Analysis in Mechanical Engineering

Selected Notes

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Dedication

To my much suffering students.

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Preface

To the Student

This is a collection of notes on partial differential equations. What standard engineering mathematics texts tell you about these equations is largely useless. You learn how to solve specific simple problems using methods like separation of variables and Laplace transforms. After you have spend a lot of time on that, you still know absolutely nothing about partial differential equations. This book covers the missing parts.

Sections marked with "[NC]" in their title are not part of the material covered in "Analysis in Mechanical Engineering II." Read them if you are interested.

The usual "Why this book?" blah-blah will eventually be found in a note at the back of this book, {N.1} A version history is in note {N.2}.

This is a living document. At this stage it is a rough initial concept. That is reflected in the version number. Maybe at some later time it becomes something useful for a general audience to read.

Acknowledgments

Many examples originate from the book by DuChateau & Zachmann, [3], that I use as basic text book in my class.

This book owes its formatting to my book "Quantum Mechanics for Engineers."

Typos have been pointed out by Bronek Gepner.

Thank you all.

Comments and Feedback

If you find an error, please let me know. There seems to be an unending supply of them. As one author described it brilliantly, "the hand is still writing though the brain has long since disengaged."

Also let me know if you find points that are unclear to the intended readership, mechanical engineering graduate students with a typical exposure to mathematics, or equivalent.

The same for sections that cannot be understood without delving back into earlier material. All within reason of course. If you pick a random starting word somewhere in the book and start reading from there, you most likely will be completely lost. But sections are intended to be fairly self-contained, and you should be able read one without backing up through all of the text.

General editorial comments are also welcome.

Part I

Calculus

Chapter 1

Graphs

1.1 Introduction

Graphs are important for engineers for a number of reasons:

- Understanding relationships between variables.
- Summarizing data.
- Representing data (like in a Moody diagram).
- Interpolating data.
- Understanding the overal nature of data. See warming.jpg[1] for an example that you simply could not appreciate by looking at a list of numbers.
- ...

Look for:

- Intercepts. Intercepts with the x-axis satisfy $y = 0$. Intercepts with the y-axis satisfy $x = 0$.
- A symmetry line exists if the curve is the same at both sides of the line. More precisely, a symmetry line acts as a mirror that mirrors the curve into itself. The y-axis is a symmetry line if the sign of x does not make a difference. The x-axis is one if the sign of y does not make a difference. The 45° line $y = x$ is one if swapping x and y does not make a difference.
- Symmetry points. Every point on the curve must have match at the exact opposite side of a symmetry point. Mathematically, if $\vec{r}_1$ is on the curve, then so must be $\vec{r}_S - (\vec{r}_1 - \vec{r}_S)$. The origin is a symmetry point if $y(-x) = -y(x)$, i.e. if function $y(x)$ is antisymmetric.

[1] `http://www.eng.fsu.edu/~dommelen/calculus/warming.jpg`

- Singular points:
  - corners where the direction of the curve changes by an angle less than 180°,
  - cusps where it changes 180°,
  - crossings where the curve crosses itself,
  - positions of infinite curvature,
  - ...

  If y or any of its derivatives is infinite or not uniquely defined, the curve has a singularity at that point.
- A vertical asymptote $x_{\text{va}} = A$ exists if $y \to \pm\infty$ for $x \to A$.
- A horizontal asymptote $y = A$ exists if $A = \lim_{x\to\pm\infty} y$ exists.
- Behavior for $x \to \pm\infty$ (e.g. $y \sim |x|^p$ for some p).
- An oblique asymptote $y_{\text{oa}} = Ax + B$ exists if $A = \lim_{x\to\pm\infty} y'$ and $b = \lim_{x\to\pm\infty} y - Ax$ exist. (Or more simply if $\lim_{x\to\pm\infty} y - Ax - B = 0$.)
- Extent in x (the range of x-values of the curve) and extent in y (the range of y-values of the curve). If y is a given function of x, then the x-extent is the x-values for which y can be computed, but the y extent may not be so simple.
- Minima and maxima. A global maximum/minimum is the highest/lowest value of y that can be found anywhere. You should find both the value of the maximum/minimum and its location(s). A local maximum/minimum is the highest/lowest value that can be found in a small vicinity around the localtion of the local maximum/minimum. Normally, you first find the local maxima/minima, and then, based on consideration of the *entire* graph, decide whether they are also global ones. The derivative changes sign at a maximum/minimum *if* defined at both sides of the maximum/minimum. So look for both zero derrivatives and singular points.
- Concavity is upward if $y'' > 0$, downward if $y'' < 0$.
- Inflection points are points where the concavity changes sign.

See [1, Chapters 13-15]

1.2 Example

From [1, p. 128, 13a]

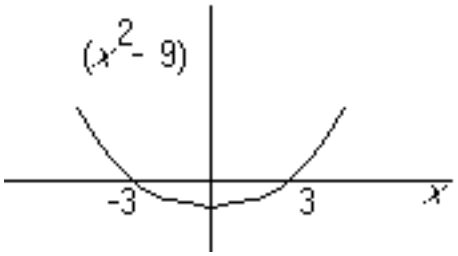
Asked: Draw the graph of

$$xy = \left(x^2 - 9\right)^2$$

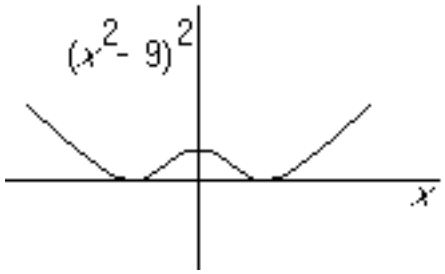
1.2.1 Using reasoning

$$xy = \left(x^2 - 9\right)^2$$

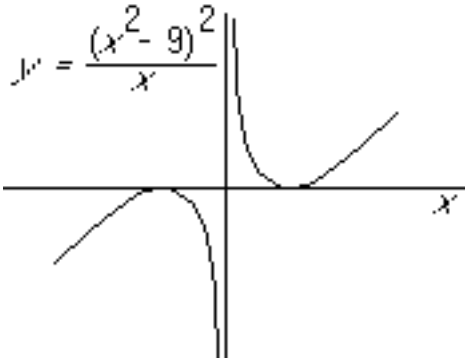
Instead of starting to crunch numbers, look at the pieces first:
Factor $x^2 - 9 = (x - 3)(x + 3)$ is a parabola with zeros at $x = \pm 3$:



Squaring gives a quartic with double zeros at $x = \pm 3$:



Dividing by x will produce a simple pole at $x = 0$ and also a sign change at negative x:



Function $y(x)$:

- has an x-extent $x \neq 0$ and a y-extend $-\infty < y < \infty$;
- is odd (symmetric with respect to the origin);
- has a relative maximum at -3 of finite curvature: $y \propto (x + 3)^2$;
- has a relative minimum at 3 of finite curvature: $y \propto (x - 3)^2$;

- has a vertical asymptote at $x = 0$ with asymptotic behavior: $y \sim 81/x$ for $|x| \to 0$;
- behaves asymptotically as $y \sim x^3$ for $x \to \pm\infty$;
- is concave up for $x > 0$, down for $x < 0$. (Should really prove this, I guess.)

1.2.2 Using brute force

$$y = \frac{(x^2 - 9)^2}{x}$$

Hence
- intercepts with x-axis are at $x = \pm 3$;
- no intercepts with the y axis;
- y is an odd function of x (symmetric about the origin);
- for $x \downarrow 0$, $y \to \infty$ (vertical asymptote);
- for $x \uparrow 0$, $y \to -\infty$ (singularity is an odd, simple pole);
- for $x \to \pm\infty$, $y \sim x^3 \to \pm\infty$.

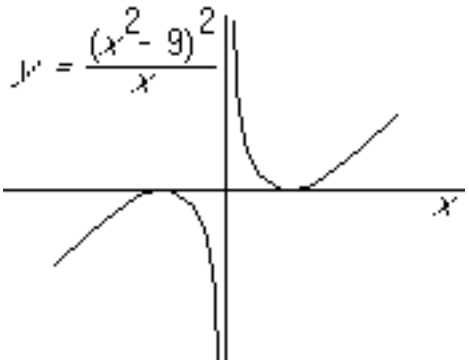
$$y' \equiv \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x^2 - 9)\,(3x^2 + 9)}{x^2}$$

Hence,
- $y' > 0$ for $-\infty < x < -3$ (y increases from $-\infty$);
- $y' = 0$ for $x = -3$ (local maximum, $y = 0$);
- $y' < 0$ for $-3 < x < 0$ (y decreases towards $-\infty$);
- $y' = -\infty$ for $x = 0$ (singular point, vertical asymptote);
- $y' < 0$ for $0 < x < -3$ (decreases from ∞);
- $y' = 0$ for $x = 3$ (local minimum, $y = 0$);
- $y' > 0$ for $3 < x < \infty$ (increases to ∞).

Also,
- $y' \to \infty$ when $x \to \pm\infty$ (no horizontal or oblique asymptotes);
- all derivatives exist, except at $x = 0$, which has no point on the curve (no corners, cusps, infinite curvature, or other singular points);
- probably no inflection points.

$$y'' = \frac{6x^4 + 162}{x^3}$$

Hence
- really no inflection points (since there is no point at $x = 0$);
- cocave downward for $x < 0$, upward for $x > 0$.



Hence the x- and y-extends are as before.

1.3 Example

From [1, p. 128, 13g]

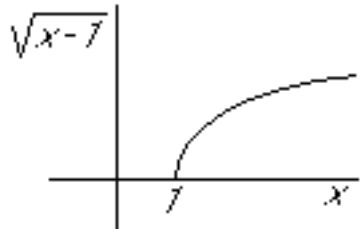
Asked: Graph

$$y = x\sqrt{x-1}$$

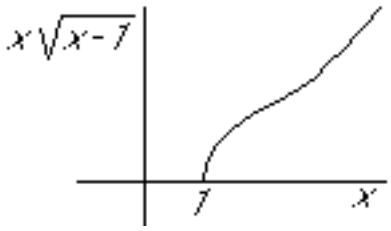
1.3.1 Solution

$$y = x\sqrt{x-1}$$

Factor $\sqrt{x-1}$ is $\sqrt{x}$ shifted one unit towards the right.



Multiplying by x magnifies it by a factor ranging from 1 to ∞:



Function $y(x)$:

- has an x-extent $x \geqslant 1$ and a y-extent $y \geqslant 0$;
- behaves asymptotically as $y \sim x^{3/2}$ for $x \to \infty$;
- is monotonous:

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{x-1} + \frac{x}{2\sqrt{x-1}} = \frac{2x-2+x}{2\sqrt{x-1}} = \frac{3x-2}{2\sqrt{x-1}} > 0;$$

- has vertical slope at $x = 1$;
- is concave down for smaller x, concave up for larger x;
- the inflection point is at

$$y'' = \frac{3x-4}{4(x-1)^{3/2}} = 0$$

giving $x = 4/3$.

Chapter 2

Optimization

2.1 Introduction

Optimization is important for engineers for a number of reasons:

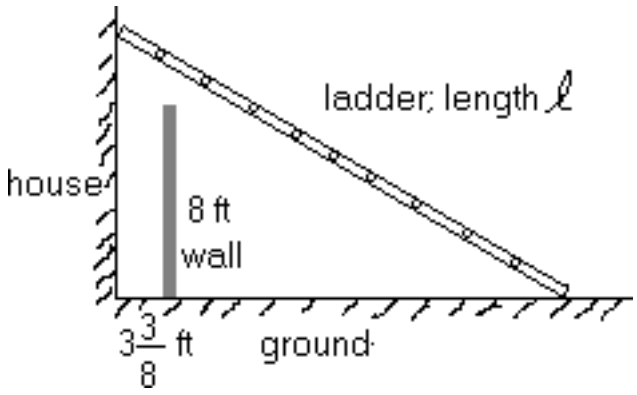
- A best design finds the maximum of some benefit function.
- Drag reduction minimizes drag.
- Potential energy minimization finds the stationary state of a system. That is the basis for true finite element methods.
- Much of economics is based on finding the extrema of cost or benefi functions.
- ...

Key ideas:

- zero partial derivatives at an interior extremum
- Lagrangian multipliers can account for constraints

2.2 Example

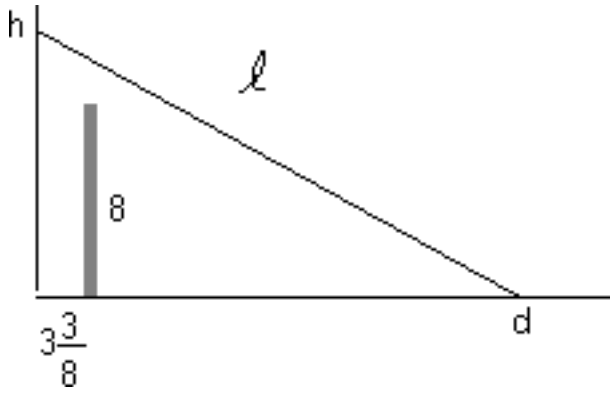
From [1, p. 116, 30]



Given: A free standing wall, located $3\frac{3}{8}$ ft from the side of a house.

Asked: What is the length ℓ of the shortest ladder that can reach the house (over the free standing wall).

2.2.1 Definition

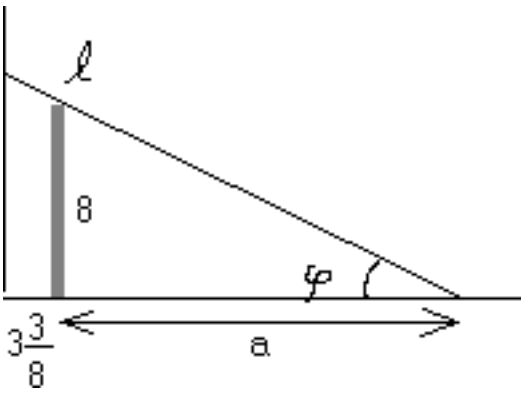


Two degrees of freedom: say h and d

One *inequality* constraint: the ladder must be above the free standing wall.

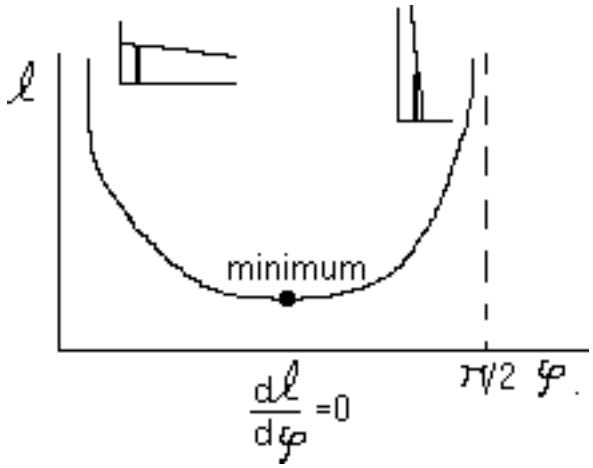
2.2.2 Reduction

The shortest ladder hits the free standing wall:



One degree of freedom left: φ.

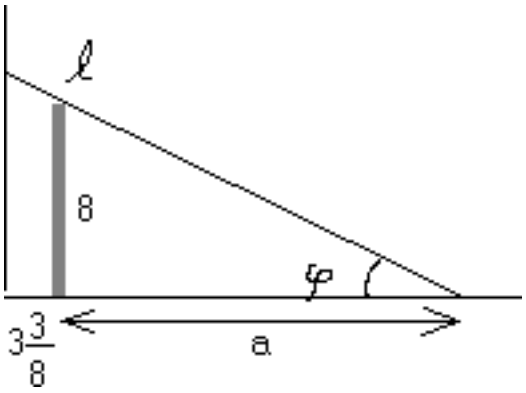
2.2.3 Further reduction



At the minimum:

$$\frac{\mathrm{d}\ell}{\mathrm{d}\varphi} = 0$$

2.2.4 Finding the length



First find a:

$$a = \frac{8}{\tan\varphi}.$$

Then:

$$\ell = \frac{3\frac{3}{8} + a}{\cos\varphi} = \frac{3\frac{3}{8}}{\cos\varphi} + \frac{8}{\sin\varphi} \tag{2.1}$$

2.2.5 Finding the optimum angle

$$\frac{\mathrm{d}\ell}{\mathrm{d}\varphi} = \frac{3\frac{3}{8}}{\cos^2\varphi}\sin\varphi - \frac{8}{\sin^2\varphi}\cos\varphi = 0.$$

$$\frac{27}{8\cos^2\varphi}\sin\varphi = \frac{8}{\sin^2\varphi}\cos\varphi$$

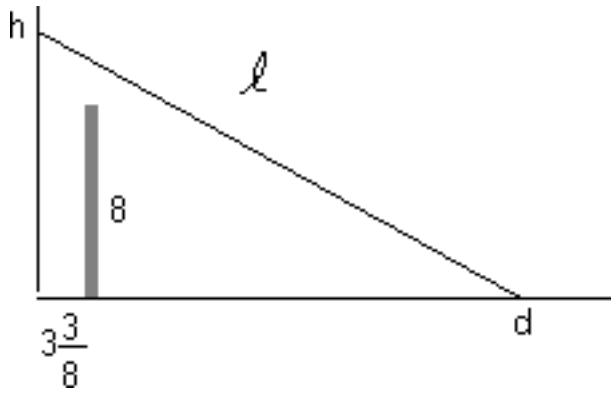
$$\tan^3\varphi = \frac{64}{27} \qquad \Rightarrow \qquad \varphi_{\min} = 0.9273 \text{ radians}$$

2.2.6 Finding the optimum length

From (2.1)

$$\ell_{\min} = 15.625 \text{ ft}$$

2.3 General Approach



If you do not know a priori that the ladder hits the wall, you can follow the general approach.

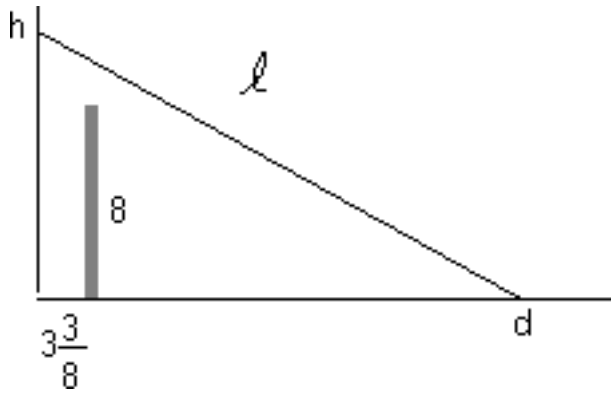
There are now two degrees of freedom. They are conveniently taken to be h and d.

Then the length of the ladder is, (from Pythagoras),

$$\ell(h, d) = \sqrt{h^2 + d^2}$$

We now need to figure out what values of h and d produce the shortest ladder.

2.3.1 Formulation



Note that by the definition of the problem, $h > 8$ and $d > 3\frac{3}{8}$. But these constraints are not precise. For example, $h = 8.001$ and $d = 3\frac{3}{8} + 0.001$ would obviously have the ladder go through the wall.

There is a precise constraint, that the ladder cannot pass through the wall. If b is the height of the point on the ladder straight above the wall, then similar

triangles give:

$$\frac{b}{d-3\frac{3}{8}} = \frac{h}{d} \ (= \tan\phi)$$

The constraint is that $b \geqslant 8$, so:

$$h\frac{d-3\frac{3}{8}}{d} \geq 8 \qquad \Rightarrow \qquad h[d-3\frac{3}{8}] - 8d \geq 0$$

Note that this is in general an *inequality* constraint. Equality occurs when the ladder hits the wall.

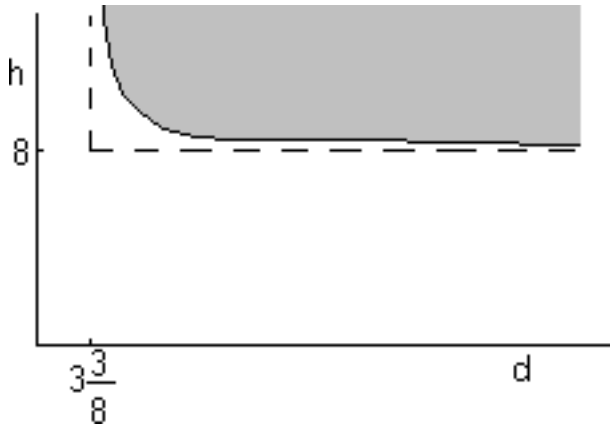
So the problem is to minimize

$$\ell(h,d) = \sqrt{h^2+d^2}$$

(from Pythagoras), subject to the inequality constraint

$$h[d-3\tfrac{3}{8}] - 8d \geq 0$$

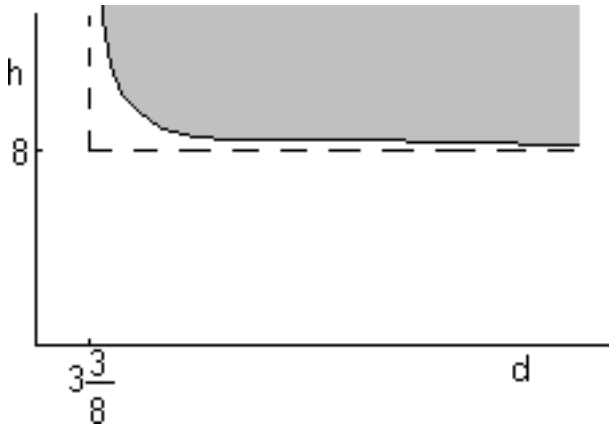
To solve this, first plot the possible h and d values:



Plotting the curve where equality occurs in the constraint gives the bottom of the grey region above. On that curve, the ladder hits the wall. If you go above the curve, into the grey region, h becomes bigger and the ladder then moves above the wall.

