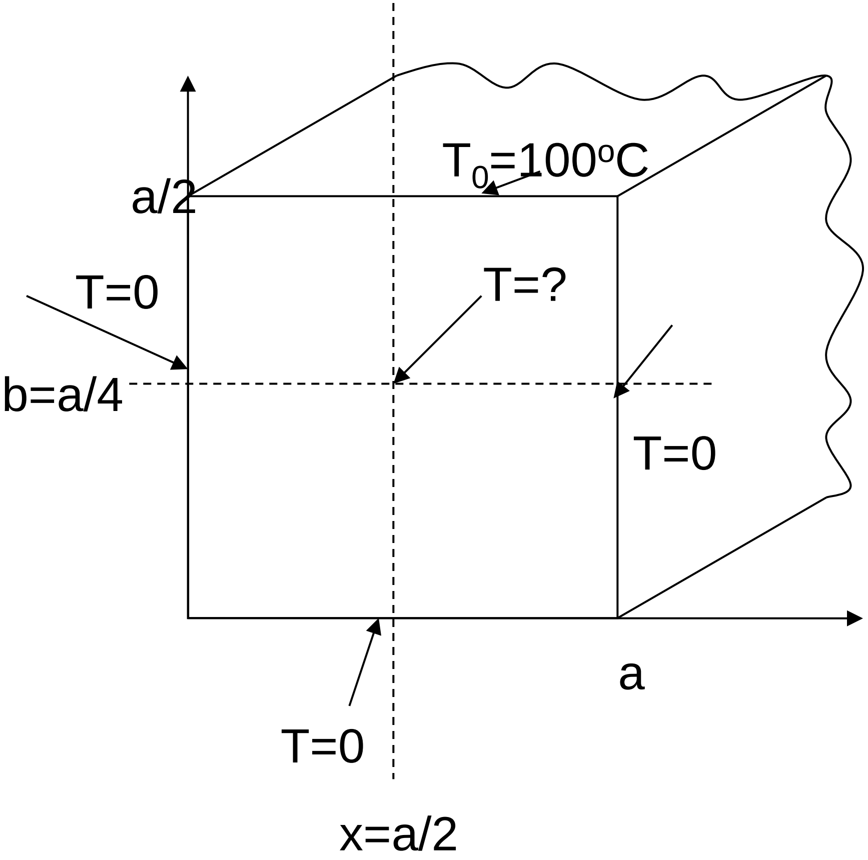
$$\frac{T(x,y)}{T_0} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(1)^n]}{n}\,\frac{\sin\frac{n\pi}{a}x\;\sinh\frac{n\pi}{a}y}{\sinh\frac{n\pi}{a}b}$$

Note :

Method of separation of variables is applicable to steady 2D problems if and when

(a) the differential equation is linear and homogenous

(b) the 4 boundary conditions are linear and 3 of them are homogeneous

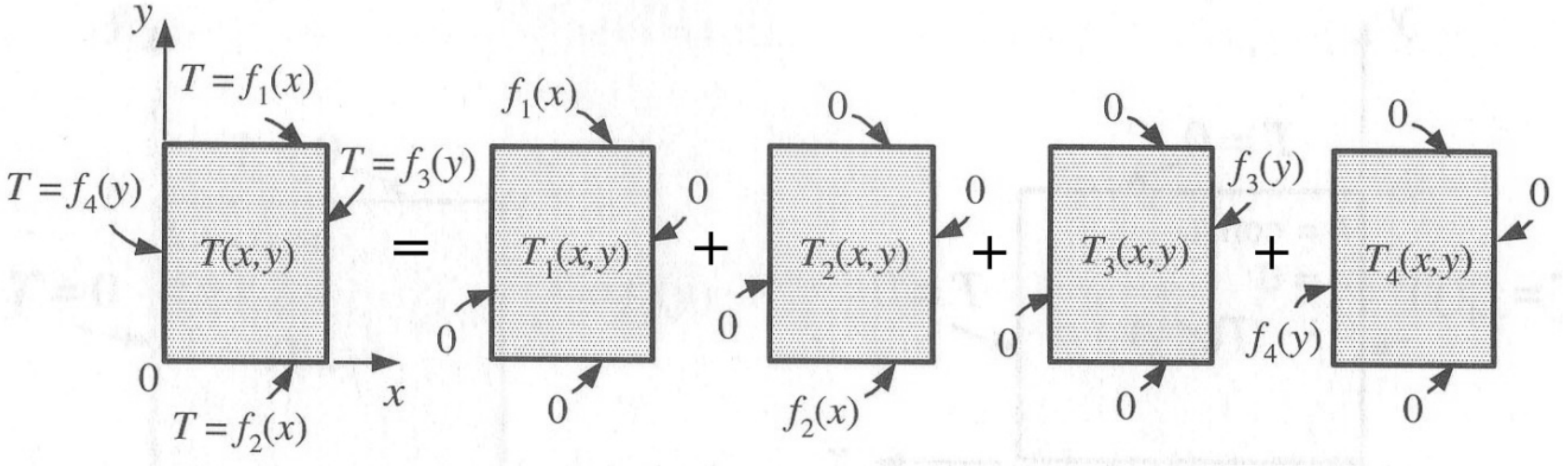


$$\frac{T(x,y)}{T_0} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(1)^n]}{n}\;\frac{\sin\frac{n\pi}{a}x\;\;\sinh\frac{n\pi}{a}y}{\sinh\frac{n\pi}{a}b}$$

