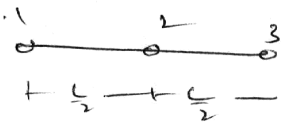
Solution

$$u = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \quad -(1)$$



$$u = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \quad -(A)$$

$x = 0, \; u = u_1$

Sub in (1) $\Rightarrow \quad u_1 = \alpha_1 \quad -(2)$

$x = \frac{L}{2}, \quad u = u_2$

Sub in (1) $\quad u_2 = \alpha_1 + \alpha_2 \frac{L}{2} + \alpha_3 \frac{L^2}{4} \quad -(3)$

$x = L, \; u = u_3$

sub in (1) $\Rightarrow \quad u_3 = \alpha_1 + \alpha_2 L + \alpha_3 L^2 \quad -(4)$

$(4) - (3) \Rightarrow \quad u_3 - u_2 = \alpha_2 \frac{L}{2} + 3\alpha_3 \frac{L^2}{4} \quad -(5)$

$(4) + (3) \Rightarrow \quad u_2 + u_3 = 2u_1 + \frac{3}{2}\alpha_2 L + \frac{5}{4}\alpha_3 L^2 \quad -(6)$

From (6) $\quad \alpha_2 \frac{L}{2} = \frac{1}{3}\left(u_2 + u_3 - 2u_1 - \frac{5}{4}\alpha_3 L^2\right) \quad -(7)$

sub (7) in (5)

$$u_3 - u_2 = \frac{1}{3}\left(u_2 + u_3 - 2u_1 - \frac{5}{4}\alpha_3 L^2\right) + 3\alpha_3 \frac{L^2}{4}$$

$$\Rightarrow \boxed{\alpha_3 = \frac{1}{L^2}\left(2u_3 - 4u_2 + 2u_1\right)} \quad -(8)$$

sub (8) in (7)

$$\alpha_2 \frac{L}{2} = \frac{1}{3}\left(u_2 + u_3 - 2u_1 - \frac{5}{4}\left(2u_3 - 4u_2 + 2u_1\right)\right)$$

$$\alpha_2 = \frac{1}{L}\left(4u_2 - u_3 - 3u_1\right)$$

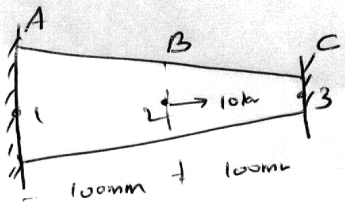
sub $\alpha_1, \alpha_2 \& \alpha_3$ in A

$$u = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ \frac{1}{L}(4u_2 - u_3 - 3u_1 \\ \frac{2}{L^2}(2u_3 - 4u_2 + 2u_1 \end{Bmatrix}$$

$$u = \begin{bmatrix} (1 - \frac{3x}{L} + \frac{2x^2}{L^2}) & \frac{4x}{L}(1 - \frac{x}{L}) & \frac{x}{L}(2\frac{x}{L} - 1) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$[N] = \begin{bmatrix} (1 - \frac{3x}{L} + \frac{2x^2}{L^2}) & \frac{4x}{L}(1 - \frac{x}{L}) & \frac{x}{L}(\frac{2x}{L} - 1) \end{bmatrix}$$

Find stresses in bars AB & BC and reaction at A & C



Section	Area (mm^2)
A	30
B	20
C	10

$\Delta T = +100^\circ C$, $E = 200\ GPa$, $\alpha = 2 \times 10^{-6}$

~~for ele~~ $$\begin{Bmatrix} f_i \\ f_j \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - EA\alpha\Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

for element ① $\quad \bar{A} = \frac{A_A + A_B}{2} = \frac{30+20}{2} = 25$

$$\begin{Bmatrix} f_{AB} \\ f_{BA} \end{Bmatrix} = \frac{25 \times 200 \times 10^3}{100}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - 200\times10^3 \times 25 \times 2\times10^{-6}\times100 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