Now you must figure out whether the shortest ladder occurs in the strict *interior* of the grey region or on its boundary. Try the interior first.

2.3.2 Interior minima

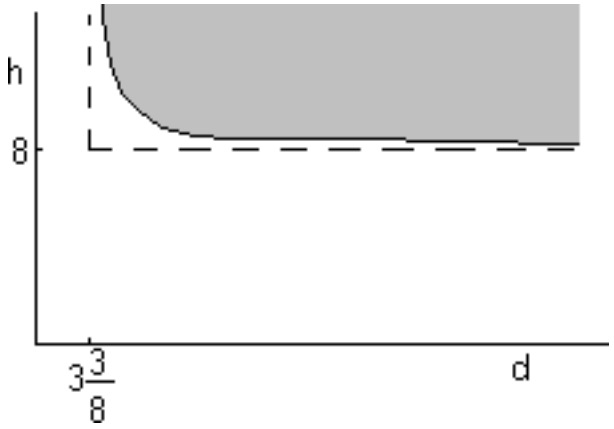


For a minimum, at least locally, in the strict *interior* of the grey region, the partial derivatives must be zero.

$$\frac{\partial \ell}{\partial d} = (h^2 + d^2)^{-1/2} d = 0 \qquad \frac{\partial \ell}{\partial h} = (h^2 + d^2)^{-1/2} h = 0$$

But these two requirements can only be true if $d = h = 0$, and that point is not in the grey region. So there is *no* interior minimum (or maximum, for that matter).

2.3.3 Boundary minima



Since the minimum is not in the interior, it must be on the boundary of the grey region. Now obviously for infinite h or d or both you do not have the shortest ladder. So the minimum cannot be on the boundary at infinity. It must be on the curve where the ladder just hits the wall.

But how do we find the minimum on this line? The partial derivatives of ℓ are not zero at this point. (Just check that out in the previous subsection.)

The trick is to define an artificial third variable λ, called a "Lagrangian multiplier," corresponding to the constraint. (In the most general case, this Lagrangian multiplier has no particular physical meaning.) Then define a new function f to replace ℓ in the minimization:

$$f = \sqrt{h^2 + d^2} + \lambda(h[d - 3\tfrac{3}{8}] - 8d).$$

Note that λ multiplies whatever is zero according to the constraint.

Now it turns out that you can find the desired minimum by finding an unconstrained stationary point to this function f:

$$\partial f/\partial d = (h^2 + d^2)^{-1/2} d + \lambda(h - 8) = 0$$

$$\partial f/\partial h = (h^2 + d^2)^{-1/2} h + \lambda(d - 3\tfrac{3}{8}) = 0$$

$$\partial f/\partial \lambda = h[d - 3\tfrac{3}{8}] - 8d = 0$$

From the first two equations

$$(h^2 + d^2)^{-1/2} d = -\lambda(h - 8) \qquad (h^2 + d^2)^{-1/2} h = -\lambda(d - 3\tfrac{3}{8})$$

or taking the ratio of these two equations,

$$\frac{d}{h} = \frac{h - 8}{d - 3\frac{3}{8}}$$

Solving the constraint for h and putting it in the above gives after simplification:

$$(d - 3\tfrac{3}{8})^3 = 8 \times 27 \qquad \Rightarrow \qquad d = \tfrac{75}{8} \qquad \Rightarrow \qquad h = \tfrac{75}{6} \qquad \Rightarrow \qquad \ell = \tfrac{125}{8}$$

If you would have more than one constraint, there is one separate Lagrangian multiplier for each one. For example, if you take care of the boundary conditions in a finite element computation this way, you will get one for each boundary condition at each boundary point.

Chapter 3

Approximations

3.1 Introduction

Why use approximation:

- Because it is needed. In fact *everything* you do in real life is an approximation. Real life is proably determined by some sort of quantum mechanics. But we do not know what it is. And we would definitely not know how to solve it even if we knew what it was.

- To reduce effort.

- To increase accuracy. You might be able to solve approximate equations more accurately than more accurate equations, producing a better result.

- To get more insight in the problem. If you have computed say an incompressible flow of interest, the only real thing you can say about it is that indeed, it is free from singularities. You can wax about "vortices," "fingers," "intestines," etcetera, that you seem to see in the flow, but that you can also see in the clouds in the sky. If you start computing the flow for various parameters, you may start getting somewhere to an inkling of insight. However, doing that is limited by what the computer can do. And the interpretation will always have ambiguity. However, if you start looking at limiting processes of your parameters, you are suddenly getting somewhere. You can now define meaningful "boundary layers", "vortex layers", "shocks." etcetera that are not just arbitrary interpretations but have rigorous mathematical definitions.

- ...

This chapter looks at simple approximations using Taylor series (which of course always includes linearization.)

3.2 Example

From [1, p. 402, 10b]

Asked: The Maclaurin series of $\sin^2 x$.

3.2.1 Identification

General Taylor series:

$$\begin{aligned} f(x) &= f(a) + f'(a)\frac{x-a}{1!} + f''(a)\frac{(x-a)^2}{2!} + \dots \\ &= \sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!} \end{aligned}$$

This is a power series (a is a given constant.) Maclaurin series: $a = 0$.

Approach:

- note that $a = 0$;
- identify the derivatives;
- evaluate them at $a = 0$;
- put in the formula;
- identify the terms for any value of n.

3.2.2 Results

$$\begin{array}{ll} f(x) = \sin^2 x & f(0) = 0 \\ f'(x) = 2\sin x \cos x & f'(0) = 0 \\ f''(x) = 2\cos^2 x - 2\sin^2 x = 2 - 4\sin^2 x & f''(0) = 2 \\ f'''(x) = -8\sin x \cos x = -4f'(x) & f'''(0) = 0 \\ f''''(x) = -4f''(x) & f''''(0) = -8 \\ f^{(5)}(x) = -4f'''(x) & f^{(5)}(0) = 0 \\ f^{(6)}(x) = -4f''''(x) = (-4)^2 f''(x) & f^{(6)}(0) = 32 \\ \vdots & \vdots \end{array}$$

$$\begin{aligned} \sin^2 x &= f(0) + f'(0)\frac{x-a}{1!} + f''(0)\frac{(x-a)^2}{2!} + \dots \\ &= 2\frac{x^2}{2!} - 8\frac{x^4}{4!} + 32\frac{x^6}{6!} + \dots \end{aligned}$$

General expression:

When $n = 2k$ with $k \geq 1$: $f^{(n)} = 2(-4)^{ek-1}$ Otherwise: $f^{(n)}e = 0$

$$\sin^2 x = \sum_{k=1}^{\infty} 2(-4)^{k-1}\frac{x^{2k}}{(2k)!}$$

3.2.3 Other way

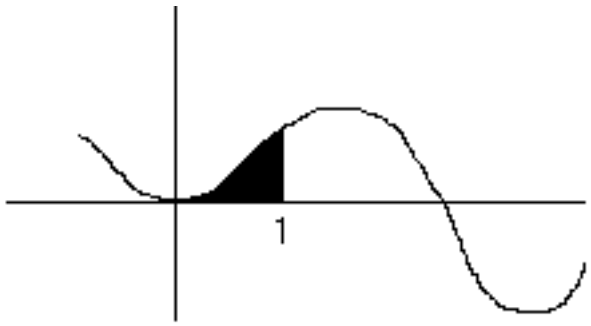
Write $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$ and look up the Maclaurin series for the cosine. (No fair.)

3.3 Example

From [1, p. 404, 30]

Asked: The area below $y = \sin x^2$ for $0 \leqslant x \leqslant 1$.

3.3.1 Identification



$$\int_0^1 \sin x^2 \, \mathrm{d}x$$

Analytically? Actually, the integral is equivalent to $\int(\sin(x)/x)\,\mathrm{d}x$, which cannot be written in terms of elementary functions.

But since the x range is not large, we will try approximating $\sin x^2$ using a Taylor series.

3.3.2 Finish

The Taylor series of $\sin x^2$ is that of $\sin x$ with x replaced by x^2. So:

$$\begin{aligned} \int_0^1 \sin x^2 \, \mathrm{d}x &= \int_0^1 \frac{x^2}{1!} - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \dots \\ &= \frac{1}{3} - \frac{1}{3!7} + \frac{1}{5!11} \\ &= .3103 \pm 0.0008 \end{aligned}$$

The error estimate is rigorous since the series is an *alternating* one whose terms get smaller monotoneously.

Chapter 4

Limits

4.1 Introduction

Taylor series often do not work because the functions involved are not analytic at the point of interest. For example, this is common if the behavior of interest is at large time, or for large values of some other parameter, like the Reynolds number Re of a flow. (In fact, for flows in infinite domains it normally also occurs for small Reynolds numbers.

Finding nonanalytic limits is then needed. Applications are very similar to those of Taylor series:

- Because it is needed.
- To reduce effort.
- To increase accuracy.
- For making estimates of how importants something is.
- To get more insight. For example, consider the laminar flow past a flat plate if the plate is aligned with the incoming flow velocity U. For finite Reynolds numbers, there is little more you can say than that the flow velocity will be zero at the plate, and U far away from the plate. To get more insight than that, you can ask: "What is the limit of the velocity for infinite Reynolds number Re, assuming that you keep the streamwise location x fixed, as well as keep the ratio $\eta = y\sqrt{\mathrm{Re}}/x$ fixed, where y the distance from the wall?" The answer is $Uf'(\eta)$ where f' is the Blasius function tabulated in any real book on fluid mechanics. (If you instead keep y itself fixed at a nonzero value, the limit is U, which is not very interesting.) Most of my theoretical research in fluids (as opposed to in numerical methods) really simply finds limits like this.
- ...

4.2 Example

From [1, p. 227, 10v]

Asked:

$$\lim_{x\to-\infty} x^2e^x$$

4.2.1 Observations

$$\lim_{x\to-\infty} x^2e^x$$

You must first look whether the limit is trivial:

$$x^2 \to \infty \qquad e^x \to 0$$

Since the product of infinity times zero is unknown, this limit is nontrivial.

4.2.2 L'Hopital

L'Hopital can be used if you create a ratio of quantities that both become zero or both become infinite. (For example, you would not want to apply L'Hopital on $\lim_{x\to0} 3/2$.)

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \lim_{x\to-\infty} \frac{(x^2)'}{(e^{-x})'}$$

Now both top and bottom become infinite. So L'Hopital can be applied, by differentiating top and bottom separately:

$$\lim_{x\to-\infty} \frac{(x^2)'}{(e^{-x})'} = \lim_{x\to-\infty} \frac{2x}{-e^{-x}} =$$

Still infinity over infinity, so differentiate once more

$$\lim_{x\to-\infty} \frac{2x}{-e^{-x}} = \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0$$

4.2.3 Better

Using some insight is always better than just crunching it out. First simplify things for yourself by defining $u = -x$. Then u goes to plus infinity instead of minus infinity like x. Then

$$\lim_{x\to-\infty} x^2e^x \lim_{u\to\infty} \frac{u^2}{e^u}$$

and that is zero because e^u is much greater than any power of u for large positive u. (To see that, just look at the Taylor series:

$$e^u = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots$$

The u^3 term is much larger than u^2 for large u and the other terms make e^u larger still.)

So you could replace x^2 by x^{100} and the limit would still be zero.

4.3 Example

From [1, p. 228, 10z]

Asked:

$$\lim_{x\to 0}(x - \arcsin x)\csc^3 x$$

4.3.1 Grinding it out

In

$$\lim_{x\to 0}(x - \arcsin x)\csc^3 x$$

x and $\arcsin x$ become zero, but $\csc x$ becomes infinite. The total is undefined.

The simplest way to make a ratio suitable for l'Hopital is to use that $\csc x \equiv 1/\sin x$:

$$\lim_{x\to 0}(x - \arcsin x)\csc^3 x = \lim_{x\to 0}\frac{x - \arcsin x}{\sin^3 x}$$

Differentiate top and bottom

$$\lim_{x\to 0}\frac{1 - (1-x^2)^{-1/2}}{3\sin^2 x\cos x}$$

Still zero over zero, so differentiate again

$$\lim_{x\to 0}\frac{-x(1-x^2)^{-3/2}}{6\sin x\cos^2 x - 3\sin^3 x}$$

Still zero over zero, so differentiate again

$$\lim_{x\to 0}\frac{-(1-x^2)^{-3/2} - 3x^2(1-x^2)^{-3/2}}{6\cos^3 x - 21\sin^2 x\cos x} = \tfrac{1}{6}$$

If I did not make any mistakes, I guess.

4.3.2 Using insight

Since $\sin x \approx x$ for small x, $\sin^3 x \approx x^3$. Also looking at a mathematical handbook, $\arcsin x \approx x + \frac{1}{6}x^3 + \ldots$. So:

$$\lim_{x\to 0} \frac{x - \arcsin x}{\sin^3 x} \approx \frac{-\frac{1}{6}x^3}{x^3} = -\frac{1}{6}$$

(Note that we needed to keep the cubic term in the Taylor series for $\arcsin x$ since the term x dropped out.)

Chapter 5

Combined Changes in Variables

5.1 Introduction

Combined changes in variables are common. For example:

- Error estimates.
- Changes for a moving particle in a field.
- Changes in scalar quantities depending on several variables,
- ...

The key concept is the total differential. For any function $f = f(x, y, z)$,

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z$$

Of course, f could be a vector.

5.2 Example

From [1, p. 422, 27a]

Given:

$$\omega = \sqrt[3]{\frac{g}{b}}$$

The maximum error in g is 1%, the maximum error in b is 0.5%.

Asked: The maximum percentage error in ω.

5.2.1 Identification

Given is that the maximum error in g is 1% and the maximum error in b is 0.5%. That means that the *relative* errors are:

$$\frac{\delta g}{g} \leq 0.01 \qquad \frac{\delta b}{b} \leq 0.005$$

where δg and δb are the *absolute* errors. Errors are always positive.

Error manipulation rules:

1. During addition and substraction of variables, add their absolute errors;
2. During multiplication or division, add their relative errors;
3. During exponentiation, multiply the relative error by the absolute power.

5.2.2 Results

Consider first the relative change in g/b due to changes $\mathrm{d}g$ in g and $\mathrm{d}b$ in b. The rule for differentiating a ratio implies:

$$\frac{\mathrm{d}(g/b)}{(g/b)} = \frac{b}{g}\left(\frac{b\mathrm{d}g - g\mathrm{d}b}{b^2}\right) = \frac{\mathrm{d}g}{g} - \frac{\mathrm{d}b}{b}$$

Note that if you do not know the sign of the errors, you can only say that the final result is no bigger than

$$\left|\frac{\mathrm{d}g}{g}\right| + \left|\frac{\mathrm{d}b}{b}\right|$$

which is simply the rule for adding relative errors if you take a ratio or product of variables.

Hence the greatest possible relative error in (g/b) is:

$$\frac{\delta(g/b)}{(g/b)} = 0.01 + 0.005 = 0.015$$

But we need the relative error in $\sqrt[3]{g/b}$ instead of in g/b. Denoting g/b by u for now, the rule for differentiating a power gives

$$\frac{\mathrm{d}u^{1/3}}{u^{1/3}} = \frac{\frac{1}{3}u^{-2/3}\mathrm{d}u}{u^{1/3}} = \frac{1}{3}\frac{\mathrm{d}u}{u}$$

That is simply the rule of multiplying the relative error by the absolute power when exponentiating.

Hence

$$\frac{\delta\omega}{\omega} = \frac{1}{3} \times 0.015 = 0.005 = 0.5\%$$

5.3 Example

From [1, p. 423, 29]

Given: A circular cylinder of changing radius r and height h. At a given time, $r = 6$ inch, $\dot{r} = 0.2$ in/sec, $h = 8$ in, $\dot{h} = -0.4$ in/sec.

Asked: $\dot{V}$ and $\dot{A}$ at that time.

5.3.1 Solution

$$V = \pi r^2 h \qquad A = 2\pi r h + 2\pi r^2$$

The total differential gives

$$\mathrm{d}V = \frac{\partial V}{\partial h}\mathrm{d}h + \frac{\partial V}{\partial r}\mathrm{d}r$$

where differential changes become time derivatives if you divide by $\mathrm{d}t$. So identifying the partial derivatives gives:

$$\dot{V} = \pi r^2 \dot{h} + \pi 2 r h \dot{r} = 15.08 \text{ in}^3/\text{sec}$$

Similarly:

$$\dot{A} = 2\pi r \dot{h} + (2\pi h + 4\pi r)\,\dot{r} = 10.05 \text{ in}^2/\text{sec}$$

Chapter 6

Curvilinear Motion

6.1 Introduction

Curvilinear motion:

- Dynamics of vehicles (cars, planes, ...)
- Ballistics,
- Forces,
- Vortex lines,
- ...

$$\vec{r} = \vec{r}(t) \qquad \vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \qquad \vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$$

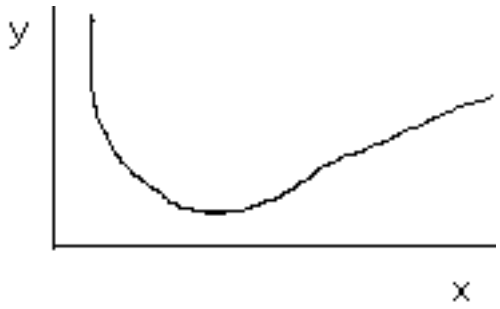
6.2 Example

From [1, p. 338, 14]

Given: A particle moves along a curve described by

$$x = \tfrac{1}{2}t^2 \qquad y = \tfrac{1}{2}x^2 - \tfrac{1}{4}\ln x \tag{6.1}$$

Asked: The velocity and acceleration at $t = 1$



6.2.1 Position

At $t = 1$:

$$x = \tfrac{1}{2}t^2 = \tfrac{1}{2} \qquad y = \tfrac{1}{2}x^2 - \tfrac{1}{4}\ln x = 0.298 \tag{6.2}$$

hence

$$\vec{r} = \begin{pmatrix} 0.5 \\ 0.298 \end{pmatrix} = 0.5\hat{\imath} + 0.298\hat{\jmath} \tag{6.3}$$

6.2.2 Velocity

Velocity:

$$\vec{v} = \begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}x}\dfrac{\mathrm{d}x}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} t \\ (x - \frac{1}{4}x^{-1})t \end{pmatrix} = \begin{pmatrix} t \\ \frac{1}{2}t^3 - \frac{1}{2}t^{-1} \end{pmatrix} \tag{6.4}$$

Velocity at $t = 1$:

$$\vec{v}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\hat{\imath} + 0\hat{\jmath} = \hat{\imath} \tag{6.5}$$

Components at $t = 1$:

$$v_x \equiv \frac{\mathrm{d}x}{\mathrm{d}t} = 1 \qquad v_y \equiv \frac{\mathrm{d}y}{\mathrm{d}t} = 0 \tag{6.6}$$

Magnitude at $t = 1$:

$$|\vec{v}| = v \equiv \frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{v_x^2 + v_y^2} = 1 \tag{6.7}$$

Angle with the positive x-axis at $t = 1$:

$$\tau = \arctan\frac{v_y}{v_x} = 0 \text{ (not } \pi). \tag{6.8}$$

6.2.3 Acceleration

Acceleration:

$$\vec{a} = \begin{pmatrix} \frac{\mathrm{d}v_x}{\mathrm{d}t} \\ \frac{\mathrm{d}v_y}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{2}t^2 + \frac{1}{2}t^{-2} \end{pmatrix} \tag{6.9}$$

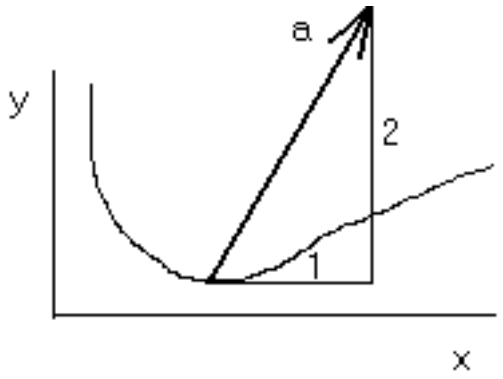
from (6.4).

Acceleration at $t = 1$:

$$\vec{a}(1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1\hat{\imath} + 2\hat{\jmath} \tag{6.10}$$

Components at $t = 1$:

$$a_x \equiv \frac{\mathrm{d}v_x}{\mathrm{d}t} = 1 \qquad a_y \equiv \frac{\mathrm{d}v_y}{\mathrm{d}t} = 2 \tag{6.11}$$



Magnitude at $t = 1$:

$$|\vec{a}| = a = \sqrt{a_x^2 + a_y^2} = \sqrt{5} \tag{6.12}$$

Angle with the positive x-axis at $t = 1$:

$$\phi = \arctan\frac{a_y}{a_x} = 63^\circ \text{ (not } 243^\circ\text{)}. \tag{6.13}$$

Component tangential to the motion:

$$a_t \equiv \frac{\mathrm{d}v}{\mathrm{d}t} \equiv \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{\vec{a}\cdot\vec{v}}{|\vec{v}|} = \frac{a_x v_x + a_y v_y}{|\vec{v}|} = 1 \tag{6.14}$$

Component normal to the motion:

$$a_n \equiv \frac{v^2}{R} = \sqrt{a^2 - a_t^2} = 2 \tag{6.15}$$

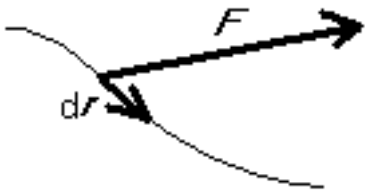
Chapter 7

Line Integrals

7.1 Introduction

Line integrals:

- work;



- potential energy;
- velocity potential
- ...