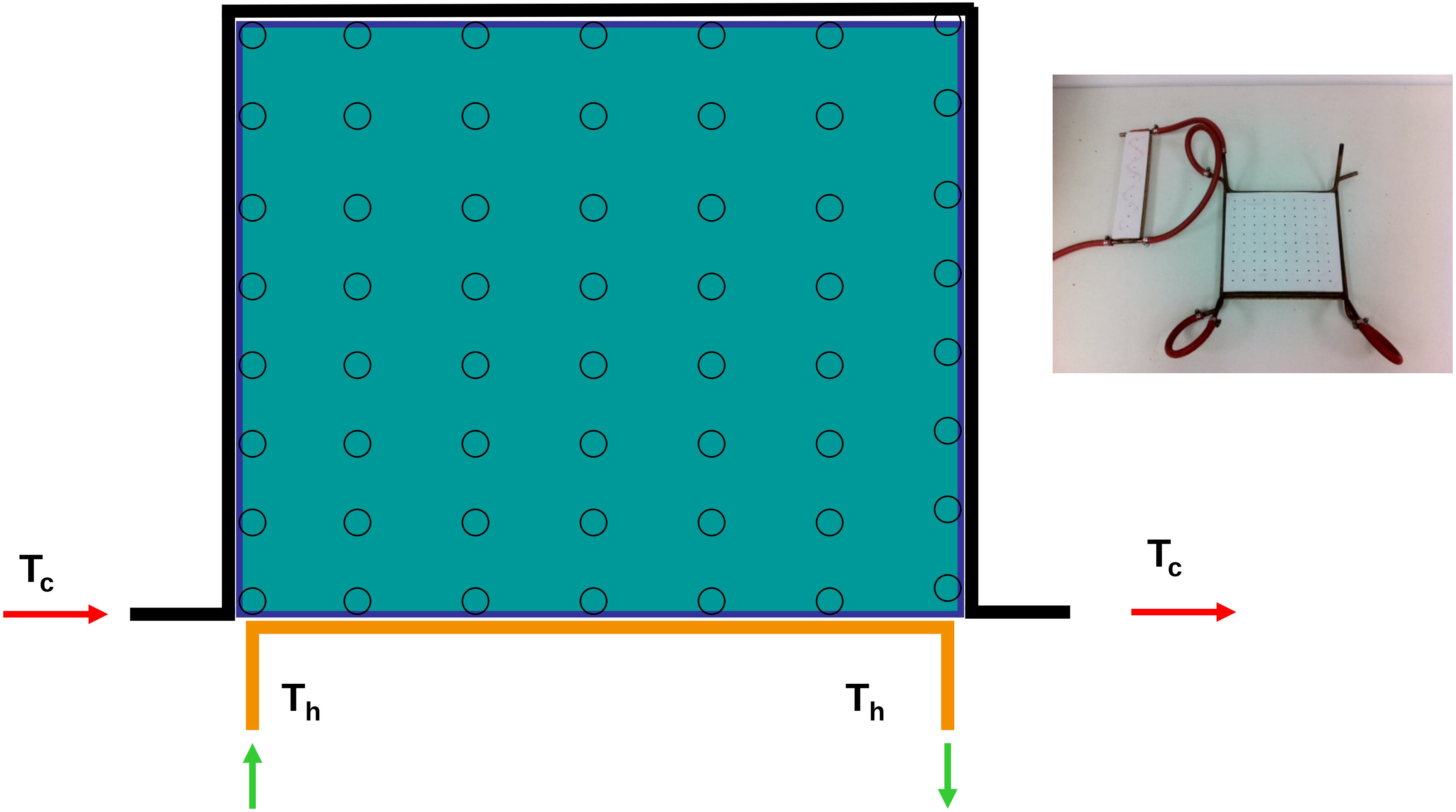
$$= 0.48061 - 0.03987 + 00502$$

$$T(a/2, a/4) = 44.576^o C$$

Method of superposition for non homogenous boundary conditions



$$\mathrm{T}(x,y) = \mathrm{T}_1(x,y) + \mathrm{T}_2(x,y) + \mathrm{T}_3(x,y) + \mathrm{T}_4(x,y)$$



Schematic of 2D metal plate used for temperature distribution measurement

The tubing at the boundary are so arranged that a desired thermal boundary conditions can be given to the plate