~~for element ②~~ ~~$\bar{A} = 15\ mm^2$~~

$$\begin{Bmatrix} f_{BC} \\ f_{CB} \end{Bmatrix} = \begin{Bmatrix} f_{B2} \\ f_{2B} \end{Bmatrix} = \begin{bmatrix} 50000 & -50000 \\ -50000 & 50000 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} - \begin{Bmatrix} -1000 \\ 1000 \end{Bmatrix}$$

for element ② $\quad A = 15\ mm^2$

$$\begin{Bmatrix} f_{BC} \\ f_{CB} \end{Bmatrix} = \begin{bmatrix} 30,000 & -30000 \\ -30000 & 30000 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} - \begin{Bmatrix} -600 \\ 600 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} 50000 & -50000 & 0 \\ -50000 & 50000 + 30000 & -30000 \\ 0 & -30000 & 30000 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} - \begin{Bmatrix} -1000 \\ 1000 - 600 \\ 600 \end{Bmatrix}$$

$$\begin{bmatrix} 50000 & -50000 & 0 \\ -50000 & 80000 & -30000 \\ 0 & -30000 & 30000 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10000 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -1000 \\ 400 \\ 600 \end{Bmatrix} = \begin{Bmatrix} -1000 \\ 10400 \\ 600 \end{Bmatrix}$$

$$80000\, u_2 = 10400 \quad , \quad \boxed{u_2 = 0.13\text{mm}}$$

Reaction

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} 50000 & -5000 & 0 \\ -5000 & 80,000 & -30000 \\ 0 & -30000 & 30000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.13 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -1000 \\ 400 \\ 600 \end{Bmatrix}$$

$$= \begin{Bmatrix} -5500 \\ 10,000 \\ -4500 \end{Bmatrix}$$

Reaction at 1 = $F_1 = -5500$ N ←

Reaction at C = $F_3 = -4500$ ←

Stresses $\sigma = D(\varepsilon - \varepsilon_0) = E[B]\{d\} - E\varepsilon_0$

$$\sigma = E[B]\{d\} - E\alpha\Delta T$$

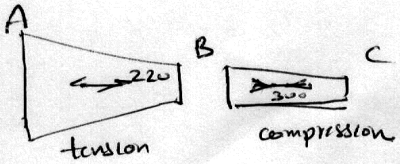
$$B = \frac{1}{L}[-1 \quad 1]$$

$$\sigma^{(1)} = \frac{200\times10^3}{100}[-1 \quad 1]\begin{Bmatrix} 0 \\ 0.13 \end{Bmatrix} - 200\times10^3 \times 2\times10^{-6} \times 100 = 220\ \text{N/mm}^2$$

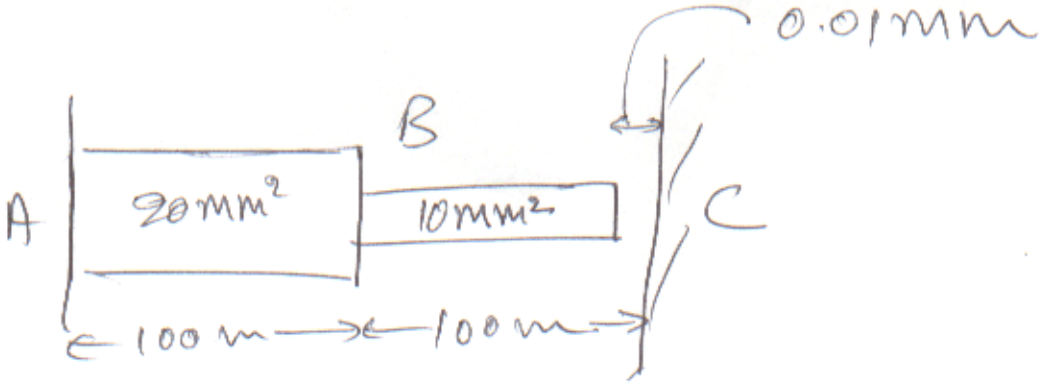
tension

$$\sigma^{(2)} = \frac{200\times10^3}{100}[-1 \quad 1]\begin{Bmatrix} 0.13 \\ 0 \end{Bmatrix} - 200\times10^3 \times 2\times10^{-6} \times 100 = -300\ \text{N/mm}^2$$