Path independence:

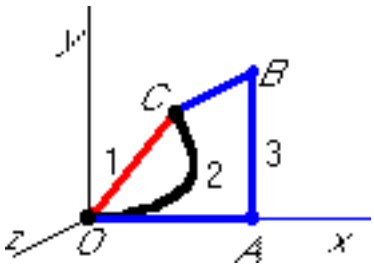
$$\int_A^B \vec{F} \cdot \mathrm{d}\vec{r}$$

is independent of the path between A and B when $\mathrm{curl}\vec{F} \equiv \mathrm{rot}\vec{F} \equiv \nabla \times \vec{F} = 0$.

7.2 Example

From [1, p. 487, 14e]

Given: $\vec{F} = x\hat{\imath} + 2y\hat{\jmath} + 3x\hat{k}$



Asked: The work done by this force going from O to C along (1) the connecting line; (2) the curve $x = t$, $y = t^2$, $z = t^3$; (3) path OABC.

7.2.1 Identification

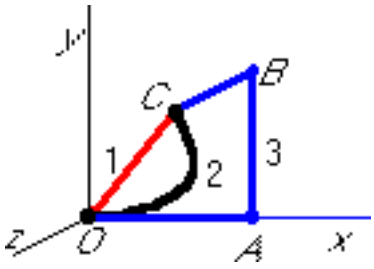
Find the curl of the vector to see whether the three integrals are going to be the same:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3x \end{vmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}$$

Nonzero, so the integrals along the three paths need not be the same.

7.2.2 Solution

$$\int_O^C \vec{F} \mathrm{d}\vec{r} = \int_O^C x\mathrm{d}x + 2y\mathrm{d}y + 3x\mathrm{d}z$$



1. Along the line $y = x$ and $z = x$:

$$\int_{x=0}^{1} 6x\mathrm{d}x = 3$$

2. Along the curve $x = t$, $y = t^2$, $z = t^3$:

$$\int_{t=0}^{1} F_x \frac{\mathrm{d}x}{\mathrm{d}t} + F_y \frac{\mathrm{d}y}{\mathrm{d}t} + F_z \frac{\mathrm{d}z}{\mathrm{d}t} = \int_0^1 t\mathrm{d}t + 2t^2\, 2t\mathrm{d}t + 3t\, 3t^2\mathrm{d}t = \frac{15}{4}$$

3. Along OABC:

$$\int_{x=0}^{1} x\mathrm{d}x + \int_{y=0}^{1} 2y\mathrm{d}y + \int_{z=0}^{1} 3\,1\mathrm{d}z = \frac{9}{2}$$

Chapter 8

Surface and Volume Integrals

8.1 Introduction

Multiple integrals are used to find various engineering quantities:

- Areas (cost, heat losses, ...):

$$\text{2D Cartesian:} \quad \mathrm{d}A = \mathrm{d}x\mathrm{d}y$$

$$\text{2D polar:} \quad \mathrm{d}A = \rho\mathrm{d}\rho\mathrm{d}\theta$$

- Volumes (weight, ...):

$$\text{3D Cartesian:} \quad \mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\text{Cylindrical:} \quad \mathrm{d}V = \rho\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}z$$

$$\text{Spherical:} \quad \mathrm{d}V = r^2 \sin\phi \mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta$$

- Centroids (center of gravity, center of pressure, ...)

$$\bar{x} = \int x\mathrm{d}A \Big/ \int \mathrm{d}A \qquad \bar{x} = \int x\mathrm{d}V \Big/ \int \mathrm{d}V$$

- Moments of inertia (solid body dynamics, center of pressure, ...)

$$I_x = \int y^2\mathrm{d}A \qquad I_0 = \int x^2 + y^2\mathrm{d}A$$

$$I_x = \int y^2 + z^2\mathrm{d}V \qquad I_{xy} = -\int xy\mathrm{d}V$$

- ...

If the material is not homogenous, you may have to put an additional density-like factor in those integrals.

Procedure:

- Draw the region to be integrated over.
- When integrating, say $\int\int\int f(a,b,c)\ \mathrm{d}a\mathrm{d}b\mathrm{d}c$, you have to decide whether you want to do a, b, or c first. Usually, you do the coordinate with the easiest limits of integration first.
- If you decide to do, say, b first, (i.e. you want to integrate

$$\int_{b_1}^{b_2} f(a,b,c)\ \mathrm{d}b$$

  first), the limits of integration b_1 and b_2 must be identified from the graph at *arbitrary* a and c, and are normally functions of a and c: $b_1 = b_1(a,c)$, $b_2 = b_2(a,c)$.
- After integrating over, say, b, the remaining double integral should no longer depend on b in any way. Nor does the region of integration: redraw it without the b coordinate. In other words, project it onto the a,c-plane. Then integrate over the next easiest coordinate in the same way.
- If you change integration variables from a,b,c to p,q,r, the integral becomes $\int\int\int f(p,q,r)J\ \mathrm{d}p\mathrm{d}q\mathrm{d}r$ with the "Jacobian"

$$J = \left|\begin{vmatrix} \frac{\partial a}{\partial p} & \frac{\partial a}{\partial q} & \frac{\partial a}{\partial r} \\ \frac{\partial b}{\partial p} & \frac{\partial b}{\partial q} & \frac{\partial b}{\partial r} \\ \frac{\partial c}{\partial p} & \frac{\partial c}{\partial q} & \frac{\partial c}{\partial r} \end{vmatrix}\right|$$

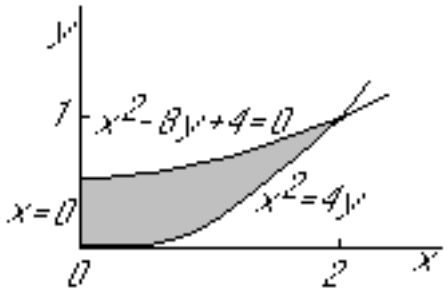
  Here the inner bars indicate the determinant of the matrix of derivatives and the outer bars the absolute value of that. (Sometimes it is easier to take the inverse of the Jacobian of the inverse transformation.)

8.2 Example

From [1, p. 487, 14e]

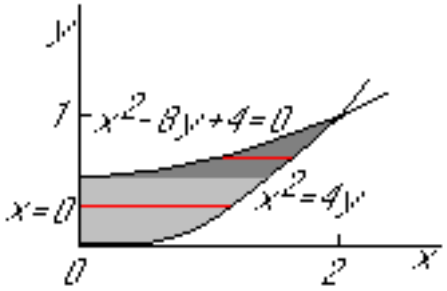
Asked: Find the centroid of the first-quadrant area bounded by $x^2 - 8y + 4 = 0$ and $x^2 = 4y$ and $x = 0$.

8.2.1 Region



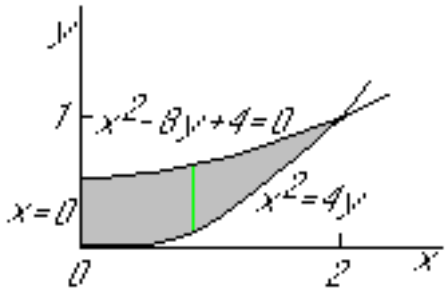
8.2.2 Approach

Integrate x first?



The integral would have to be split up into the light and dark areas since the lower boundary of integration is $x_1 = 0$ in the light region and $x_1 = \sqrt{8y - 4}$ in the dark region.

So integrate y first!



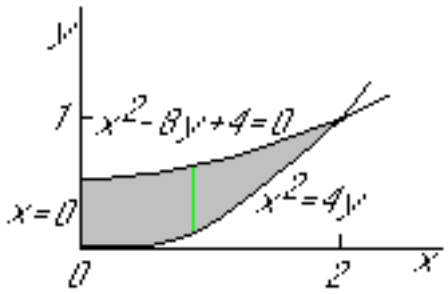
The boundaries of integration will be

$$y_1 = \tfrac{1}{4}x^2 \qquad y_2 = \tfrac{1}{8}x^2 + \tfrac{1}{2}$$

After integration over y, the remaining region of integration over x will be a line segment:

$$x_1 = 0 \qquad x_2 = 2$$

8.2.3 Results



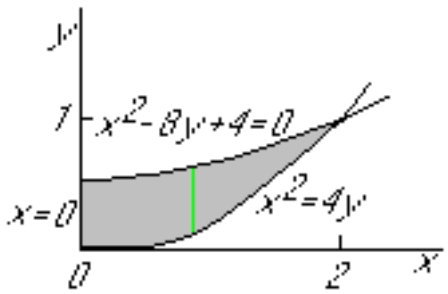
For $A = \int \mathrm{d}A = \int\int \mathrm{d}x\mathrm{d}y$:

$$A = \int_{x=0}^{x=2} \left[\int_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} \mathrm{d}y \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[y \Big|_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{8}x^2 + \tfrac{1}{2}) - (\tfrac{1}{4}x^2) \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{2} - \tfrac{1}{8}x^2 \right] \mathrm{d}x = \tfrac{2}{3}$$



For $A\bar{x} = \int x \mathrm{d}A = \int\int x \mathrm{d}x\mathrm{d}y$:

$$A = \int_{x=0}^{x=2} \left[\int_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} x \, \mathrm{d}y \right] \mathrm{d}x$$

where x is constant in the integration;

$$= \int_{x=0}^{2} \left[xy \Big|_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{8}x^3 + \tfrac{1}{2}x) - (\tfrac{1}{4}x^3) \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{2}x - \tfrac{1}{8}x^3 \right] \mathrm{d}x = \tfrac{1}{2}$$

Hence $\bar{x} = \frac{1}{2}/\frac{2}{3} = \frac{3}{4}$.
For $A\bar{y} = \int y \mathrm{d}A = \int\int y \mathrm{d}x\mathrm{d}y$:

$$A = \int_{x=0}^{x=2} \left[\int_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} y \, \mathrm{d}y \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[\tfrac{1}{2}y^2 \Big|_{y=\frac{1}{4}x^2}^{y=\frac{1}{8}x^2+\frac{1}{2}} \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[\tfrac{1}{2}(\tfrac{1}{8}x^2 + \tfrac{1}{2})^2 - \tfrac{1}{2}(\tfrac{1}{4}x^2)^2 \right] \mathrm{d}x$$

$$= \int_{x=0}^{2} \left[(\tfrac{1}{8} + \tfrac{1}{16}x^2 - \tfrac{3}{128}x^2 \right] \mathrm{d}x = \tfrac{4}{15}$$

Hence $\bar{y} = \frac{4}{15}/\frac{2}{3} = \frac{2}{5}$.

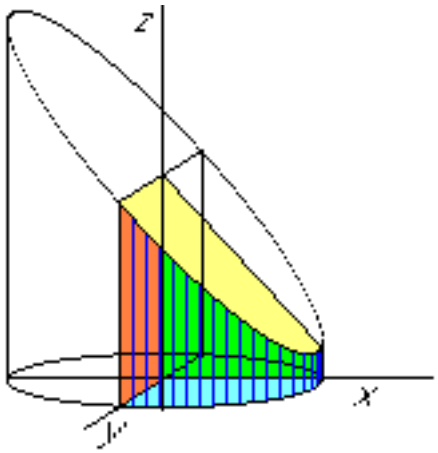
8.3 Example

From [1, p. 507, 21c]

Asked: Find the centroid of the first octant region inside $x^2 + y^2 = 9$ and below $x + z = 4$.

8.3.1 Approach

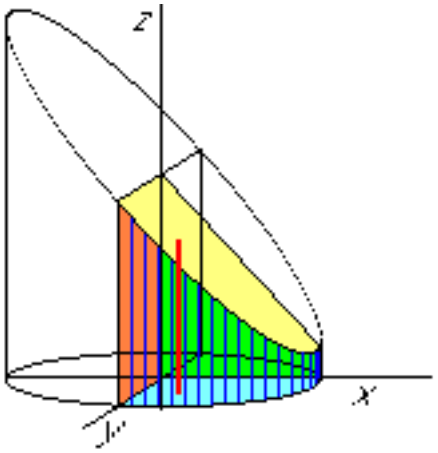
The region inside $x^2 + y^2 = 9$ is the inside of a cylinder of radius 3 around the z-axis. The equation $x + z = 4$ describes a plane through the y-axis under 45 degrees with the x-axis:



Use cylindrical coordinates r, θ, and z:

$$x = r\cos\theta \qquad y = r\sin\theta$$

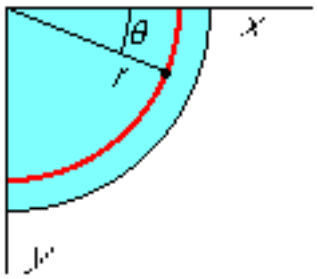
Integrate z first:



(Why not r first? Why not θ?). Boundaries are

$$z_1 = 0 \qquad z_2 = 4 - x = 4 - r\cos\theta$$

Next integrate θ and r:



$$\theta_1 = 0 \qquad \theta_2 = \tfrac{1}{2}\pi$$

$$r_1 = 0 \qquad r_2 = 3$$

8.3.2 Results

For the volume $V = \int\int\int \mathrm{d}V = \int\int\int r\ \mathrm{d}z\mathrm{d}r\mathrm{d}\theta$:

$$V = \int_{\theta=0}^{\pi/2}\int_{r=0}^{3}\left[\int_{z=0}^{4-r\cos\theta} r\ \mathrm{d}z\right]\mathrm{d}r\mathrm{d}\theta$$

$$= \int_{\theta=0}^{\pi/2}\left[\int_{r=0}^{3}(4-r\cos\theta)r\mathrm{d}r\right]\mathrm{d}\theta$$

$$= \int_{\theta=0}^{\pi/2} 18 - 9\cos\theta\mathrm{d}\theta = 9(\pi-1)$$

For $V\bar{x} = \int\int\int x\ \mathrm{d}V = \int\int\int xr\mathrm{d}z\mathrm{d}r\mathrm{d}\theta$:

$$V\bar{x} = \int_{\theta=0}^{\pi/2}\int_{r=0}^{3}\left[\int_{z=0}^{4-r\cos\theta} r^2\cos\theta\ \mathrm{d}z\right]\mathrm{d}r\mathrm{d}\theta$$

$$= \int_{\theta=0}^{\pi/2}\left[\int_{r=0}^{3} 4r^2\cos\theta - r^3\cos^2\theta\mathrm{d}r\right]\mathrm{d}\theta$$

$$= \int_{\theta=0}^{\pi/2} 36\cos\theta - \frac{81}{4}\cos^2\theta\mathrm{d}\theta = \frac{9}{16}(64-9\pi)$$

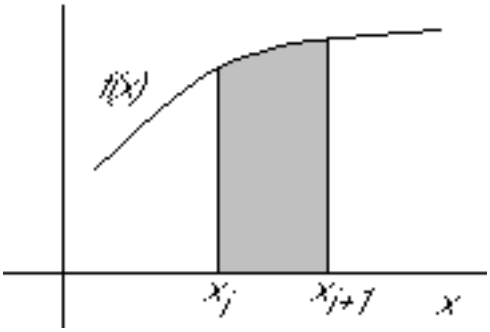
hence $\bar{x} = (64-9\pi)/16(\pi-1)$

Etcetera.

Chapter 9

Numerical Integration

9.1 Introduction

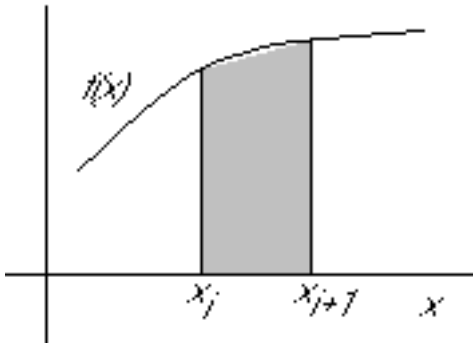


Numerical integration using Newton formulae:

- can handle any function;
- simple;
- can handle measured data easily.

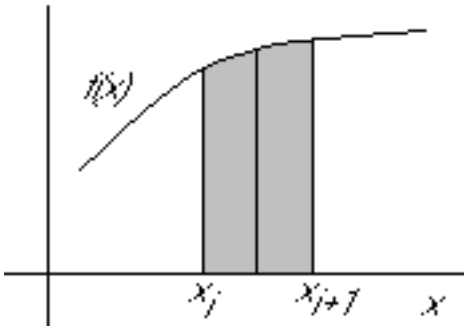
Trapezium rule for an interval from $x = x_i$ to x_{i+1}:

$$\int_{x_i}^{x_{i+1}} f(x)\,\mathrm{d}x \approx (x_{i+1} - x_i)\,\frac{f(x_i) + f(x_{i+1})}{2}$$



Simpson rule for an interval from $x = x_i$ to x_{i+1}:

$$\int_{x_i}^{x_{i+1}} f(x)\,\mathrm{d}x \approx (x_{i+1} - x_i)\,\frac{f(x_i) + 4f(x_{i+\frac{1}{2}}) + f(x_{i+1})}{6}$$



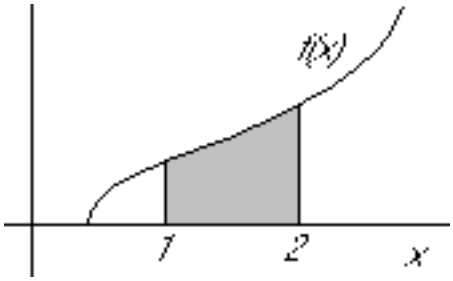
These rules are accurate if the interval from x_i to x_{i+1} is sufficiently small. To integrate over an interval that is not small, divide it into small ones, then integrate over each small interval and add the results.

9.2 Example

From [1, p. 205, 44], modified.

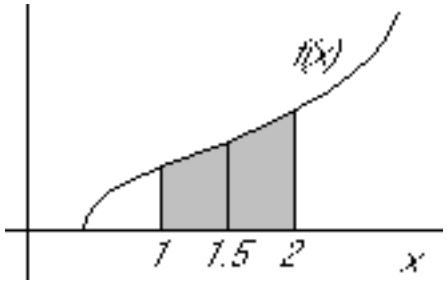
Asked:

$$\int_1^2 x\sqrt[3]{x^5 + 2x^2 - 1}\,\mathrm{d}x$$



9.2.1 Solution

Divide into n=2 intervals and use the trapezium rule:



If

$$f(x) = x\sqrt[3]{x^5 + 2x^2 - 1}$$

then the trapezium rule gives

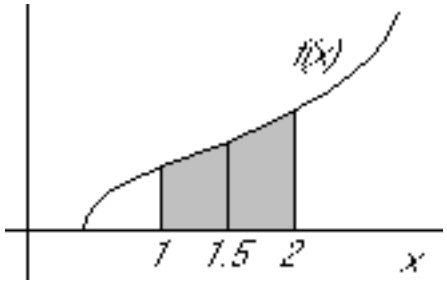
$$\int_1^{1.5} f \, \mathrm{d}x = 0.5\frac{f(1) + f(1.5)}{2} = 0.5\frac{1.259921 + 3.345421}{2} = 1.151336$$

$$\int_{1.5}^{2} f \, \mathrm{d}x = 0.5\frac{f(1.5) + f(2)}{2} = 0.5\frac{3.345421 + 6.782423}{2} = 2.531961$$

$$\int_1^2 f \, \mathrm{d}x = 1.151336 + 2.531961 = 3.683297$$

Exact is 3.571639.

Now divide into n=2 *half* intervals and use the Simpson rule:



$$\int_1^2 f \, \mathrm{d}x = 1\frac{f(1) + 4f(1.5) + f(2)}{6}$$

$$= 1\frac{1.259921 + 4 * 3.345421 + 6.782423}{6} = 3.570671$$

Closer to the exact value 3.571639.

Chapter 10

Geometry using vectors

10.1 Introduction

Vectors for geometry:

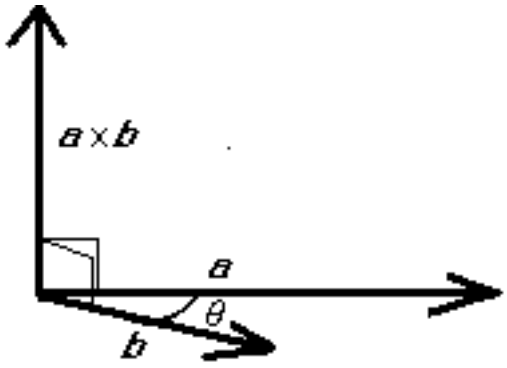
- straight line trajectories;
- surfaces;
- ...
- Dot (scalar) product:

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = |\vec{a}||\vec{b}| \cos \vartheta$$

- Cross (vector) product:

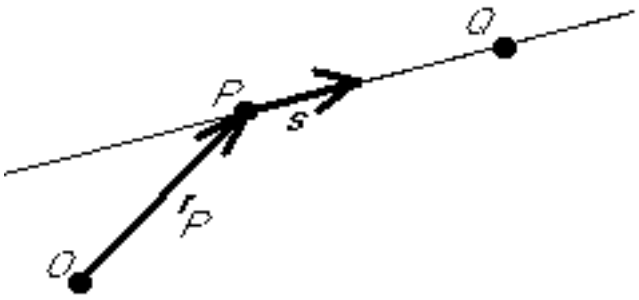
$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \vartheta$$

  and normal to both vectors. Seen from below:



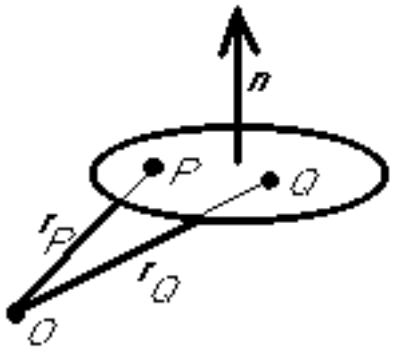
- Line through point P parallel to vector $\vec{s}$:

$$\vec{r} = \vec{r}_P + \lambda\vec{s}$$



- Plane through point P normal to vector $\vec{n}$:

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_P$$

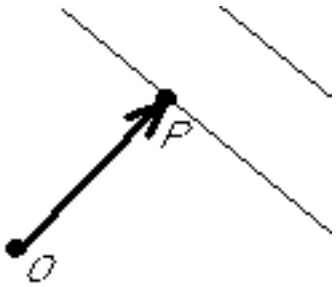


- Each equation ordinarily reduces the dimensionality by one: 3D (space) $\rightarrow$ 2D (plane) $\rightarrow$ 1D (line) $\rightarrow$ 0D (point) $\rightarrow$ nothing.

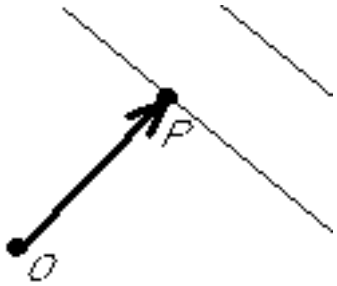
10.2 Example

From [1, p. 439, 35(b)].

Asked: The line through point P_0, (2,-3,5), and parallel to the line $x - y + 2z + 4 = 0$, $2x + 3y + 6z - 12 = 0$.



10.2.1 Identification



- I need a vector in the direction of the desired line.
- This is the same direction as the given line.
- The two equations give me vectors $\vec{n}_1$ and $\vec{n}_2$ normal to the given line
- Cross the two vectors!

10.2.2 Solution

$$x - y + 2z + 4 = 0 \qquad \Rightarrow \qquad \vec{n}_1 = (1, -1, 2)$$

$$2x + 3y + 6z - 12 = 0 \qquad \Rightarrow \qquad \vec{n}_2 = (2, 3, 6)$$

$$\vec{s} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 2 \\ 2 & 3 & 6 \end{vmatrix} = \begin{pmatrix} -12 \\ -2 \\ 5 \end{pmatrix}$$

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -12 \\ -2 \\ 5 \end{pmatrix}$$

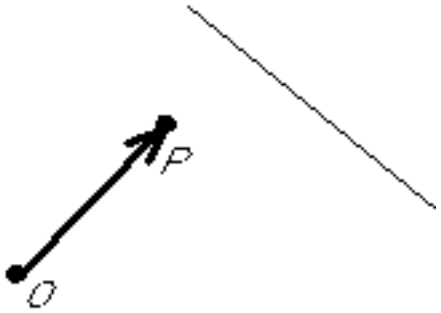
Alternatively:

$$\frac{x-2}{-12} = \frac{y+3}{-2} = \frac{z-5}{5} (= \mu)$$

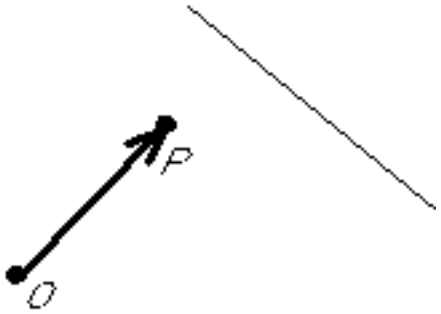
10.3 Example

From [1, p. 440, 36(b)].

Asked: The plane through point P_0, (2,-3,2), and the line $6x+4y+3z+5 = 0$, $2x + y + z - 2 = 0$.



10.3.1 Identification



- I need a vector normal to the plane.
- I can get this by crossing two vectors in the plane.
- One such vector is $\vec{n}_1 \times \vec{n}_2$.
- To find another, find *any* point Q on the line, then $r_Q - r_{P_0}$ is in the plane.

10.3.2 Solution

$$6x + 4y + 3z + 5 = 0 \qquad \Rightarrow \qquad \vec{n}_1 = (6, 4, 3)$$

$$2x + y + z - 2 = 0 \qquad \Rightarrow \qquad \vec{n}_2 = (2, 1, 1)$$

$$\vec{s} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 6 & 4 & 3 \\ 2 & 1 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

When $x = 0$ on the line,

$$4y + 3z + 5 = 0, \quad y + z - 2 = 0 \qquad \Rightarrow \qquad x = 0, \quad y = -11, \quad z = 13$$

$$\vec{n} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \times \left[\begin{pmatrix} 0 \\ -11 \\ 13 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} -2 \\ -8 \\ 11 \end{pmatrix} = \begin{pmatrix} -16 \\ -7 \\ -8 \end{pmatrix}$$

$$\begin{pmatrix} 16 \\ 7 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16 \\ 7 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

$$16x + 7y + 8z = 27$$

Chapter 11

Vector Analysis

11.1 Coordinate Changes

11.1.1 General

Suppose that starting at Cartesian coordinates, we make a switch to new coordinates, like cylindrical or spherical ones? What is involved such a change in coordinates?

To find out, as always take the Cartesian coordinates (x, y, z) to form a vector

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

in physical space. Let the new coordinates be called (u_1, u_2, u_3). You can take these to form a vector $\vec{u}$ in an artificial "parameter space". Note that that vector has no physical meaning; it is just a concise way of writing the three coordinates.

The "Jacobian matrix" is defined as the matrix of derivatives

$$\boxed{\frac{\partial \vec{r}}{\partial \vec{u}} \equiv \begin{pmatrix} \dfrac{\partial x}{\partial u_1} & \dfrac{\partial x}{\partial u_2} & \dfrac{\partial x}{\partial u_3} \\ \dfrac{\partial y}{\partial u_1} & \dfrac{\partial y}{\partial u_2} & \dfrac{\partial y}{\partial u_3} \\ \dfrac{\partial z}{\partial u_1} & \dfrac{\partial z}{\partial u_2} & \dfrac{\partial z}{\partial u_3} \end{pmatrix}}$$

Note that often it is easier to differentiate the new coordinates with respect to the physical ones, rather than vice versa. If so, you can find the Jacobian matrix as the inverse:

$$\boxed{\frac{\partial \vec{r}}{\partial \vec{u}} = \left(\frac{\partial \vec{u}}{\partial \vec{r}} \right)^{-1}}$$

This is based on the fact that the product of the two matrices above equals $\partial\vec{r}/\partial\vec{r}$, which is the unit matrix.

Any small change $\mathrm{d}\vec{u}$ in the artificial vector $\vec{u}$ corresponds to a small change in physical position $\mathrm{d}\vec{r}$ given by:

$$\boxed{\mathrm{d}\vec{\vec{r}} = \frac{\partial \vec{r}}{\partial \vec{u}} \mathrm{d}\vec{u}}$$

So if the Jacobian matrix is well defined, *and its determinant nonzero*, any small change in physical space $\mathrm{d}\vec{r}$ will correspond to a *unique* displacement $\mathrm{d}\vec{u}$ in coordinate space, and vice versa. If however the coefficients of the Jacobian matrix are undefined at a point, *or the determinant is zero*, the mapping is called locally singular at that point.

So the determinant of the Jacobian matrix is very important. This determinant is called the "Jacobian" J:

$$\boxed{J \equiv \left| \frac{\partial \vec{r}}{\partial \vec{u}} \right| \equiv \begin{vmatrix} \dfrac{\partial x}{\partial u_1} & \dfrac{\partial x}{\partial u_2} & \dfrac{\partial x}{\partial u_3} \\ \dfrac{\partial y}{\partial u_1} & \dfrac{\partial y}{\partial u_2} & \dfrac{\partial y}{\partial u_3} \\ \dfrac{\partial z}{\partial u_1} & \dfrac{\partial z}{\partial u_2} & \dfrac{\partial z}{\partial u_3} \end{vmatrix}}$$

If you would rather differentiate the new coordinates with respect to the old, the determinant will give the inverse of J.

Now suppose you take a little "block" $\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3$ in the artificial parameter space. In *physical* space, that little block will correspond to a little parallelepiped with sides

$$\boxed{\mathrm{d}\vec{r}_1 \equiv \frac{\partial \vec{r}}{\partial u_1}\mathrm{d}u_1 \qquad \mathrm{d}\vec{r}_2 \equiv \frac{\partial \vec{r}}{\partial u_2}\mathrm{d}u_2 \qquad \mathrm{d}\vec{r}_3 \equiv \frac{\partial \vec{r}}{\partial u_3}\mathrm{d}u_3}$$

The volume of this small parallelepiped, call it $\mathrm{d}V$ is given by the scalar triple product of its three sides, which equals the determinant of these three sides:

$$\mathrm{d}V = \mathrm{d}\vec{r}_1 \cdot (\mathrm{d}\vec{r}_2 \times \mathrm{d}\vec{r}_3) = \left| \frac{\partial \vec{r}}{\partial u_1} \frac{\partial \vec{r}}{\partial u_2} \frac{\partial \vec{r}}{\partial u_3} \right| \mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3$$

But the determinant above is just the Jacobian J! So it follows that if you would rather integrate some function f in new coordinates instead of physical ones, you must simply add a Jacobian:

$$\boxed{\iiint f \, \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint f|J| \, \mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3}$$

11.1.2 Orthogonal coordinates

Orthogonal coordinates are a special case of the new coordinates discussed in the previous subsection. For orthogonal coordinates, the little parallelepiped in physical space corresponding to a little block $\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3$ in coordinate space has orthogonal sides. So it too is a little block, rather than just a little parallelepiped.

Since the three sides are proportional to the derivatives $\partial\vec{r}/\partial u_1$, $\partial\vec{r}/\partial u_2$, and $\partial\vec{r}/\partial u_3$, these derivatives must be orthogonal for orthogonal coordinates. That means that if you write them as magnitudes h_1, h_2, and h_3 times unit vectors $\hat{\imath}_1$, $\hat{\imath}_2$, and $\hat{\imath}_3$,

$$\boxed{\frac{\partial\vec{r}}{\partial u_1} \equiv h_1\hat{\imath}_1 \qquad \frac{\partial\vec{r}}{\partial u_2} \equiv h_2\hat{\imath}_2 \qquad \frac{\partial\vec{r}}{\partial u_3} \equiv h_3\hat{\imath}_3}$$

then these three unit vectors are orthogonal at each point. So you can use them as a local orthogonal coordinate system. And write any vector $\vec{v}$ at the point in the form

$$\boxed{\vec{v} = v_1\hat{\imath}_1 + v_2\hat{\imath}_2 + v_3\hat{\imath}_3}$$

You can then convert the gradient, divergence, and curl operators to the new coordinates. These formulas will involve the magnitudes, called "metric indices" h_1, h_2, and h_3. Since you can find these formulae in any mathematical handbook, they will not be discussed here.

But often you want to find other derivatives in the new coordinates. To do so, you must know how to find the derivatives of the unit vectors $\hat{\imath}_1$, $\hat{\imath}_2$, and $\hat{\imath}_3$. These formulae are not soeasy to find, so they will be given here. For any i equal to 1, 2, or 3, and any j equal to 1, 2, or 3,

$$\boxed{\frac{\partial\hat{\imath}_i}{\partial u_j} = \frac{1}{h_i}\frac{\partial h_j}{\partial u_i}\hat{\imath}_j \quad \text{if} \quad j \neq i \qquad \frac{\partial\hat{\imath}_i}{\partial u_i} = \frac{1}{h_i}\frac{\partial h_i}{\partial u_i}\hat{\imath}_i - \sum_{k=1}^{3}\frac{1}{h_k}\frac{\partial h_i}{\partial u_k}\hat{\imath}_k}$$

The derivation is in {D.1}.

Part II

Linear Algebra

Chapter 12

Gaussian Elimination

12.1 Elimination Procedure

To do Gaussian (or forward) elimination in this class, the general procedure consists of four basic steps. Take the initial "submatrix" to look at to be the complete given matrix. Then:

GE I If there is only one row left in the submatrix currently being looked at, you are done with Gaussian elimination. And so you are if there are no more nonzero coefficients in the submatrix.

GE II In the submatrix currently being looked at, identify the *first* column that still has a nonzero coefficient. Any earlier columns no longer appear anywhere, and should be dropped from the submatrix being looked at.

GE III In the submatrix currently being looked at, use the *first* row to eliminate the *first* coefficient from the *subsequent* rows. Normally, you do that by subtracting suitable multiples of the first row from the subsequent ones. Step GE III can be described as "creating zeros below the pivot." Note that this is only possible if the coefficient in the first column of the first row, called the *pivot*, is nonzero. In general, before doing the actual step GE III, you may need to exchange the current *first* row with a *later* one that will produce a better pivot. The next section will explain that in more detail.

GE IV Drop the first row and the first column from the submatrix being looked at. With the so-reduced submatrix, repeat the process starting from step GE I.

This process will give rise to a so-called "echelon form" of the matrix. That is a special case of an upper triangular (if square) or upper trapezoidal (if not square) matrix.

Additional note: the eliminations in GE II can be done in a single step for all rows below the pivot. You do not have to show each row using a separate matrix. You must however explicitly show the multiplier(s) used.

12.2 Partial Pivoting

Partial pivoting is defined as interchanging the *first* row of the submatrix currently being looked at with a subsequent one. Its purpose is to replace the current pivot by a more desirable one. In particular, partial pivoting *must* be used if otherwise the pivot would be zero.

More generally, partial pivoting must be used to select the equation with the most desirable pivot. In particular

- In a *numerical* solution of systems of equations on a computer, the main objective is to minimize the effect of round-off errors. Here you should select the row having the pivot with the largest *absolute* value. (This assumes that the equations are scaled in a consistent, reasonable way.)
- In hand computations with integer matrices in homework or an exam in this class, the objective is to avoid fractions in the algebra. So you try to select a pivot which is ± 1.

  Note that there may not be such a pivot. In that case, you may be still be able to avoid fractions by partial pivoting. Or else you might be able to avoid them by multiplying the row being replaced by an integer greater than 1 instead of 1.

  If that works, it should *normally* be done. However, if you want to identify matrices L and U in the LU theorem, you can only multiply the row being replaced by 1. Otherwise the theorem does not work. So you may have to live with fractions.

12.3 Back substitution

To solve an echelon system resulting from Gaussian elimination, start with the last equation and work backwards to the first. At any stage,

- If all the coefficients of the unknowns are zero, and the right hand side too, the equation is trivial. Ignore it.
- If all the coefficients of the unknowns are zero, and the right hand side is not, there is no way to satisfy it. So there is no solution to the given system of equations. Note so explicitly.
- In all other cases, solve the equation for the pivot variable (the variable with the pivot, i.e. with the first nonzero coefficient.) Take the other terms to the right hand side and substitute in anything you already learned in solving the previous equations below it.

12.4 LU Theorem

If you put minus the multipliers in a Gaussian elimination without partial pivoting in a lower triangular matrix L, with values 1 on the main diagonal, then

$$\boxed{LU = A} \tag{12.1}$$

That is the LU theorem.

After L and U have been found through Gaussian elimination, you can solve $A\vec{x} = \vec{b}$ for any given right hand side vector $\vec{b}$ as follows: The system to solve is according to the theorem $LU\vec{x} = \vec{b}$. Temporarily call $U\vec{x} = \vec{y}$. Solve $L\vec{y} = \vec{b}$ using simple forward substitution. With $\vec{y}$ now known, $U\vec{x} = \vec{y}$ can be solved using back substitution as before:

$$\boxed{L\vec{y} = \vec{b} \qquad U\vec{x} = \vec{y}} \tag{12.2}$$

In case partial pivoting was needed, remember the row order interchanges you did and do the same interchanges with the coefficients of $\vec{b}$ before solving as above. You also need to do the row exchanges on the forming matrix L while doing the GE. LU will then be the matrix A^{pp}, equal to matrix A except with its rows in the final order produced by all the row interchanges.

For large $n \times n$ matrices the number of computer "operations" (defined as 1 multiplication and 1 addition) to find L and U is approximately $\frac{1}{3}n^3$. After that, to find a $\vec{x}$ given a $\vec{b}$ takes only about n^2 operations.

Some warnings. Normally speaking:

1. Never ever use Cramer's rule for anything but the tiniest of matrices. Multiplying out a determinant takes about $(n+1)!$ operations, which is gigantically larger than $\frac{1}{3}n^3$ for everything except the tiniest n. Not to mention the possible round-off error growth with so many operations. And the risk of overflow and underflow (i.e. numbers getting outside the range that they can be stored on a computer.)
2. If you find the inverse matrix A^{-1}, you can simply find any $\vec{x}$ as $A^{-1}\vec{b}$. But that is normally a bad idea. One reason is that it takes n^3 operations to find A^{-1}, three times more than to find L and U. And to evaluate $A^{-1}\vec{b}$ still takes n^2 operations. Also the additional operations tend to increase round-off error, [2].
3. A *band matrix* is a matrix in which the nonzero elements are restricted to a relatively narrow band around the main diagonal. Never ever use an LU decomposition library routine designed for a full matrix to solve a system with a band matrix. The waste in storage and computer effort to store all these zeros, and compute with them, would be gigantic. Use an LU decomposition subroutine designed for a band matrix.

4. Never ever find the inverse of a band matrix if you can help it. The inverse will be a full matrix. However, L and U will still be band matrices. (Partial pivoting may increase U a bit to outside the band of A, by up to the width of L.)

12.5 Row Canonical Form

To reduce a matrix to "row canonical" form, (AKA "row reduced echelon" form, or "reduced row-echelon" form, or "Gauss-Jordan" form), first reduce it to echelon form using Gaussian elimination as described in section 12. Then, starting from the last equation with a nonzero coefficient of an unknown,

1. Use the equation to eliminate the first unknown with a nonzero coefficient in the equation, (i.e. the pivot unknown), from the previous equations (i.e. create zeros above the pivot). Do not worry about fractions anymore; they cannot normally be avoided.
2. Divide the equation by the pivot so that the pivot becomes 1.
3. Go to the previous equation, if any, and repeat the process.

Note: you can do both steps 1 and 2 above in each stage at the same time. You do not have to show each row nor the final scaling using a separate matrix. You must however explicitly show the multiplier(s) used.

Note: in this class you may *not* reduce a matrix to row canonical unless you are explicitly told so or the given class procedure requires it. Normally you must stop after you did the Gaussian elimination (producing normal echelon form with pivots that in general are not 1). That is because this is the standard and most efficient way to do it.

You may however proceed to row-canonical when finding the inverse of a matrix using the Gaussian Elimination method described later. You must proceed to row-canonical to find a simplified basis of the row space of a matrix, or of the column space, (starting from the transpose matrix), as described later.

12.6 Null Spaces and Solution Spaces

To find the null space of a matrix, reduce it to echelon form as described earlier. To refresh your memory, the first nonzero elements in the rows of the echelon form are the pivots. Solve the homogeneous system by back substitution as also described earlier. To refresh your memory, you solve for the *pivot* variables. The variables *without* pivots cannot be solved for and become parameters with arbitrary values in the null space, multiplying "basis vectors". The coefficients inside the basis vectors come from the solved variables or from writing trivialities.

For example, if your unknowns are $(x_1, x_2, x_3, x_4, x_5, x_6)$ and your echelon matrix is

$$\begin{pmatrix} 0 & 2 & 4 & 6 & 8 & 6 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

then the last equation is trivial and you get from the third

$$2x_5 + 4x_6 = 0 \quad \Longrightarrow \quad x_5 = -2x_6$$

then from the second

$$x_4 = 0$$

then from the first

$$2x_2 + 4x_3 + 6(0) + 8(-2x_6) + 6x_6 = 0 \quad \Longrightarrow \quad x_2 = -2x_3 + 5x_6$$

To get the null *space* (i.e. the full set of vectors $(x_1, x_2, x_3, x_4, x_5, x_6)$ that produce zero when premultiplied by the original A), the variables (x_1, x_3, x_6) *without pivots* go in the right hand side as arbitrary constants that can be anything:

$$\text{null space:} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}_{\text{null space}} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 0 \\ 5 \\ 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

The coefficients for the pivot variables (x_2, x_4, x_5) in the vectors in the right hand side come from the solved equations, and those for (x_1, x_3, x_6) from trivialities.

To get *a basis* for the null space, you can use the constant vectors in the right hand side:

$$\text{a basis of the null space:} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 \\ 5 \\ 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

(By definition any vector in the null space is a linear combination of the above three vectors. And it is easy to see that the three are linearly independent.)

If the above basis would contain fractions, you should consider multiplying them by some nonzero constants to clean up. Note that that will of course affect the given expression for the null space.

To find a solution space is almost the same as finding the null space, except that you will use an augmented matrix to include the given nonzero right hand side. The right hand side will produce an additional vector in the solution space that is not multiplied by any unknown. Therefore the solution space is not a vector space (assuming that the given right hand side is not zero) and has no basis.

Chapter 13

Inverse Matrices

13.1 Finding inverses using GE

Note: if you are required to find an inverse using minors in this class, do not use *any* GE, even for attractive simplifications.

Note: If a matrix is orthonormal, its inverse *must* be found using a transpose in this class.

Finding the inverse of a matrix is usually a bad idea. But if you do need it, you may use the following trick:

1. Create an augmented matrix in which the n right hand side column vectors are the columns of the unit matrix.
2. Reduce the matrix to row canonical form using Gaussian elimination as described earlier. In other words

$$(A|I) \quad \xrightarrow{\text{GE}} \quad (A_{\text{RowCan}}|B_{\text{Red}})$$

3. For a nonsingular matrix A, A_{RowCan} will be a unit matrix and B_{Red} will be the desired A^{-1}. (For a singular matrix, one of the pivots will be found to be zero in the reduction to echelon form, and there is no inverse matrix.)

Note that what you are really doing in the above is solving the system of equations $AA^{-1} = I$ for the columns of A^{-1}. The only trick is that if you reduce the matrix all the way to a unit matrix, the solution A^{-1} becomes the same as the right hand side of the reduced system.

Also note that the most efficient way to find the inverse, in terms of operations, is still LU decomposition. Normally this would take $\frac{1}{3}n^3$ operations to find L and U and then n^3 operations to find the n columns of A^{-1}. But the right hand sides to the $U\vec{y} = \vec{b}$ systems are unit vectors, which are mostly zeros. If you use this to avoid computing coefficients in y that are automatically zero, you can reduce the operations by $\frac{1}{3}n^3$.