Compression



2B



$$\begin{bmatrix} f_{AB} \\ f_{BA} \end{bmatrix} = \frac{20 \times 200 \times 10^3}{100} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - 200 \times 10^3 \times 20 \times 2 \times 10^{-6} \times 100 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

for Element ①

$$\begin{bmatrix} 40,000 & -40,000 \\ -40,000 & 40000 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} -800 \\ 800 \end{bmatrix} = \begin{bmatrix} f_{AB} \\ f_{BA} \end{bmatrix}$$

for. Elemt ②

$$\begin{bmatrix} 20\,000 & -20,000 \\ -20,000 & 20,000 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} -400 \\ 400 \end{bmatrix} = \begin{bmatrix} f_{BC} \\ f_{CB} \end{bmatrix}$$

$$\begin{bmatrix} f_{AB} \\ f_B \\ f_C \end{bmatrix} = \begin{bmatrix} 40{,}000 & -40{,}000 & 0 \\ -40{,}000 & 60{,}000 & -20{,}000 \\ 0 & -20{,}000 & 20{,}000 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} -800 \\ 400 \\ 400 \end{bmatrix}$$

$u_1 = 0$

$u_2 = u_2$

$u_3 = 0.01\ mm.$

$$\begin{bmatrix} f_A \\ f_B \\ f_C \end{bmatrix} = 20{,}000 \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ 0.01 \end{bmatrix} - \begin{bmatrix} -800 \\ 400 \\ 400 \end{bmatrix}$$

$$20000\ (-2u_2) + 800 = f_A$$

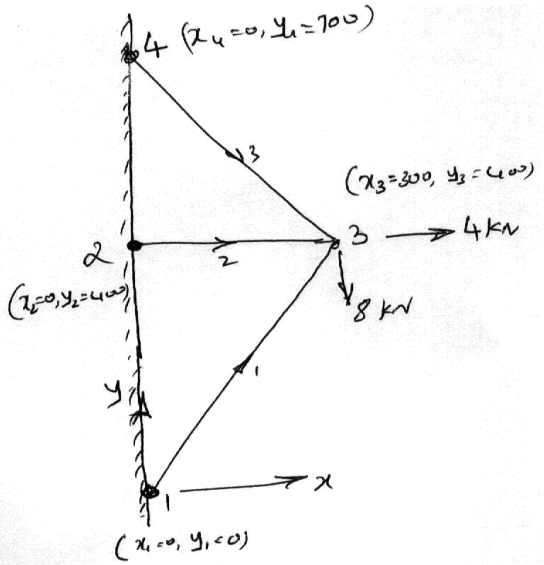
$$60{,}000\ u_2 - 20{,}000(0.01) - 400. = f_B = 0$$

$$u_2 = \frac{600}{60,000} = 0.01 \text{ mm}$$

$$20,000(-u_2) + 20,000(0.01) - 400 = f_c$$

$$f_c = -400 \text{ N}$$

$$f_A = 400 \text{ N}.$$



$$[k] = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

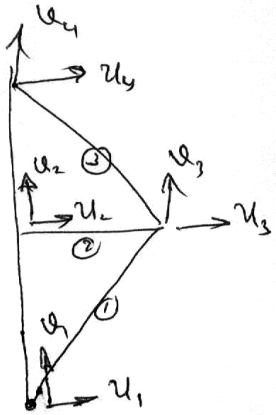
$$L = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$$

Element	i-node (Backnode)	coordinate x	coordinate y	j node (Fore node)	coordinate x	coordinate y	L	$c = \frac{x_j - x_i}{L}$	$s = \frac{y_j - y_i}{L}$
1	①	0	0	③	300	400	500	0.6	0.8
2	②	0	400	③	300	400	300	1	0
3	4	0	700	③	300	400	424.26	0.707	-0.707

Element ①

$$[k^1] = \frac{40 \times 200 \times 10^3}{500}\begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 \\ 0.48 & 0.64 & -.48 & -.64 \\ -0.36 & -0.48 & 0.36 & 0.48 \\ -0.48 & -0.64 & 0.48 & 0.64 \end{bmatrix}$$

$$= \begin{array}{c} \begin{matrix} u_1 & u_1 & u_3 & u_3 \end{matrix} \\ \begin{bmatrix} 5760 & 7680 & -5760 & -7680 \\ 7680 & 10240 & -7680 & -10240 \\ -5760 & -7680 & 5760 & 7680 \\ -7680 & 10240 & 7680 & 10240 \end{bmatrix} \end{array} \begin{matrix} u_1 \\ u_1 \\ u_3 \\ u_3 \end{matrix}$$



Element ②

$$[K_2] = \frac{10 \times 200 \times 10^3}{300} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{matrix} \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{matrix} \begin{matrix} \begin{matrix} u_2 & v_2 & u_3 & v_3 \end{matrix} \\ \begin{bmatrix} 6666.67 & 0 & -6666.67 & 0 \\ 0 & 0 & 0 & 0 \\ -6666.67 & 0 & 6666.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Element ③

$$[K_3] = \frac{20 \times 200 \times 10^3}{424.26} \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} = \begin{matrix} \\ u_4 \\ v_4 \\ u_3 \\ v_3 \end{matrix} \begin{matrix} \begin{matrix} u_4 & v_4 & u_3 & v_3 \end{matrix} \\ \begin{bmatrix} 4714.1 & -4714.1 & -4714.1 & 4714.1 \\ -4714.1 & 4714.1 & 4714.1 & -4714.1 \\ -4714.1 & 4714.1 & 4714.1 & -4714.1 \\ 4714.1 & -4714.1 & -4714.1 & 4714.1 \end{bmatrix} \end{matrix}$$

$$[K_3] = \begin{matrix} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{matrix} \begin{matrix} \begin{matrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{matrix} \\ \begin{bmatrix} 5760 & 7680 & & & -5760 & -7680 & & \\ 7680 & 10240 & & & -7680 & -10240 & & \\ & & 6666.67 & & -6666.67 & & & \\ & & & 0 & 0 & & & \\ -5760 & -7680 & -6666.67 & & 5760 & 7680 & & \\ & & & & 6666.67 & & & \\ & & & & 4714.1 & -4714.1 & -4714.1 & 4714.1 \\ -7680 & -10240 & & & 7680 & 10240 & & \\ & & & & 0 & & & \\ & & & & -4714.1 & 4714.1 & 4714.1 & -4714.1 \\ & & & & & & & \\ & & & & -4714.1 & 4714.1 & 4714.1 & -4714.1 \\ & & & & & & & \\ & & & & 4714.1 & -4714.1 & -4714.1 & 4714.1 \end{bmatrix} \end{matrix}$$

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3}=4000 \\ F_{y3}=8000 \\ F_{x4} \\ F_{y4} \end{Bmatrix} = \begin{bmatrix} 5760 & 7680 & 0 & 0 & -5760 & 7680 & 0 & 0 \\ 7680 & 10240 & 0 & 0 & -7680 & -10240 & 0 & 0 \\ 0 & 0 & 6666.67 & 0 & -6666.67 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5760 & -7680 & -6666.67 & 0 & 17140.77 & 2965.9 & -4714.1 & 4714.1 \\ -7680 & -10240 & 0 & 0 & 2965.9 & 14954.1 & 4714.1 & -4714.1 \\ 0 & 0 & 0 & 0 & -4714.1 & 4714.1 & 4714.1 & -4714.1 \\ 0 & 0 & 0 & 0 & 4714.1 & -4714.1 & -4714.1 & 4714.1 \end{bmatrix} \begin{Bmatrix} u_1=0 \\ v_1=0 \\ u_2=0 \\ v_2=0 \\ u_3 \\ v_3 \\ u_4=0 \\ v_4=0 \end{Bmatrix}$$

$$\begin{Bmatrix} 4000 \\ 8000 \end{Bmatrix} = \begin{bmatrix} 17140.77 & 2965.9 \\ 2965.9 & 14954.1 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix}$$