13.2 Finding inverses using minors

Note: if you are required to find an inverse using GE in this class, do not use *any* minors, even on 2×2 matrices.

Note: If a matrix is orthonormal, its inverse *must* be found using a transpose in this class.

In this class, if you need to find an inverse matrix using the method of minors,

- Write the transpose matrix.
- Form a matrix of the minors of each of the elements of the transpose matrix, with a checkerboard sign pattern. (Positive on the main diagonal.)
- Divide by the determinant.
- Evaluate.

13.3 Finding inverses using transposing

A matrix is orthonormal if its columns are mutually orthogonal and of length 1. For an orthonormal square matrix, its inverse *must* be found using a transpose in this class. This is mainly important, in this class, for the transformation matrices that diagonalize symmetric matrices.

Chapter 14

Eigenvalues and Eigenvectors

14.1 Finding Eigenvalues

In this class, to find the eigenvalues of an $n \times n$ matrix,

1. Form the matrix $A - \lambda I$. That means, add $-\lambda$ to each diagonal elements. (Don't forget zero diagonal elements.)
2. Find the determinant of that matrix using *the method of minors*. (Gaussian elimination is too messy here and should not be used.)
3. Set this determinant to zero. For an $n \times n$ matrix, the determinant can *always* be written in the form

$$|A| = \pm(\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n)$$

   where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the *eigenvalues*. They are found as the roots (or zeros) of the determinant. There are always n eigenvalues. But these n eigenvalues do not necessarily correspond to n different numbers. For example, for some matrix λ_2 might be the same number as λ_4. In that case, that number is called a "double eigenvalue". If three of the eigenvalues are the same number, that number is called a "triple eigenvalue," etcetera.

For an $n \times n$ matrix, the determinant is always a polynomial of degree n, call it $p_n(\lambda)$. Now finding the roots of quadratics, $n = 2$, is easy. But if the dimension of the matrix $n = 3$, you have to solve a cubic equation. For that the general solution is very and hard to apply, especially if you do not know complex variables. For $n = 4$, the general solution is even worse, and for $n = 5$ or more, there is no general expression for the roots at all. (It has in fact been proved that such an expression is impossible to find.) To deal with such problems, here are some tricks:

- Do not be too quick to multiply out. Maybe you can recognize a common factor $\lambda - \lambda_i$ in all terms before multiplying out. In that case, λ_i is one of your eigenvalues. To find the remaining eigenvalues,

take the factor $\lambda - \lambda_i$ out of the entire expression and look at what is left.

- If you can guess a root λ_i (by trying, say, 0, ± 1, ± 2, ..., and seeing whether the determinant is zero for that value of λ), then you can write the original characteristic polynomial $p_n(\lambda)$ as $(\lambda - \lambda_i)p_{n-1}(\lambda)$ where $p_{n-1}(\lambda)$ is one degree less that $p_n(\lambda)$. So its roots are easier to find. You find $p_{n-1}(\lambda)$ by long division of $p_n(\lambda)$ by $(\lambda - \lambda_i)$.
- Sometimes a fourth order polynomial is really just a quadratic when written in terms of λ^2, like $(\lambda^2)^2 - a(\lambda^2) - b$. In that case, first find the values of λ^2, then from those the values of λ.

14.2 Eigenvectors of nonsymmetric matrices

To find the eigenvectors of a nonsymmetric matrix,

1. Find the eigenvalues of the matrix as described.
2. For each *distinct* eigenvalue λ_i, find the basis of the null space of $A - \lambda_i I$. This basis forms a complete set of eigenvectors $\vec{e}_i$ for the eigenvalue. The method for finding nullspaces was discussed earlier.

There is always at least one eigenvector. However, for a multiple eigenvalue, the number of eigenvectors might be less than the multiplicity of the eigenvalue. If so, the matrix is *defective*. (This is not to be confused with singular. A matrix is singular if an eigenvalue is zero, making its determinant zero.) For example, a matrix is defective if a double eigenvalue has only one eigenvector. Or a triple eigenvector only one or two.

14.3 Eigenvectors of symmetric matrices

Real symmetric matrices (or more generally, complex Hermitian matrices) always have real eigenvalues, and they are never defective. Their eigenvectors can, and in this class must, be taken orthonormal. (Mutually orthogonal and of length 1.)

For real symmetric matrices, initially find the eigenvectors like for a nonsymmetric matrix. However, after that, you *must* make the eigenvectors orthonormal in this class.

- For eigenvectors belonging to single eigenvalues, orthogonality to all the other eigenvectors is automatic. Therefore it is enough to just divide the eigenvector by its length:

$$\boxed{\vec{e}_i = \vec{e}_i^{\,\mathrm{BAD}}/|\vec{e}_i^{\,\mathrm{BAD}}|} \tag{14.1}$$

Clean up, and that is it.

- However, eigenvectors belonging to a multiple eigenvalue are usually not mutually orthogonal if correctly found using the class procedure (as basis of the $A-\lambda_i I$ null space). You must *make* them orthogonal.

  The most general method to do that is using the Gram-Schmidt procedure. This works as follows. Start with the nonorthogonal eigenvectors corresponding to the multiple eigenvalue,

  $$\vec{e}_1^{\,\mathrm{BAD}}, \vec{e}_2^{\,\mathrm{BAD}}, \vec{e}_3^{\,\mathrm{BAD}}, \ldots$$

  obtained using the same procedure as for nonsymmetric matrices. For an eigenvalue of multiplicity m, there will be exactly m of these eigenvectors (since symmetric matrices can never be defective). I am assuming here that you have numbered these eigenvectors as 1, 2, ... If not, just replace 1, 2, ... by i_1, i_2, ...

  You create your first good eigenvector $\vec{e}_1$ as before, by simply dividing $\vec{e}_1^{\,\mathrm{BAD}}$ by its length:

  $$\vec{e}_1 = \vec{e}_1^{\,\mathrm{BAD}}/|\vec{e}_1^{\,\mathrm{BAD}}|$$

  Now you need to make the remaining vectors in the set of m orthogonal to $\vec{e}_1$. You do that by removing from each vector its vector component in the direction of $\vec{e}_1$, as follows:

  $$\vec{e}_2^{\,\mathrm{BAD},2} = \vec{e}_2^{\,\mathrm{BAD}} - (\vec{e}_2^{\,\mathrm{BAD}} \cdot \vec{e}_1)\vec{e}_1 \quad \vec{e}_3^{\,\mathrm{BAD},2} = \vec{e}_3^{\,\mathrm{BAD}} - (\vec{e}_3^{\,\mathrm{BAD}} \cdot \vec{e}_1)\vec{e}_1 \quad \ldots$$

  Vectors $\vec{e}_2^{\,\mathrm{BAD},2}, \vec{e}_3^{\,\mathrm{BAD},2}, \ldots$ are now orthogonal to $\vec{e}_1$.

  Now you create your second good eigenvector $\vec{e}_2$ by simply dividing $\vec{e}_2^{\,\mathrm{BAD},2}$ by its length

  $$\vec{e}_2 = \vec{e}_2^{\,\mathrm{BAD},2}/|\vec{e}_2^{\,\mathrm{BAD},2}|$$

  If there are still vectors left, you need to make each orthogonal to $\vec{e}_2$. In the same way as before

  $$\vec{e}_3^{\,\mathrm{BAD},3} = \vec{e}_3^{\,\mathrm{BAD},2} - (\vec{e}_3^{\,\mathrm{BAD},2} \cdot \vec{e}_2)\vec{e}_2 \quad \ldots$$

  Now you create your third good eigenvector $\vec{e}_3$ by simply dividing $\vec{e}_3^{\,\mathrm{BAD},3}$ by its length

  $$\vec{e}_3 = \vec{e}_3^{\,\mathrm{BAD},3}/|\vec{e}_3^{\,\mathrm{BAD},3}|$$

  If there are still vectors $\vec{e}_4^{\,\mathrm{BAD},3}, \ldots$ left, you now have to make them orthogonal to $\vec{e}_3$. Then you can find $\vec{e}_4$ by dividing $\vec{e}_4^{\,\mathrm{BAD},4}$ by its length.

  Continue like this for multiplicities greater than 4. The general formula is

  $$\boxed{\vec{e}_j^{\,\mathrm{BAD,new}} = \vec{e}_j^{\,\mathrm{BAD,old}} - (\vec{e}_j^{\,\mathrm{BAD,old}} \cdot \vec{e}_i)\vec{e}_i \quad \text{for} \quad i = j{+}1, j{+}2, \ldots, m} \tag{14.2}$$

  where e_i is the last good eigenvector obtained so far and $j > i$.

Chapter 15

Change of Basis

15.1 General Procedure

Normally, you use "Cartesian coordinates" when dealing with vectors. That is based on "Cartesian basis vectors" $\hat{\imath}, \hat{\jmath}, \hat{k}, \ldots$. For example, in two dimensions,

$$\hat{\imath} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \hat{\jmath} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These Cartesian basis vectors are shown in red in figure 15.1*(a)*. Using these two basis vectors, you can write *any* vector $\vec{v}$ in the form

$$\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath}$$

This is illustrated graphically in figure 15.1*(b)*. Note how, starting from the origin, if you first move over $v_1\hat{\imath}$ and then over $v_2\hat{\jmath}$ you reach the end point of vector $\vec{v}$. Now v_1 and v_2 are *ordinary numbers* that are called the "components", or "coefficients," or "coordinates" of vector $\vec{v}$.

Therefore, in this way, you are never have to deal with more vectors that $\hat{\imath}$ and $\hat{\jmath}$. All the rest is just ordinary numbers.

But sometimes it is convenient to use a different basis than the obvious $\hat{\imath}, \hat{\jmath}$ one. For example, you might know that in dealing with plane stresses, it is often convenient to rotate the coordinate system to the "principal axes." In principal axes, there are no shear stresses, just normal stresses. But if you rotate the coordinate system, $\hat{\imath}$ and $\hat{\jmath}$ become different vectors, call them $\hat{\imath}'$ and $\hat{\jmath}'$. The point however is that in using these *new* basis vectors $\hat{\imath}'$ and $\hat{\jmath}'$, your physical problem has simplified.

As we will see later, to simplify a problem, the desired new vectors are not always orthonormal (orthogonal and of length 1) like $\hat{\imath}'$ and $\hat{\jmath}'$ in the example above. In general, the new basis vectors, we will call them $\vec{p}_1$ and $\vec{p}_2$, can be anything, as long as they are linearly independent. As long as that is true, you can still write any arbitrary vector $\vec{v}$ as

$$\vec{v} = v_1'\vec{p}_1 + v_2'\vec{p}_2$$

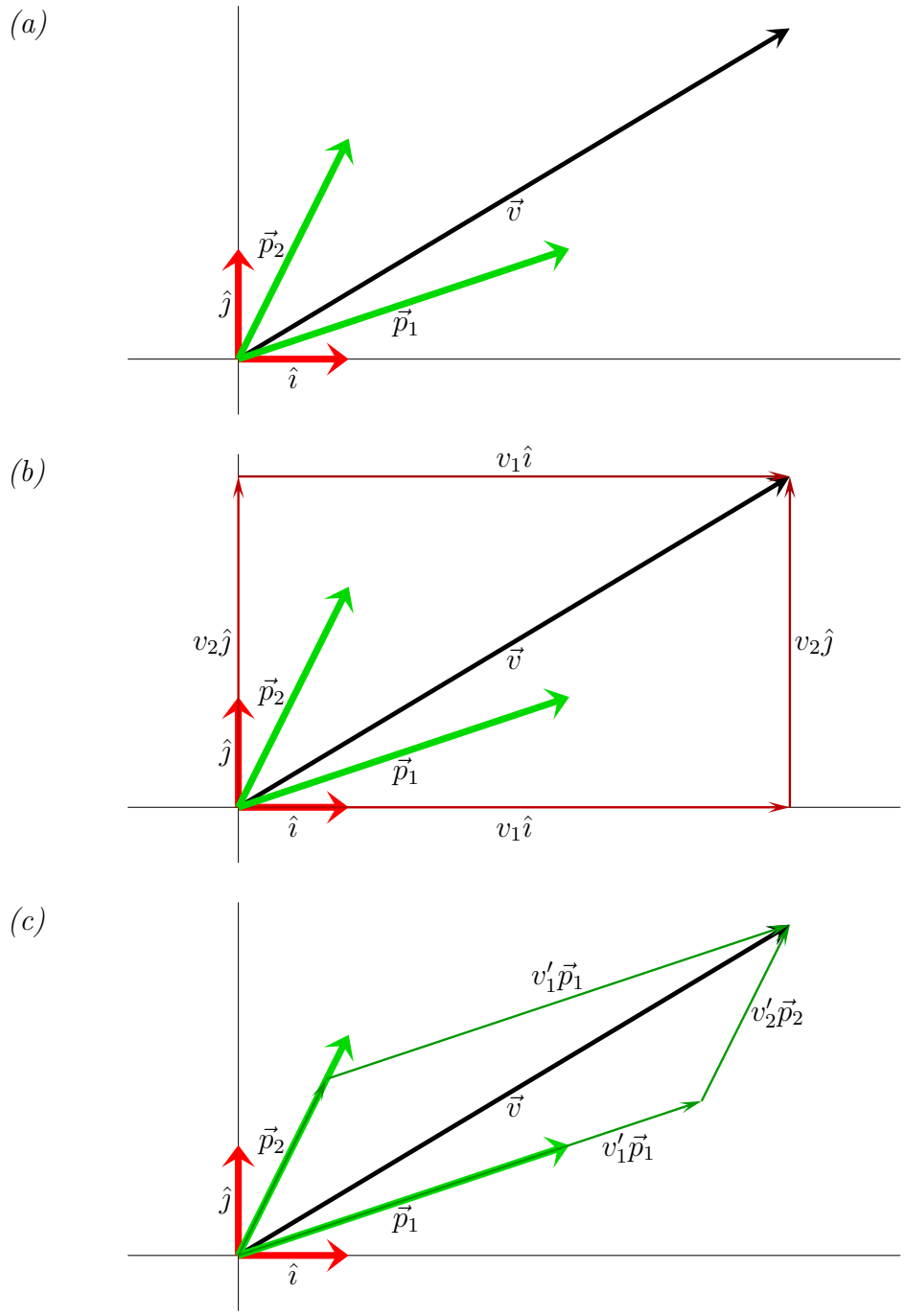


Figure 15.1: Illustration of a change of basis in two dimensions. *(a)* The standard basis $\hat{\imath}, \hat{\jmath}$, a new basis $\vec{p}_1, \vec{p}_2$, and any arbitrary vector $\vec{v}$. *(b)* Any vector $\vec{v}$ can be written as $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath}$. *(c)* However, the same vector can also be written as $\vec{v} = v_1'\vec{p}_1 + v_2'\vec{p}_2$. That uses the new basis.

where v'_1 and v'_2 are still ordinary numbers. They are called the coordinates of $\vec{v}$ in the new basis $\vec{p}_1, \vec{p}_2$. It is illustrated graphically in figure 15.1*(c)*.

However, in general the coordinates v'_1 and v'_2 in the new basis are not the same as the coordinates v_1 and v_2 in the old basis $\hat{\imath}, \hat{\jmath}$. So, if you want to use the new basis to your advantage, you will normally have to know how to compute v'_1 and v'_2 if you know v_1 and v_2 and/or vice-versa. That is the problem that this section will address.

First, to find the old coordinates v_1 and v_2 given the new ones v'_1 and v'_2 is easy. Just write the above equation as a row-column multiplication:

$$\vec{v} = \begin{pmatrix} \vec{p}_1 & \vec{p}_2 \end{pmatrix} \begin{pmatrix} v'_1 \\ v'_2 \end{pmatrix}$$

Then write that out in terms of the old Cartesian coordinates:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \vec{p}_1 & \vec{p}_2 \end{pmatrix} \begin{pmatrix} v'_1 \\ v'_2 \end{pmatrix}$$

or more concisely,

$$\vec{v} = \begin{pmatrix} \vec{p}_1 & \vec{p}_2 \end{pmatrix} \vec{v}'$$

where a prime on a vector means that its coordinates are written out in terms of the new basis, and no prime means they are written out in terms of the old basis. According to the above, to get the old coordinates from the new ones, just put $\vec{p}_1$ and $\vec{p}_2$ in a matrix, call it P,

$$\boxed{P = \begin{pmatrix} \vec{p}_1 & \vec{p}_2 \end{pmatrix}} \tag{15.1}$$

and multiply by that matrix. Do note that the $\vec{p}_1$ and $\vec{p}_2$ that you put in P must be written out in terms of the old Cartesian coordinates. But what else could you do?

So you get the following transformation formulae between coordinates

$$\boxed{\vec{v} = P\vec{v}' \qquad \vec{v}' = P^{-1}\vec{v}} \tag{15.2}$$

with P as above. Note that while P computes the *old* coordinates *from* the *new* ones, it is called "the transformation matrix from *old to new*". It does not make any sense, but that is what mathematicians call it. Just remember, "*old to new*" really means *old from new*.

The final thing you need to know is what happens to matrices. If a matrix A converts a vector $\vec{v}$ to a vector $\vec{w}$ in terms of Cartesian coordinates, then A' should convert $\vec{v}'$ to $\vec{w}'$ in terms of the new coordinates. Since

$$A\vec{v} = \vec{w} \quad \Longrightarrow \quad AP\vec{v}' = P\vec{w}' \quad \Longrightarrow \quad P^{-1}AP\vec{v}' = \vec{w}'$$

the desired matrix A' is seen to equal $P^{-1}AP$.

So the transformation rules for matrices are

$$\boxed{A = PA'P^{-1} \qquad A' = P^{-1}AP} \tag{15.3}$$

That is much like the ones for vectors, except there is an additional trailing P^{-1}, respectively P.

While we used a two dimensional example, you can do all of the above in any number of dimensions. You just add more basis vectors to transformation matrix P.

15.2 Diagonalization of nonsymmetric matrices

If a matrix A is not defective, you can use its eigenvectors as new basis. It turns out that in that basis the matrix simplifies to a *diagonal* matrix

$$\boxed{A' = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}} \tag{15.4}$$

Since this diagonal matrix has the eigenvalues on the main diagonal, (in the order that you arranged the corresponding eigenvectors), it is often written as Λ instead of A'.

Needless to say, this simplification is a tremendous help if you are doing analytical or numerical work involving the matrix.

The quickest was to see why A' is diagonal like above is to note that in terms of the new basis, $A\vec{v}$ produces a new vector $\vec{w}'$ according to

$$\vec{w}' = A'\vec{v}' = A'(v'_1\vec{e}_1 + v'_2\vec{e}_2 + \ldots + v'_n\vec{e}_n) = v'_1\lambda_1\vec{e}_1 + v'_2\lambda_2\vec{e}_2 + \ldots + v'_n\lambda_n\vec{e}_n$$

since $\vec{e}_1, \vec{e}_2, \ldots$ are eigenvectors of A. So the coefficients of $\vec{w}'$ are related to those of $\vec{w}'$ as $v'_1\lambda_1, v'_2\lambda_2, \ldots, v'_n\lambda_n$. And that is just what multiplying by the diagonal matrix A' above accomplishes.

Recall from section 15.1 that the transformation matrix P for change of basis to the eigenvectors must equal the matrix E of eigenvectors. You therefore have for any vector $\vec{v}$ and matrix M that you want to transform from new coordinates to old or vice-versa:

$$\boxed{\vec{v} = E\vec{v}' \quad \vec{v}' = E^{-1}\vec{v} \qquad M = EM'E^{-1} \quad M' = E^{-1}ME} \tag{15.5}$$

Here the primes mean the vector or matrix as it appears in the new basis of eigenvectors.

In summary, a nondefective matrix becomes diagonal when its eigenvectors are used as basis. The main diagonal contains the eigenvalues, ordered like the corresponding eigenvectors in E.

15.3 Diagonalization of symmetric matrices

For symmetric matrices the same observations apply as for nonsymmetric matrices in the previous section. But there are some further considerations.

Most importantly, the eigenvectors, if found using the class procedure, are orthonormal. So you can consider the eigenvectors $\vec{e}_1, \vec{e}_2, \ldots$ to be a rotated *Cartesian* basis. To make this clearer to other people, you should rename $\vec{e}_1, \vec{e}_2, \ldots$ to $\hat{\imath}', \hat{\jmath}', \ldots$.

The most important other thing to remember is that the transformation matrix

$$E = (\hat{\imath}' \; \hat{\jmath}' \; \ldots)$$

is now orthonormal. So be sure to use section 13.3 to find its inverse.

Note further that if the determinant of the transformation matrix E is negative, the rotated coordinate system is also left-handed instead of right-handed. It corresponds to a mirroring of a coordinate besides the rotation. If this bothers you, multiply one of the eigenvectors by minus one.

Part III

Ordinary Differential Equations

Chapter 16

Laplace Transformation

16.1 Partial Fractions

Partial fraction expansion simplifies fractions of the form

$$\frac{T(s)}{B(s)}$$

where $T(s)$ and $B(s)$ are polynomials in some variable s. This is of importance, for example, in integration of such fractions and in Laplace transformation. This section explains the procedure.

First, in case $T(s)$ has a degree equal or higher than $B(s)$, you should perform long division to take the ratio apart into powers of s plus a remainder. That remainder is again a fraction of the form $T(s)/B(s)$, but the new $T(s)$ is of a degree less than $B(s)$.

Now the only thing you still have to know is now how to deal with a ratio $T(s)/B(s)$ when the degree of $T(s)$ is less than that of $B(s)$.

The first thing you will need to do is factor $B(s)$. Assume it is a polynomial of some degree n, in other words

$$B(s) = C(s^n + b_{n-1}s^{n-1} + \ldots b_2 s^2 + b_1 s + b_0)$$

where C is some nonzero constant and the b_i for $i = n-1, \ldots, 2, 1, 0$ are n additional constants. You can simply take C out of the entire ratio $T(s)/B(s)$ before proceeding. From now on, it will be assumed that you have done that, so that there is no longer a C in the above expression to worry about. It is then known from complex variable theory that $B(s)$ can always be written in the form

$$B(s) = (s - s_1)(s - s_2)(s - s_3)\ldots(s - s_n)$$

Here the n constants s_i, $(i = 1, 2, \ldots, n)$, are the locations, or roots, where $B(s)$ is zero. So $B(s_i) = 0$ for $i = 1, 2, \ldots, n$. Note that some of these roots may

coincide. In other words some of the s_i may be equal to each other. For example

$$s^3 + s^2 - 5s + 3 = (s-1)(s-1)(s+3)$$

has $s_1 = s_2 = 1$ and $s_3 = -3$.

There is a further complication. Some of the roots s_i may be complex. For example, a quadratic with a negative discriminant has complex roots instead of real ones. However, assuming that the polynomial $B(s)$ is real, (for real s), you would probably not want to deal with complex numbers. You can avoid that because for real $B(s)$, the complex roots come in "complex-conjugate pairs." That means that for every complex root s_i there is a second root s_j so that $(s-s_i)(s-s_j)$ is a *real* quadratic with roots s_i and s_j. So a real $B(s)$ can always be written as a product of real factors linear in s and real factors quadratic in s.

For example, you might have a $B(s)$ that can be written as

$$B(s) = (s-4)(s-3)^3(s^2-2s+5)(s^2+4s+13)^2$$

That would be a polynomial of degree 10, $(1+3+2+4=10)$. For this $B(s)$, any desired ratio $T(s)/B(s)$ can be written as

$$\begin{aligned}\frac{T(s)}{B(s)} \;=\; & \frac{C_1}{s-4} + \\ & \frac{C_2}{s-3} + \frac{C_3}{(s-3)^2} + \frac{C_4}{(s-3)^3} + \\ & \frac{C_5 s + C_6}{s^2-2s+5} + \\ & \frac{C_7 s + C_8}{s^2+4s+13} + \frac{C_9 s + C_{10}}{(s^2+4s+13)^2}\end{aligned}$$

The right hand side is the partial fraction expansion of the ratio.

Note that the individual terms in the partial fraction expansion are *much* simpler than the original ratio. The original ratio has a polynomial of degree 10 in the bottom. And a polynomial of a degree up to 9 in the top. So if you want to do anything with the given ratio, it will become much easier if you use the right hand side above to do it. That is why you want to do partial fraction expansions.

That leaves two key questions to be answered:

1. In general, how do you know what terms there are in the right hand side?
2. How do you find the values of all these coefficients $C_1, C_2, C_3, \ldots$?

The answer to the first question is as follows;

1. For every factor $(s - s_i)$ that appears k times in $B(s)$, there are terms in the right hand side of the form

$$\frac{C_{i,1}}{s - s_i} + \frac{C_{i,2}}{(s - s_i)^2} + \ldots + \frac{C_{i,k}}{(s - s_i)^k}$$

   Check it out for the factors $(s - 4)$ (single) and $(s - 3)$ (triple) in the example given earlier. (What you want to call the constants is of course up to you, as long as each has a unique name.)

2. For every factor $(s^2 + a_i s + b_i)$ that appears k times in $B(s)$, there are terms in the right hand side of the form

$$\frac{C_{i,1}s + C_{i,2}}{s^2 + a_i s + b_i} + \frac{C_{i,3}s + C_{i,4}}{(s^2 + a_i s + b_i)^2} + \ldots + \frac{C_{i,2k-1}s + C_{i,2k}}{(s^2 + a_i s + b_i)^k}$$

   Check it out for the factors (s^2-2s+5) (single) and $(s^2+4s+13)$ (double) in the example above.

The second question was how to find all these constants. The method you must use in this class is to crunch it out:

1. Bring all the terms in the partial fraction expansion over the common denominator $B(s)$.

2. Multiply out the top. This must equal the given $T(s)$. So the net coefficient of each power of s must match the corresponding coefficient in $T(s)$. That gives you your equations for your unknown coefficients. Use Gaussian elimination to solve them.

For the example, its partial fraction expansion becomes, when brough over the common denominator $B(s)$:

$$\begin{array}{l}
C_1(s-3)^3(s^2-2s+5)(s^2+4s+13)^2 \\
\quad +C_2(s-4)(s-3)^2(s^2-2s+5)(s^2+4s+13)^2 \\
\quad\quad +C_3(s-4)(s-3)(s^2-2s+5)(s^2+4s+13)^2 \\
\quad\quad\quad +C_4(s-4)(s^2-2s+5)(s^2+4s+13)^2 \\
\quad\quad\quad\quad +(C_5s+C_6)(s-4)(s-3)^3(s^2+4s+13)^2 \\
\quad\quad\quad\quad\quad +(C_7s+C_8)(s-4)(s-3)^3(s^2-2s+5)(s^2+4s+13) \\
\quad\quad\quad\quad\quad\quad +(C_9s+C_{10})(s-4)(s-3)^3(s^2-2s+5) \\
\hline
(s-4)(s-3)^3(s^2-2s+5)(s^2+4s+13)^2
\end{array}$$

Multiply out the top, (don't make any mistakes, of course), then equate the net coefficients of the s^0, s, s^2, ..., s^8 and s^9 powers to the corresponding coefficients in the given $T(s)$. That gives 10 equations for the 10 unknowns C_1, C_2, ... C_{10}. Solve using Gaussian elimination. (Cramer's rule is *not* recommended.) That will be fun!

Note: Of course, if $B(s)$ is not real, there is no point in using quadratics. Just expand in linear factors of the form $s - s_i$ (with s_i now complex) only.

Note: there are more intelligent ways of finding the coefficients than crunching it out as we must do in this class. For example, consider once more the example:

$$\begin{aligned}\frac{T(s)}{B(s)} &= \frac{C_1}{s-4} + \\ &\quad \frac{C_2}{s-3} + \frac{C_3}{(s-3)^2} + \frac{C_4}{(s-3)^3} + \\ &\quad \frac{C_5 s + C_6}{s^2 - 2s + 5} + \\ &\quad \frac{C_7 s + C_8}{s^2 + 4s + 13} + \frac{C_9 s + C_{10}}{(s^2 + 4s + 13)^2}\end{aligned}$$

If you multiply the expansion in the right hand side by the single factor $(s-4)$ and then evaluate it at $s = 4$, you get C_1. (All the other terms are zero at $s = 4$ because of the multiplication by $(s-4)$.) The bottom line is therefore that if you multiply the *left* hand side by $(s-4)$ and then evaluate it at $s = 4$, you get C_1 too. And that is doable because the left hand side is given. (To evaluate at $s = 4$, you either need to divide out the common factor $(s-4)$ from top and bottom or use l'Hopital once. Dividing out the common factor from top and bottom is a simple matter of a long division of the bottom if you have it in unfactored form.) To find C_4, multiply by $(s-3)^3$ and evaluate at $s = 3$. To find C_3, multiply by $(s-3)^3$, differentiate the result once, and evaluate at $s = 3$. Etcetera. Especially if you want just a single coefficient, this can be much more convenient. Or you can use it to test the correctness of a few sample coefficients.

16.2 Completing the square

Completing the square simply means that you write a quadratic

$$ax^2 + bx + c$$

as

$$a\Big[\Big(\underbrace{x + \frac{b}{2a}}_{\substack{\text{shifted}\\ \text{x}}}\Big)^2 + \underbrace{\frac{c}{a} - \frac{b^2}{4a^2}}_{\substack{\text{new}\\ \text{constant}}}\Big]$$

If you multiply out, you see that it is the same.

One place where you often need this is in Laplace transforms. Laplace transforms are often given in terms of a quadratic $s^2 + K$ where K is an arbitrary

Properties of the Laplace Transform

Property	$f(t)$	$\widehat{f}(s)$
P1: Inversion	$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widehat{f}(s)e^{st}\,\mathrm{d}s$	$\int_0^\infty f(t)e^{-st}\,\mathrm{d}t$
P2: Linearity	$C_1 f_1(t) + C_2 f_2(t)$	$C_1\widehat{f}_1(s) + C_2\widehat{f}_2(s)$
P3: Dilation	$f(\omega t)$	$\omega^{-1}\widehat{f}(s/\omega)$
P4: Differentiation	$f^{(n)}(t)$	$s^n\widehat{f}(s) - s^{n-1}f(0^+) - \ldots - f^{(n-1)}(0^+)$
P5: Differentiation	$t^n f(t)$	$(-1)^n\widehat{f}^{(n)}(s)$
P6: Shift	$H(t-\tau)f(t-\tau)$ $H(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$	$e^{-\tau s}\widehat{f}(s)$
P7: Shift	$e^{\sigma t}f(t)$	$\widehat{f}(s-\sigma)$
P8: Convolution	$\int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau$ Do *not* write as $f * g$	$\widehat{f}(s)\widehat{g}(s)$

Special Laplace Transform Pairs

	$f(t)$	$\widehat{f}(s)$		$f(t)$	$\widehat{f}(s)$
S1:	1	$\frac{1}{s}$	**S8:**	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
S2:	t^n	$\frac{n!}{s^{n+1}}$	**S9:**	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
S3:	$e^{\sigma t}$	$\frac{1}{s-\sigma}$	**S10:**	$t\sin(\omega t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
S4:	$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$	**S11:**	$t\cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
S5:	$\frac{1}{\sqrt{\pi t}}e^{-k^2/4t}$	$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}}$	**S12:**	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
S6:	$\frac{k}{\sqrt{4\pi t^3}}e^{-k^2/4t}$	$e^{-k\sqrt{s}}$	**S13:**	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
S7:	$\operatorname{erfc}\left(k/2\sqrt{t}\right)$	$\frac{1}{s}e^{-k\sqrt{s}}$	**S14:**	$\delta(t-\tau)$	$e^{-\tau s}$

Table 16.1: Properties of the Laplace Transform. ($k, \tau, \omega > 0$, $n = 1, 2, \ldots$)

constant. But you might have $as^2 + bs + c$ instead of something of the form $s^2 + K$. However, you can write the part inside the square brackets above as s^2+K if you use the shift theorem to account for the $b/2a$ inside the parentheses. And the additional factor a is trivial to account for.

Chapter 17

More on Systems

17.1 Solution of systems using diagonalization

You should know by now how to solve a system of ordinary differential equations of the form

$$\dot{\vec{u}} = A\vec{u} + \vec{f} \qquad \vec{u}(0) = \vec{g}$$

where unknown vector $\vec{u}$ and given vector $\vec{g}$ depend on time, but $\vec{f}$ is a given constant vector and A a given constant matrix. (Yes, I will use $\vec{u}$ instead of $\vec{x}$ here.) The dot of course indicates the time derivative.

However, suppose that $\dot{\vec{u}}$ would be replaced by the second order derivative $\ddot{\vec{u}}$? Like in

$$\ddot{\vec{u}} = A\vec{u} + \vec{f} \qquad \vec{u}(0) = \vec{g} \qquad \dot{\vec{u}}(0) = \vec{h}$$

That happens in mechanics when the forces only depend on position (no friction). Note that for this second order system we also need the initial velocities $\dot{\vec{u}}(0)$.

Of course, you can convert the above system to a double-size first order one. But suppose you want to keep the system size the same? Well, you can solve the system directly using the basis of eigenvectors of matrix A, assuming it is not defective. And the relevant matrix A is typically a real symmetric one in these applications, so never defective.

I will now show how the solution procedure works. First of course you must find the eigenvalues and eigenvectors of A:

$$\begin{array}{cccc} \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \vec{e}_1 & \vec{e}_2 & \ldots & \vec{e}_n \end{array}$$

But you always needed to do that.

Next you write *every* vector in the problem in terms of the eigenvectors:

$$\vec{u} \;=\; u_1'\vec{e}_1 + u_2'\vec{e}_2 + \ldots + u_n'\vec{e}_n$$

$$\begin{aligned}
\vec{f} &= f'_1\vec{e}_1 + f'_2\vec{e}_2 + \ldots + f'_n\vec{e}_n \\
\vec{g} &= g'_1\vec{e}_1 + g'_2\vec{e}_2 + \ldots + g'_n\vec{e}_n \\
\vec{h} &= h'_1\vec{e}_1 + h'_2\vec{e}_2 + \ldots + h'_n\vec{e}_n
\end{aligned}$$

Here the primes indicate coefficients of the vectors in the basis of eigenvectors. Note that the u'_i and f'_i in general depend on time but the g'_i and h'_i are constants, for any i from 1 to n.

You will need to figure out what the coefficients of the given vectors $\vec{f}$ and $\vec{g}$ are now. Note that the above equations can be written in matrix form as

$$E\begin{pmatrix} f'_1 \\ f'_2 \\ \vdots \\ f'_n \end{pmatrix} = \vec{f} \qquad E\begin{pmatrix} g'_1 \\ g'_2 \\ \vdots \\ g'_n \end{pmatrix} = \vec{g} \qquad E\begin{pmatrix} h'_1 \\ h'_2 \\ \vdots \\ h'_n \end{pmatrix} = \vec{h} \qquad E \equiv \left(\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n\right)$$

Matrix E, of course, is our transformation matrix to the basis of eigenvectors. In any case, the above equations must be solved to find the f'_i, g'_i, and h'_i. (In doing that, remember that for a real symmetric matrix, you take the eigenvectors orthonormal, after which the inverse matrix E^{-1} is just the transpose one, E^{T}.)

Next remember that in the basis of the eigenvectors, matrix A becomes a diagonal one, with diagonal values equal to the eigenvalues. Therefore the original system of ordinary differential equations simplifies to decoupled equations:

$$\begin{aligned}
\ddot{u}'_1 &= \lambda_1 u'_1 + f'_1 \qquad u'_1(0) = g'_1 \quad \dot{u}'_1(0) = h'_1 \\
\ddot{u}'_2 &= \lambda_2 u'_2 + f'_2 \qquad u'_2(0) = g'_2 \quad \dot{u}'_n(0) = h'_n \\
&\vdots \\
\ddot{u}'_n &= \lambda_n u'_n + f'_n \qquad u'_n(0) = g'_n \quad \dot{u}'_n(0) = h'_n
\end{aligned}$$

You should be able to solve each of these scalar second order equations easily.

Finally you can find the solution vector $\vec{u}$ at any desired time by summing:

$$\vec{u} = u'_1\vec{e}_1 + u'_2\vec{e}_2 + \ldots + u'_n\vec{e}_n \equiv \sum_{i=1}^{n} u'_i\vec{e}_i$$

Of course, you could also solve the *first* order system that way. Compared to the class procedure, that has one big advantage. In the class procedure, we solved a system of equations for the variation of parameters, and one for the initial conditions. In the above method, the matrix of the two systems of equations to solve is the same, E, so you can use a single augmented matrix with two right hand sides (being $\vec{f}$ and $\vec{g}$). (And if A is symmetric. it is easier still, because you only need to multiply by E^{T}.)

17.2 Solving Partial Differential Equations

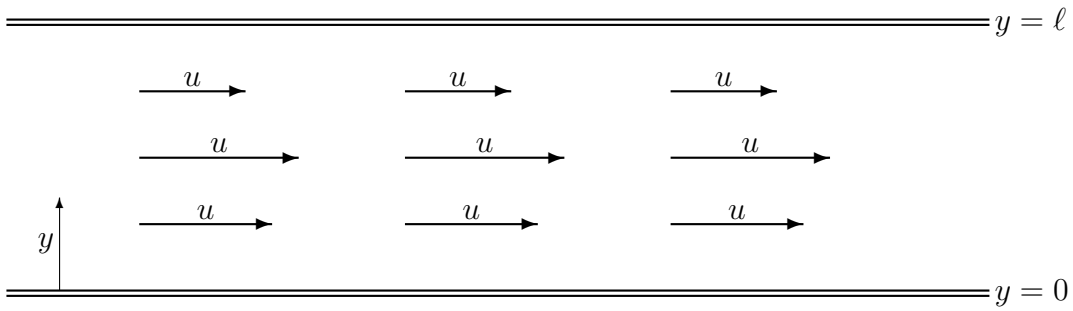


Figure 17.1: Laminar viscous flow in a long duct.

Consider unsteady viscous laminar flow of, say, water, in a long and thin horizontal two-dimensional duct. The velocity u depends on the time and the vertical position y, so $u = u(t; y)$. However, for a very long duct, it does not depend on the streamwise coordinate x.

According to fluid mechanics, the velocity develops according to the equation

$$\dot{u} = \nu \frac{\partial^2}{\partial y^2} u + f$$

Here f is some given function of t and y accounting for forces like gravity or electromagnetic ones. The equation above is called a *partial differential equation* because there are derivatives with respect to two variables; not just t but also y. To solve it, you also need an initial condition:

$$u(0; y) = g$$

where g is some given function.

Note that so far, the above system looks almost exactly like the first order system of ordinary differential equations in the previous section. However, where the system of ordinary differential equations has vectors, the scalar partial differential equation above has functions of y. The only other difference is that where the system of ordinary differential equations had some matrix A, the partial differential equation above has an "operator"

$$L \equiv \nu \frac{\partial^2}{\partial y^2}$$

But that is no big difference: when you apply a matrix A on a vector $\vec{v}$, you get a new vector $A\vec{v}$. In exactly the same way, if you apply L above on a function $F(y)$, you get a new function of y equal to $\nu F''(y)$. It is the same thing.

There is however one thing really different for the partial differential equation; it has *boundary conditions in* y. The fluid must be at rest at the walls of

the duct. With the walls at $y = 0$ and $y = \ell$, (with ℓ the height of the duct), that means

$$y = 0: \quad u(t;0) = 0 \qquad y = \ell: \quad u(t;\ell) = 0$$

(It is like the first and the last component of vector $\vec{u}$ would have to be zero.)

Still, you can solve the partial differential equation much like the system of ordinary differential equations in the previous section. I will now show you how.

First, we need the eigenfunctions of the operator L. Now a simple second-order derivative operator has eigenfunctions that are sines and cosines. So here the eigenfunctions could be sines or cosines of y. But the eigenfunctions must satisfy the above boundary conditions for u too. And these boundary conditions better be homogenous! (I will tell you in the next section what to do if the boundary conditions for u at $y = 0$ and $y = \ell$ are not homogeneous.) Fortunately, the ones above are homogeneous; there are no terms independent of u. So we can proceed. The cosines of y are out: cosines are 1 at $y = 0$, not 0. The sines are always 0 at zero, so that is OK. But they must also be 0 at $y = \ell$, and that only happens for

$$\begin{array}{llll} Y_1 = \sin(\pi y/\ell) & Y_2 = \sin(2\pi y/\ell) & Y_3 = \sin(3\pi y/\ell) & \ldots \\ \lambda_1 = -\nu\pi^2/\ell^2 & \lambda_2 = -\nu 2^2\pi^2/\ell^2 & \lambda_3 = -\nu 3^2\pi^2/\ell^2 & \ldots \end{array}$$

You find the eigenvalues by simply computing LY_i for $i = 1, 2, 3, \ldots$. That also verifies that the Y_i are really eigenfunctions like I told you.

(Note that $\sin(-\pi x/\ell) = -\sin(\pi x/\ell)$, so that is not an additional independent eigenfunction. That is just like $-\vec{e}_1$ would not be an additional eigenvector in the previous section.)

Next you write everything in terms of these eigenfunctions:

$$\begin{aligned} u &= u_1 Y_1 + u_2 Y_2 + u_3 Y_3 + \ldots \\ f &= f_1 Y_1 + f_2 Y_2 + f_3 Y_3 + \ldots \\ g &= g_1 Y_1 + g_2 Y_2 + g_3 Y_3 + \ldots \end{aligned}$$

I can leave out the primes of the previous section, because nobody ever writes the components of a function of y. Allow me to write the first of the expansions above out showing the arguments of the functions:

$$u = u_1(t)Y_1(y) + u_2(t)Y_2(y) + u_3(t)Y_3(y) + \ldots$$

I do that so that you can see why the solution method we are using is called "separation of variables" (the one for a partial differential equation, not for an ordinary one.) Each term is separated in a function of t times a function of y.

Once again, you have to compute these coefficients $f_1, f_2, \ldots$ and $g_1, g_2, \ldots$. But how do you do that? You can hardly invert an infinite "matrix" $E =$

$(Y_1, Y_2, Y_3, ...)$ of eigenfunctions like in the previous section. Well, for an operator like L, just a constant multiple of the second derivative, there is a trick: you can integrate to find them. In particular,

$$f_i = \frac{\int_{y=0}^{\ell} Y_i(y) f(t;y)\,\mathrm{d}y}{\int_{y=0}^{\ell} Y_i^2(y)\,\mathrm{d}y} \qquad g_i = \frac{\int_{y=0}^{\ell} Y_i(y) g(y)\,\mathrm{d}y}{\int_{y=0}^{\ell} Y_i^2(y)\,\mathrm{d}y} \qquad \text{for all } i = 1, 2, 3, \ldots$$

If you are astonished by that, don't be. The second derivative operator is a real symmetric one, so in the vector case you would find the $\vec{f'} = E^{\mathrm{T}}\vec{f}$. So you would find the f_i' as the dot product of the rows of E^{T}, the eigenvectors, times the given column vector $\vec{f}$. In the eigenfunction case, the dot-product summation becomes integration over y. And the bottom factors in the ratios above are just correction factors for the fact that I did not normalize the eigenfunctions in any way. You can see why the justification for the equations above is called the *orthogonality* property.