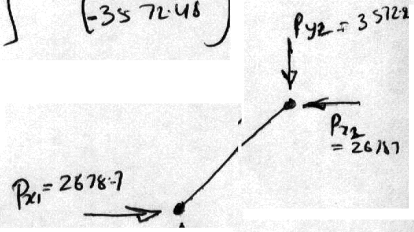
$$\begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = \frac{1}{247526511.8} \begin{bmatrix} 14954.1 & -2965.9 \\ -2965.9 & 17140.77 \end{bmatrix} \begin{Bmatrix} 4000 \\ 8000 \end{Bmatrix}$$

$$\begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0.3375 \\ -0.602 \end{Bmatrix}$$

Element forces

Element ①

$$\begin{Bmatrix} P_{x1} \\ P_{y1} \\ P_{x2} \\ P_{y2} \end{Bmatrix} = \begin{bmatrix} 5760 & 7680 & -5760 & -7680 \\ 7680 & 10240 & -7680 & -10240 \\ -5760 & -7680 & 5760 & 7680 \\ -7680 & -10240 & 7680 & 10240 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.337 \\ -0.602 \end{Bmatrix} = \begin{Bmatrix} 2678.7 \\ 3572.48 \\ -2678.7 \\ -3572.48 \end{Bmatrix}$$

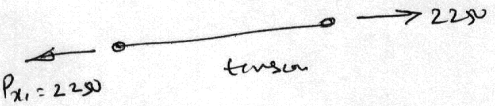


Stress $\sigma^1 = \frac{E}{L}\begin{bmatrix} -c & -s & c & s \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{Bmatrix}$

$= \frac{200000}{500}\begin{bmatrix} 0.6 & -0.8 & 0.6 & 0.8 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.3375 \\ -0.601 \end{Bmatrix} = -111.64$ N/mm² (compression)

Element ②

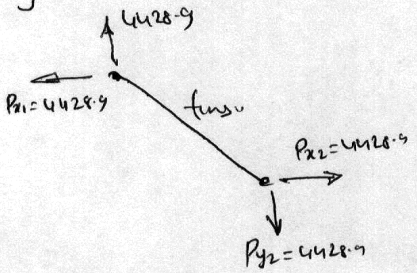
$$\begin{Bmatrix} P_{x1} \\ P_{y1} \\ P_{x2} \\ P_{y2} \end{Bmatrix} = \begin{bmatrix} 6666.67 & 0 & -6666.67 & 0 \\ 0 & 0 & 0 & 0 \\ -6666.67 & 0 & 6666.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.3375 \\ -0.602 \end{Bmatrix} = \begin{Bmatrix} -2250 \\ 0 \\ 2250 \\ 0 \end{Bmatrix}$$



$\sigma^2 = \frac{200000}{500}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.3375 \\ -0.602 \end{Bmatrix} = 225$ N/mm² tensile

Element ③

$$\begin{Bmatrix} P_{x1} \\ P_{y1} \\ P_{x2} \\ P_{y2} \end{Bmatrix} = \begin{bmatrix} 4714.1 & -4714.1 & -4714.1 & 4714.1 \\ -4714.1 & 4714.1 & 4714.1 & -4714.1 \\ -4714.1 & 4714.1 & 4714.1 & -4714.1 \\ 4714.1 & -4714.1 & -4714.1 & 4714.1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.3375 \\ -0.602 \end{Bmatrix} = \begin{Bmatrix} -4428.9 \\ 4428.9 \\ 4428.9 \\ -4428.9 \end{Bmatrix}$$



$\sigma^3 = \frac{200000}{4426}\begin{bmatrix} -0.707 & 0.707 & 0.707 & -0.707 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.3375 \\ -0.602 \end{Bmatrix}$

$= 313.17$ N/mm² tensile