Much like in the previous section, the basis of eigenfunctions makes L diagonal, with the eigenvalues on the main diagonal. So the partial differential equation becomes a system of independent equations for the coefficients of u:

$$\begin{aligned}
\dot{u}_1 &= \lambda_1 u_1 + f_1 \qquad & u_1(0) &= g_1 \\
\dot{u}_2 &= \lambda_2 u_2 + f_2 \qquad & u_2(0) &= g_2 \\
\dot{u}_3 &= \lambda_3 u_3 + f_3 \qquad & u_3(0) &= g_3 \\
&\vdots & &\vdots
\end{aligned}$$

These equations are no more difficult to solve than for the case of ordinary differential equations.

Afterwards, you can find u at any time t and position y you want by summing:

$$u(t;y) = \sum_{i=1}^{\infty} u_i(t) Y_i(y)$$

Of course, you cannot normally sum infinitely many terms, even on a computer. You will need to instruct the computer to stop at some large value of i, call it I. The same holds in case you cannot do the earlier integrals for the f_i and g_i analytically; then you will need to do them numerically, up to some large I. And you may even have to solve the ordinary differential equations numerically. (Note that a first order linear equation can be reduced to an integral, so you would not need to use an ODE solver from your library.)

17.3 More details on the extension

If your advisor actually asks you to do the above thing, your problem might of course be more complicated than my one.

One problem would be if the boundary conditions on y are not homogenous. For example, if the top wall of the duct in the previous section moved horizontally with a given speed $U(t)$, you would have the boundary conditions

$$y = 0: \quad u(t;0) = 0 \qquad y = \ell: \quad u(t;\ell) = U(t)$$

and the second one is not homogenous. The trick then is to write u as something (anything) that satisfies the boundary conditions, and a remainder $\widetilde{u}$. In this case, a good choice would be

$$u = U\frac{y}{\ell} + \widetilde{u}$$

If you replace u everywhere in the PDE and its initial and boundary conditions by the expression above, you get a problem for $\widetilde{u}$. That problem will have *homogeneous* boundary conditions, so you are back in business, now for solving for $\widetilde{u}$.

The next thing is finding those eigenfunctions. If L is a constant times the second derivative, the eigenfunctions are sines and cosines. Then you look at the boundary conditions to figure out just which ones. But suppose you have something like

$$L \equiv a\frac{\partial^2}{\partial y^2} + b\frac{\partial}{\partial y} + c$$

What then? (Note that the coefficients a, b and c cannot depend on t; otherwise the usual method of separation of variables does not work. But they could and often do, depend on y)

In the simplest case that $b = 0$ and a and c are constants, the eigenfunctions are still sines and cosines. The constant c will just change the eigenvalues. So that is relatively trivial.

In any other case, you will need to solve the basic eigenvalue problem $LY = \lambda Y$, an ordinary differential equation, symbolically. But if a, b and/or c depend on y, I never taught you how to do that! Then you will need to search through mathematical handbooks. Look under Bessel functions, Gamma function, error function, orthogonal polynomials such as those of Legendre and Hermite, etcetera. Note that you often need to play around a bit with your equation to get it in a form that you can find in the tables. Or look a bit deeper; common conversions are often mentioned somewhere.

There is now also another problem. The orthogonality property no longer applies in the form used in the previous section. There is a theorem, called the "Sturm-Liouville" theorem, that says that you have to find a positive solution r to the differential equation

$$\frac{\mathrm{d}ar}{\mathrm{d}y} = br$$

Then you have to push this r, a function of y, inside each of the orthogonality integrals in the previous section as an additional factor:

$$f_i = \frac{\int_{y=0}^{\ell} Y_i(y) f(t;y) r(y)\,\mathrm{d}y}{\int_{y=0}^{\ell} Y_i^2(y) r(y)\,\mathrm{d}y} \qquad g_i = \frac{\int_{y=0}^{\ell} Y_i(y) g(y) r(y)\,\mathrm{d}y}{\int_{y=0}^{\ell} Y_i^2(y) r(y)\,\mathrm{d}y}$$

If you are in two spatial dimensions and time, you will have separate L_y and L_z operators. Taking L_y as the simpler operator, after you switch to the basis of eigenfunctions of L_y, the equations for the u_i will still contain both t and z. You will now need to find the eigenfunctions of the L_z operator. Note that you may be forced to include the L_y eigenvalue λ_i inside the definition of the L_z operator. For example, that happens in polar coordinates, (flow in a pipe), where "y" is the angular coordinate and "z" the radial one. The net description of u then involves terms of the form $u_{ij}(t)Y_i(y)Z_{ij}(z)$ that must be summed over both i and j. And the orthogonality integrals become double integrals over both y and z. All a whole lot of work, but not fundamentally different from what I told you.

As far as I can think of right now, the above covers all there is to say about the method of separation of variables. Not extremely difficult, but it sure requires a graduate student with a lot of time to carefully get all the details right.

Let me finally warn you about some common mistakes. One mistake that I see a lot is where the student leaves out an eigenfunction with eigenvalue 0. You need *all* the eigenfunctions. Remember that say an eigenfunction 1 is indeed a function: it is the function that is 1 at every position y. Another mistake that I see a lot is that a student tries to treat $f = 1$ or $g = 1$ as a number. It is a *function*, and you still need to write it as $f_1Y_1 + f_2Y_2 + \ldots$ or $g_1Y_1 + g_2Y_2 + \ldots$. And do the integrals. Then there are the boundary conditions. If the original problem has a boundary condition at some y-boundary that $Au + B\partial u/\partial y = C$, then you should subtract a function that satisfies that boundary condition as described above. And then you should discover that the remainder $\widetilde{u}$ satisfies the boundary condition $A\widetilde{u} + B\partial\widetilde{u}/\partial y = 0$. Your eigenfunctions better satisfies that *homogeneous* boundary condition too, or forget it. Don't try to define an eigenfunction expansion for a time-like variable that has initial conditions. If you are tempted to do that, instead try a Laplace transform in time. That is another way to solve a lot of simple partial differential equations. For separation of variables as explained here, you really want boundary conditions for the eigenfunctions.

Part IV

Partial Differential Equations

Chapter 18

Introduction

18.1 Basic Concepts

18.1.1 The prevalence of partial differential equations

Partial differential equations are equations involving derivatives with respect to more than one independent variable.

Partial differential equations are the basic equations in many areas of science and engineering. Some examples:

Fluid mechanics The basic equations that govern the inviscid flow of simple idealized substances are called the Euler equations. The basic equations that govern the viscous flow of simple substances like air and water under normal conditions are called the Navier-Stokes equations. Both are partial differential equations. Nonlinear ones, unfortunately. However, they do become linear in many special cases of great interest.

Heat transfer The equations of heat conduction and convection are partial differential equations. Radiation may be described by Maxwell's equations, which are also partial differential equations. However, often radiation can be more simply described by so-called boundary integral methods. In the most basic cases, the partial differential equations describing convection are linear.

Solid mechanics The equations that govern simple solids are partial differential equations. Often the interest is in steady problems. The equations for simple relatively stiff solids are linear.

Dynamics The dynamics of flexible solids is governed by partial differential equations. The equations for simple relatively stiff solids are linear.

Electromagnetics and optics The Maxwell equations that govern basic electromagnetic phenomena are partial differential equations. They are linear

in vacuum. Simplified partial differential equations govern the special cases of electrostatics and magnetostatics.

Geometry Many geometrical issues such as minimal surfaces and developable surfaces are governed by partial differential equations. The equation for minimal surfaces may be reduced to a very simple linear one.

That did not even touch on such areas as biology and economics that are also awash in partial differential equations.

18.1.2 Definitions

Here is a list of some basic definitions used for partial differential equations:

Partial differential equations are equations that involve derivatives with respect to more than one independent variable. The simplest partial differential equation that you can write is:

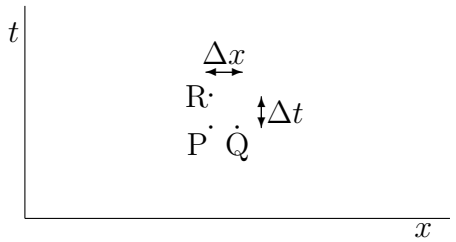
$$u_t = u_x$$

In that case $u(x,t)$ would be unknown, or dependent variable. That is the function you want to find. The independent variables would be x and t. People usually think of a spatial coordinate when they use x, and time when they use t.

It may be noted that equations that involve only derivatives with respect to one independent variable are called "ordinary differential equations." Equations that also involve integrals are called "integro-differential equations."

Partial derivatives are derivatives with respect to one independent variable, keeping the other variables constant. The figure below illustrates the definition of partial derivatives:

$$(u_x)_P \equiv \left(\frac{\partial u}{\partial x}\right)_P = \lim_{\Delta x \to 0} \frac{u_Q - u_P}{\Delta x} \qquad (u_t)_P \equiv \left(\frac{\partial u}{\partial t}\right)_P = \lim_{\Delta t \to 0} \frac{u_R - u_P}{\Delta t}$$



A simple numerical approximation of a partial derivative will just eliminate the limit process. In that case Δx and Δt are merely taken to be small compared to the typical scale of the problem.

Order The order of a partial differential equation is the order of the highest derivative. Generally speaking, the highest derivatives are most responsible for the nature of the solutions.

Degree The degree of a partial differential equation is the highest power to which the dependent variable appears in the equation.

Consider for example the Burgers' equation

$$u_t + uu_x = 0$$

It is the simplest nonlinear model for the equations of fluid mechanics, and for other systems involving shock formation. It is of order 1 and degree 2.

Linear partial differential equations are equations of first degree. The terms that are of the first degree in the unknown are called the homogeneous part. The remaining terms that do not involve the unknown are called the inhomogeneous part.

Consider the following Poisson equation governing steady heat conduction in a plate with external heat addition:

$$-\kappa(u_{xx} + u_{yy}) = x^2y^3 + \sin(x)\cos(y)$$

Here u is the temperature, κ the heat conduction coefficient, and the right hand side represents the external heat added per unit area. This equation is first degree (in u.) The left hand side, linear in u is the homogeneous part and the right hand side, independent of u, is the inhomogenous part.

The following equation describes steady heat conduction in a plate without external heat addition, but with a heat conduction coefficient that depends on temperature:

$$\kappa(u)(u_{xx} + u_{yy}) + \kappa'(u)u_x^2 + \kappa'(u)u_x^2 = 0$$

This equation is nonlinear. It is of infinite degree, not second degree, because the Taylor series of κ intruduces all powers of u. However, this equation is still linear in terms of the highest, second order, derivatives u_{xx} and u_{yy}. (Not counting lower order derivatives.) Therefore it is called "quasi-linear."

The domain Ω is usually taken to be the spatial region in which the partial differential equation applies. For unsteady heat conduction in a sphere, the domain Ω is the sphere and time is an additional coordinate.

The boundary $\delta\Omega$ is where the domain stops. Boundary points are immediately adjacent to both points inside the domain and points outside it. The boundary is also often indicated by S (for surface). Typically, Ω indicates the domain without the boundary points and $\overline{\Omega}$ the domain including the boundary points.

For unsteady heat conduction in a sphere of radius a, the boundary is all points with spherical coordinate r equal to a. It is the surface of the sphere. For this example Ω is all points $r < a$ and $\overline{\Omega}$ is all points $r \leqslant a$.

Boundary conditions are conditions on the solution that apply at points on the boundary.

Initial conditions are conditions on the solution that apply at the starting time. Almost always, the starting time is taken to be the zero of time.

Singularities You cannot do much in partial differential equations without having to deal with singularities. There might be singularities in the solution, in the initial and boundary conditions, or in the shape of the boundary. Some typical singularities, in a rough order from relatively mild to more severe, are

- Locations where all derivatives exist, but the function does not have a Taylor series with a nonzero radius of convergence. For example, that happens for the function e^{-1/x^2} at $x = 0$.
- Locations where higher order derivatives have singularities. For example, the function $|x|^{3/2}$ has an infinite second order derivative at x=0. That means it has infinite radius of curvature at $x = 0$. In general, the lower the order of the derivative involved, the stronger the singularity.
- *Kinks:* Locations where the first derivative jumps from one finite value to another. For example, the function $|x| = \sqrt{x^2}$ has a kink at $x = 0$.
- *Cusps:* where the function itself is still continuous, but the derivative jumps from $-\infty$ to ∞ or vice-versa. For example, $\sqrt[3]{x^2}$ has a cusp at $x = 0$
- *Jumps:* where the function jumps from one finite value to another. For example, the Heaviside function $H(x)$, which is 0 for negative x and 1 for positive x, has a jump at $x = 0$.
- *Spikes:* where the function is infinite at a single point. An important example is the Dirac delta function $\delta(x)$, which is the derivative of the Heaviside function. The delta function is a single infinite spike at $x = 0$ and the area under the spike is 1. If you take derivatives of the

delta function, you get increasingly singular functions. For example, the first derivative is the more singular "dipole."

- *Poles:* where the function goes to infinity. For example, the function $1/x$ has a "simple pole" at $x = 0$. The higher the negative power of x, the more singular the function. For example $1/x^2$ is more singular than $1/x$.

18.1.3 Typical boundary conditions

To get a meaningful solution to a partial differential equation, you will need initial and/or boundary conditions. Initial conditions are normally straightforward. Boundary conditions vary a lot, however.

There are some very simple types of boundary condition that you must know by heart:

Dirichlet The value of the unknown u itself is given on the boundary:

$$u = f \qquad \text{on} \qquad \delta\Omega$$

Here f is some given function of the position on the surface.

Neumann The derivative $\partial u/\partial n$ of the unknown in the direction normal to the boundary is given. Here n is the coordinate normal to the boundary:

$$\frac{\partial u}{\partial n} = f \qquad \text{on} \qquad \delta\Omega$$

If the boundary is oblique, the following formula can be used to relate the derivative in the direction normal to the boundary to the partial derivatives of u in Cartesian coordinates:

$$\frac{\partial u}{\partial n} = \vec{n} \cdot \nabla u \qquad \nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Here $\vec{n}$ is a unit vector that is normal to the boundary at the considered point.

Note that if u itself is prescribed on the boundary, it already implies the value of derivative *along* the boundary. That is why only the derivative normal to the boundary is included in this list of the most basic boundary conditions.

Mixed A linear combination of u and $\partial u/\partial n$ is given on the boundary. This is also called a "radiation" or "Robin" boundary condition.

$$\alpha u + \beta\frac{\partial u}{\partial n} = f \qquad \text{on} \qquad \delta\Omega$$

Here α, β, and f are all given functions of the position on the surface. Such boundary conditions are often used in simple wave propagation problems to indicate that no waves enter the domain from outside.

18.2 The Standard Examples

There are a few standard examples of partial differential equations. You must know these by heart.

18.2.1 The Laplace equation

The Laplace equation governs basic steady heat conduction, among much else.

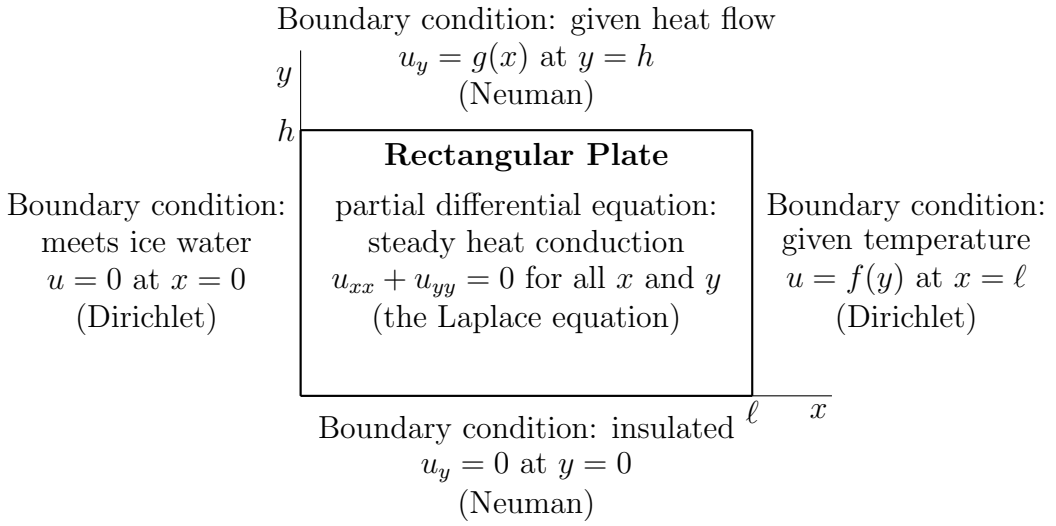


Figure 18.1: An example Laplace equation problem.

An example problem is shown in figure 18.1. Physically it is steady heat conduction in a rectangular plate of dimensions $\ell \times h$. The unknown u in this example is the temperature. The independent variables are the Cartesian coordinates x and y. The domain Ω is the two-dimensional interior of the plate. The boundary $\delta\Omega$ is the one-dimensional perimeter of the plate. (The boundary might still be indicated by S instead of $\delta\Omega$ even though here it is not a surface.)

The Laplace equation also describes ideal flows, unidirectional flows, membranes, electrostatics and magnetostatics, complex functions, and countless other problems.

In any number of dimensions, the Laplace equation reads

$$\boxed{\text{Laplace equation:} \qquad \nabla^2 u = 0} \tag{18.1}$$

In particular, in three dimensions and Cartesian coordinates

$$u_{xx} + u_{yy} + u_{zz} = 0$$

For coordinates that are not Cartesian, the Laplacian ∇^2 can be found in table books.

Some important properties of the Laplace equation are:

- *Steady state problems:* The Laplace equation normally describes processes that are in a steady state situation.

- *Boundary-value problems:* The Laplace equation needs "boundary-value problems." At every point on the boundary, one boundary condition should be prescribed. For example, consider the example problem figure 18.1. On the vertical boundaries, the temperature is given. That is a Dirichlet boundary condition. On the horizontal boundaries, the heat flow out of the boundary is given. Now the heat flow is proportional to the gradient of the temperature. In particular, the heat flow in the vertical direction is proportional to u_y. So the horizontal boundaries have Neumann boundary conditions; the derivative of u in the direction normal to these boundary is given.

- *Infinite propagation speed:* Sometimes the solution of the Laplace equation may still depend parametrically on time. For example, the Laplace equation applies to unsteady ideal flows of incompressible fluids. The reason that the Laplace equation can apply to such flows is that the incompressibility assumption implies an infinite speed of sound. If the boundary conditions are somewhere changed, the flow field instantly adapts to the new conditions everywhere.

- *Unlimited region of influence:* The Laplace equation has an unlimited region of influence. In terms of the example that means that if you change the temperature a bit somewhere on the boundary, it will affect the temperature to some extent everywhere inside the plate.

- *Smoothness:* The solutions to the Laplace equation are smooth. Even if you prescribe singular values for the solution on the boundary, the solution is still perfectly smooth in the interior of the domain. In particular, any point in the interior has infinitely many continuous derivatives, as well as a Taylor series with a finite radius of convergence. {D.2}

- *Maximum-minimum principle:* The Laplace equation has the property that the maximum and minimum of u always occur on the boundary. For example, in the problem figure 18.1 the temperature in the interior of the plate can nowhere be higher than the highest temperature on the boundary. {D.3}

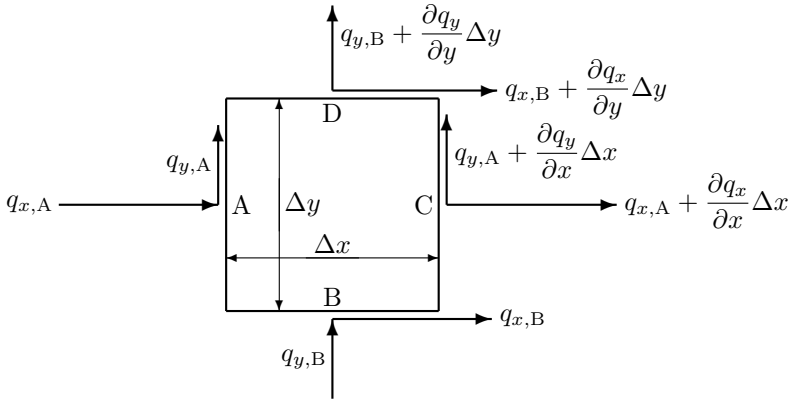
- *Mean value theorem:* Suppose u is defined on and within some spherical surface and satisfies the Laplace equation. Then the average of u on the spherical surface is the same as the value of u at the center of the sphere. {D.3}

(For domains that extent to infinity, various rules above assume that you consider the infinite domain as the limit of a finite one.)

The Laplace equation is the basic example of what is called an "elliptic" partial differential equation. Solutions of the Laplace equation are called "harmonic functions."

18.2.1 Review Questions

1. Derive the Laplace equation for steady heat conduction in a two-dimensional plate of constant thickness δ. Do so by considering a little Cartesian rectangle of dimensions $\Delta x \times \Delta y$. A sketch is shown below:



   Assume Fourier's law:

$$\vec{q} = (q_x, q_y) \qquad q_x = -k\frac{\partial u}{\partial x} \qquad q_y = -k\frac{\partial u}{\partial y}$$

   Here u is the temperature, assumed independent of z. Also, k is the heat conduction coefficient of the material. The vector $\vec{q}$ is the heat flux density. Vector $\vec{q}$ is in the direction of the heat flow. Its magnitude $|\vec{q}|$ equals the heat flowing per unit area normal to the direction of flow.

   If you want the heat flow $\dot{Q}$ through an area element $\mathrm{d}S$ that is not normal to the direction of heat flow, the expression is

$$\dot{Q} = \vec{q} \cdot \vec{n}\, \mathrm{d}s$$

   Here $\vec{n}$ is the unit vector normal to the surface element $\mathrm{d}S$. Positive $\dot{Q}$ means a heat flow through the surface element in the same direction as $\vec{n}$.

   Assume that no heat is added to the little rectangle from external sources.

   Solution stanexl-a

2. Derive the Laplace equation for steady heat conduction using vector analysis. Assume Fourier's law as given in the previous question. In vector form

$$\vec{q} = -k\nabla u$$

Assume that no heat is added to the solid from external sources.

Solution stanexl-b

3. Consider the Laplace equation within a unit circle:

$$u_{xx} + u_{yy} = 0 \qquad \text{for} \qquad x^2 + y^2 < 1$$

The boundary condition on the perimeter of the circle is

$$u = (y^2 + 1)x \qquad \text{for} \qquad x^2 + y^2 = 1$$

To find the value of u at the point (0.1,0.2), can I just plug in the coordinates of that point into the boundary condition? If not, what is the correct value of u at the point, and what would I get from the boundary condition?

Also answer the above questions for the following problem:

$$u_{xx} + u_{yy} = 0 \qquad \text{for} \qquad x^2 + y^2 < 1$$

The boundary condition on the perimeter of the circle is

$$u = 2 + 3x + 5y \qquad \text{for} \qquad x^2 + y^2 = 1$$

Find the value of u at the point (0.1,0.2). Fully defend your solution.

Solution stanexl-b1

4. Suppose you have a Laplace equation problem where the boundary is symmetric around the y-axis, like, say, in the previous two problems. In general, such a symmetric boundary means that if (x, y) is a boundary point, then so is $(-x, y)$. Also assume that u is given as an antisymmetric function of x on this boundary; $u(-x, y) = --u(x, y)$ for any boundary point. Show that in that case, u is antisymmetric function of x everywhere, i.e. $u(-x, y) = --u(x, y)$ everywhere.

Then show that this means that the solution u will be zero on the y axis.

Also explain why the above would no longer be true if you had a first order x derivative in the PDE, like for example $u_{xx} + u_{yy} + u_x = 0$.

Solution stanexl-b2

5. Consider the Laplace equation within a unit circle, but now in polar coordinates:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad \text{for} \qquad r < 1$$

The boundary condition on the perimeter of the circle is

$$u(1, \theta) = f(\theta)$$

where f is a given function.

The solution is the Poisson integral formula

$$u(r, \theta) = \frac{1 - r^2}{2\pi} \oint \frac{f(\bar{\theta})\, \mathrm{d}\bar{\theta}}{1 - 2r\cos(\bar{\theta} - \theta) + r^2}$$

Now suppose that function $f(\theta)$ is increased slightly, by an amount δf, and *only* in a very small interval $\theta_1 < \theta < \theta_2$.

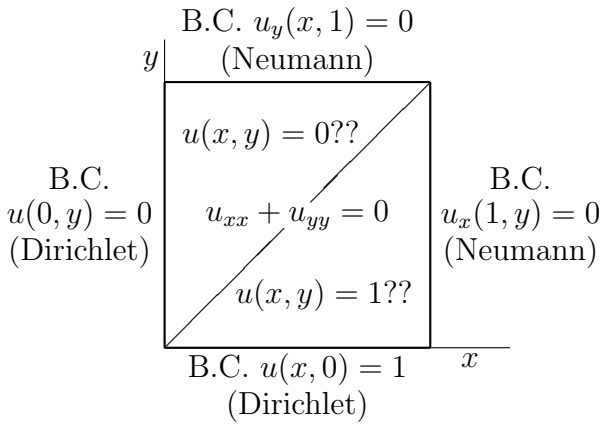
Does the solution u change everywhere in the circle, or only in the immediate vicinity of the interval on the boundary at which f was changed. What is the sign of the change in u if δf is positive?

Solution stanexl-b3

6.
- Show that if u is a harmonic function in a finite domain, and positive on the boundary, then it is positive everywhere in the domain.
- Show by example that this does not need to be true for an infinite domain.
- Let u, v, and w be harmonic functions. Show that if $u \leqslant v \leqslant w$ on the boundary of a finite domain, then $u \leqslant v \leqslant w$ everywhere inside the domain.

Solution stanexl-b5

7. Consider the following Laplace equation problem in a unit square:



The problem as shown has a unique solution. It is relevant to a case of heat conduction in a square plate, with u the temperature. Someone proposed that the solution should be simple: in the upper triangle the solution $u(x,t)$ is 0, and in the lower triangle, it is 1.

Thoroughly discuss this proposed solution. Determine whether the boundary conditions and initial conditions are satisfied. Is the partial differential equation satisfied in both triangles?

Explain why all isotherms except 0 and 1 coincide with the 45° line. And why the zero and 1 isotherms are indeterminate.

Finally discuss whether the solution is right.

Solution stanexl-c

8. If for the problem of the previous question, the proposed solution is wrong, then so are the described isotherms.

To get a clue about the correct solution and isotherms, consider the following simpler problem. In this problem the top and right boundaries

have been distorted into a quarter circle:

$$\text{BC:} \quad u(x,0) = 1 \qquad u(0,y) = 0 \qquad \frac{\partial u}{\partial n} = 0 \text{ on } x^2 + y^2 = 1$$

Solve this problem. Then neatly draw the $u = 0$, 0.25, 0.5, 0.75, and 1 isotherms for this problem.

Also neatly draw u versus the polar angle θ at $r = 0.5$. In a separate graph, draw the solution proposed in the previous section, $u = 1$ for $y < x$ and $u = 0$ for $y > x$, again against θ at $r = 0.5$.

Now go back to the problem of the previous question and very neatly sketch the correct $u = 0$, 0.25, 0.5, 0.75, and 1 isotherms for that problem. Pay particular attention to where the 0.25, 0.5, and 0.75 isotherms meet the boundaries and under what angle.

Solution stanexl-d

9. Return once again to the problem of the second-last question.

The correct solution to this problem, that you would find using the so-called method of separation of variables, is:

$$u(x,y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n \cosh(\frac{1}{2}n\pi)} \sin(\tfrac{1}{2}n\pi x) \cosh(\tfrac{1}{2}n\pi(1-y))$$

Verify that this solutions satisfies both the partial differential equation and all boundary conditions.

Now shed some light on the question why this solution is smooth for any arbitrary $y > 0$. To do so, first explain why any sum of sines of the form

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\tfrac{1}{2}n\pi x\right)$$

is smooth as long as the sum is finite. A finite sum means that the coefficients c_n are zero beyond some maximum value of n.

Next, you are allowed to make use of the fact that the function is still smooth if the coefficients c_n go to zero quickly enough. In particular, *if* you can show that

$$\lim_{n\to\infty} n^k c_n = 0$$

for every k, however large, then the function $f(x)$ is infinitely smooth.

Use this to show that u above is indeed infinitely smooth for any $y > 0$. And show that it is not true for $y = 0$, where the solution jumps at the origin.

Solution stanexl-e

18.2.2 The heat equation

The heat equation governs basic unsteady heat conduction, among much else.

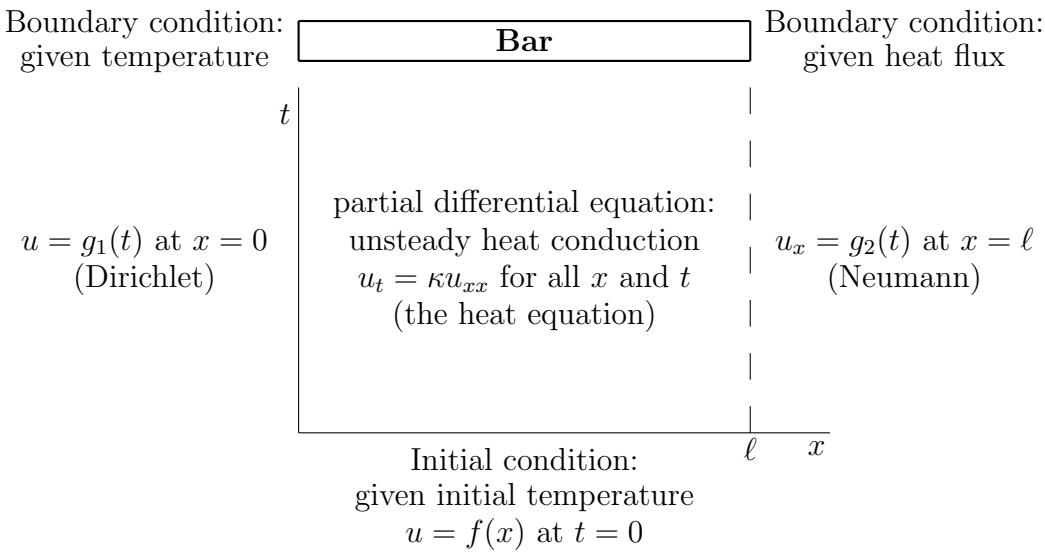


Figure 18.2: An example heat equation problem.

An example problem is shown in figure 18.2. Physically it is unsteady heat conduction in a bar of length ℓ. The unknown u is the temperature. The independent variables in this case are the coordinate x along the bar and the time t. The domain Ω in this example is the bar. Mathematically, that is the line segment $0 \leqslant x \leqslant \ell$ with ℓ the length of the bar. The boundary $\delta\Omega$ consists in this case of a mere two points: $x = 0$ and $x = \ell$.

The heat equation also describes unsteady viscous unidirectional flows and many other diffusive phenomena.

In any number of dimensions, the heat equation reads

$$\boxed{\text{Heat equation:} \qquad u_t = \kappa \nabla^2 u} \tag{18.2}$$

Here t is time and κ the heat conduction constant. In particular, in three dimensions and Cartesian coordinates

$$u_t = \kappa \left(u_{xx} + u_{yy} + u_{zz} \right)$$

Some important properties of the heat equation are:

- *Transient problems:* The heat equation normally describes processes that evolve in time.

- *Initial-value or initial/boundary-value problems:* The heat equation needs initial-value problems or initial/boundary-value problems. The example figure 18.2 is an initial/boundary-value problem. The initial temperature is given. In addition, there is a Dirichlet boundary condition, (given temperature T_0), at $x = 0$. There is also a Neumann boundary condition,

(zero heat flux out of the boundary so $u_x = 0$), at $x = \ell$. If you let the ends of the bar go to infinity, you get a pure initial-value problem. (However, in reality there are still some constraints at infinity. In particular the temperature should not become too singular at infinity.)

- *Infinite propagation speed:* If you change the initial temperature or the boundary temperature a bit, it immediately changes the solution everywhere. More precisely, at any time after the change, the temperature will be different *everywhere*. Very little different maybe, but different.

- *The region of influence is limited by time:* If the boundary conditions are changed, it only changes the solution at later times.

- *Smoothness:* The solutions are smooth. Even if you prescribe a singular initial temperature distribution, the solution will be smooth for all later times. In particular, for later times the temperature distribution will have infinitely many continuous derivatives. A similar observation holds for boundary conditions.

- *Maximum-minimum principle:* The maximum and minimum of the solution must occur initially and/or on the boundaries.

- *Dissipative:* Assuming that the boundary conditions are steady, the solution will eventually approach a steady state.

The heat equation is the basic example of what is called a "parabolic" partial differential equation.

18.2.2 Review Questions

1. This is a continuation of a corresponding question in the subsection on the Laplace equation. See there for a definition of terms.

   Derive the heat equation for unsteady heat conduction in a two-dimensional plate of thickness δ, Do so by considering a little Cartesian rectangle of dimensions $\Delta x \times \Delta y$.

   In particular, derive the heat conduction coefficient κ in terms of the material heat coefficient k, the plate thickness t, and the specific heat of the solid C_p.

   Solution stanexh-a

2. This is a continuation of a corresponding question in the subsection on the Laplace equation. See there for a definition of terms.

   Derive the heat equation for unsteady heat conduction using vector analysis.

   Solution stanexh-b

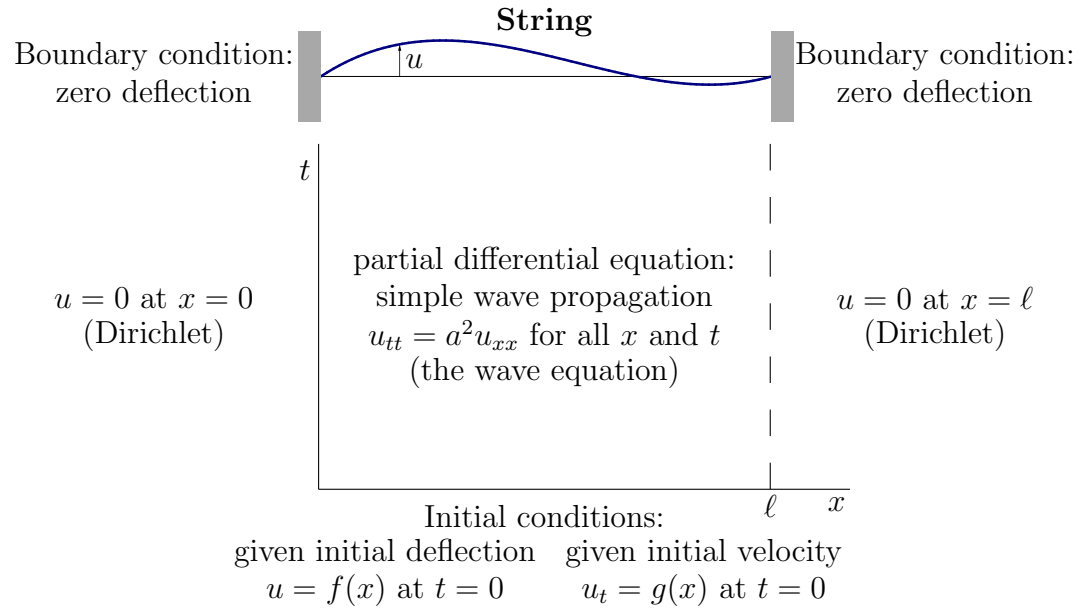


Figure 18.3: An example wave equation problem.

18.2.3 The wave equation

This equation governs basic vibrations, among much else.

An example problem, vibrations of a string, is shown in figure 18.3. The unknown u is the transverse deflection of the string. The independent variables are again x and t like for the heat equation example. The domain Ω is again the x-interval along the string and the boundary $\delta\Omega$ is the two end points.

The heat equation also describes acoustics, steady supersonic flow, water waves, optics, electromagnetic waves, and many other basic phenomena characterized by wave propagation.

In any number of dimensions, the wave equation reads

$$\boxed{\text{Wave equation:} \qquad u_{tt} = a^2 \nabla^2 u} \tag{18.3}$$

Here t is time and a the constant wave propagation speed. In particular, in three dimensions and Cartesian coordinates

$$u_{tt} = a^2 \left(u_{xx} + u_{yy} + u_{zz} \right)$$

Some important properties of the wave equation are:

- *Transient problems:* The wave equation normally describes processes that evolve in time.
- *Initial-value or initial/boundary-value problems:* Like the heat equation, the wave equation needs initial-value problems or initial/boundary-value

problems. However, it needs two initial conditions instead of one, since the equation is second order in time. For the example figure 18.2, that means that both the initial transverse deflection u and the initial transverse velocity u_t of each point of the string must be given. A string would normally be fixed at its end points, producing Dirichlet boundary conditions. However, the same equation as in the example also governs acoustics in a pipe, and either Dirichlet or Neumann boundary conditions may be relevant to the ends of the pipe.

- *Finite propagation speed:* Effects propagate with the wave speed a.
- *The region of influence is limited by the wave speed:* Suppose that a boundary or initial condition is somewhere changed a bit. The change will not affect the solution at other locations until a wave traveling from the point of the change at speed a has had time to reach them.
- *Propagation of singularities:* If singular initial or boundary conditions are prescribed, the wave equation will not smooth them out. Instead singularities will usually be propagated in one or more directions with the wave propagation speed a.
- *No maximum or minimum principles:* For the example, if the string has zero initial deflection but a nonzero initial velocity, the deflection will grow in time.
- *Energy conservation:* The wave equation preserves the sum of potential and kinetic energy of the string motion. So, if the wave equation was exact, the string would keep vibrating forever.

The wave equation is the basic example of what is called a "hyperbolic" partial differential equation.

18.2.3 Review Questions

1. Derive the wave equation for small transverse vibrations of a string by considering a little string segment of length Δx.
   Solution stanexw-a
2. Maxwell's equations for the electromagnetic field in vacuum are

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad (1) \qquad\qquad \nabla \cdot \vec{B} = 0 \qquad (2)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (3) \qquad\qquad c^2 \nabla \times \vec{B} = \frac{\vec{j}}{\epsilon_0} + \frac{\partial \vec{E}}{\partial t} \qquad (4)$$

Here $\vec{E}$ is the electric field, $\vec{B}$ the magnetic field, ρ the charge density, $\vec{j}$ the current density, c the constant speed of light, and ϵ_0 is a constant

called the permittivity of space. The charge and current densities are related by the continuity equation

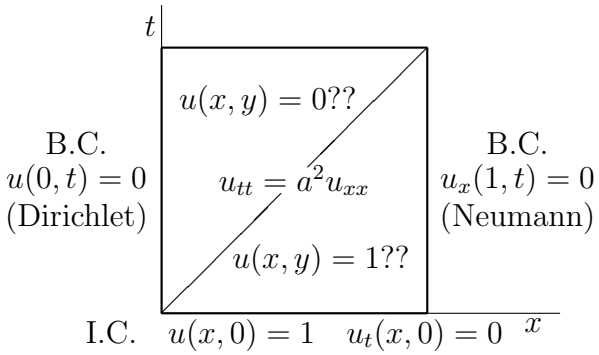
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 \qquad (5)$$

Show that if you know how to solve the standard wave equation, you know how to solve Maxwell's equations. At least, if the charge and current densities are known.

Identify the wave speed.

Solution stanexw-b

3. Consider the following wave equation problem in a unit square:



This is basically identical to a Laplace equation problem in the first subsection. Like that problem, the above wave equation problem has a unique solution. It is relevant to a case of acoustics in a tube, with u the pressure. Someone proposed that the solution should be simple: in the upper triangle the solution $u(x,t)$ is 0, and in the lower triangle, it is 1.

Thoroughly discuss this proposed solution. Determine whether the boundary conditions and initial conditions are satisfied. Is the partial differential equation satisfied in both triangles? Finally discuss whether the solution is right. Consider the value of the wave speed a in your answer.

Sketch the isobars of the correct solution. In particular, sketch the u = 0 0.25, 0.5, 0.75, and 1 isobars, if possible. Sketch both the case that $a = 1$ and that $a = \sqrt{2}$.

Solution stanexw-c

4. Return again to the problem of the last question. Assume $a = 1$.

The correct solution to this problem, that you would find using the so-called method of separation of variables, is:

$$u = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(\tfrac{1}{2} n\pi x) \cos(\tfrac{1}{2} n\pi t)$$

Verify that this solutions satisfies both the partial differential equation and all boundary and initial conditions.

Explain that it produces the moving jump in the solution as given in the previous question.

The discontinuous solution given in the previous question is right in this case. It is right because it is the proper limiting case of a smooth solution that everywhere satisfies the partial differential equation. In particular, if you sum the above sum for u up to a very high, but not infinite value of n, you get a smooth solution of the partial differential equation that satisfies all initial and boundary conditions, except that the value of u at $t = 0$ still shows small deviations from $u = 1$. The more terms you sum, the smaller those deviations become. (There will always be some differences right at the singularity, but these will be restricted to a negligibly small vicinity of $x = 0$.)

Solution stanexw-e

5. Find the possible plane wave solutions for the two-dimensional wave equation

$$u_{tt} = a^2 u_{xx} + a^2 u_{yy}$$

What is the wave speed?

Also find the possible standing wave solutions. Assume homogeneous Dirichlet or Neumann boundary conditions on some rectangle $0 < x < \ell$, $0 < y < h$. What is the frequency?

Repeat for the generalized equation

$$u_{tt} = a_1^2 u_{xx} + a_2^2 u_{yy} + b^2 u$$

where a_1, a_2, and b are positive constants.

Solution stanexw-f

18.3 Properly Posedness

Properly posedness is really quite unique to partial differential equations. Ordinary differential equations can be hard to solve if they involve very different time scales. For example, that is an issue in many chemical reactions.

But for partial differential equations, "hard to solve" becomes "impossible to solve." That happens even for apparently very simple linear partial differential equations with constant coefficients.

This section has a look at some of the issues involved.

18.3.1 The conditions for properly posedness

In words, a properly posed problem in partial differential equations can be described as follows: it is a problem that has a unique solution that is physically reasonable.

Phrased more mathematically, a problem is properly posed if:

1. A solution exists. Of course. Note it can be much harder to show that a decent solution exists for partial differential equations than for ordinary differential equations. For example, it has not yet been achieved for the unsteady three-dimensional Navier-Stokes equations of viscous flows of simple fluids.

2. The solution is unique. Of course.

3. Small changes in the data produce correspondingly small changes in the solution. The "data" are here such things as the initial or boundary conditions or an inhomogeneous term in the equation.

Requirement 3 above is the one that makes the solution physically reasonable. Physically, nothing is exactly known. There are always some errors in the data, however accurate they may be. If these negligible errors can produce a significant change in the solution, then all bets are off that the solution obtained is the right one.

You may wonder what "correspondingly small" in condition 3 really means. The true answer is that it varies. However, generally it is taken to mean that changes in the solution are no more than proportional to the changes in the data that cause these changes. And there must be some overall upper bound to the constant of proportionality that is independent of the details of the change in data.

That still requires that suitable measures of the magnitude of the changes in data and solution are defined. That however is beyond this discussion.

One thing should be emphasized. It is not partial differential equations that are properly or improperly posed. It is *problems* that are properly or improperly posed. Before you know what boundary and initial conditions are specified for your partial differential equation, you cannot say anything meaningful about properly posedness.

The following subsections give a few typical examples of how improperly posed problems arise. It illustrates that if you try to solve some partial differential equation numerically, you better know what sort of equation it is. Or you can get into major problems.

18.3.1 Review Questions

1. Show that the Dirichlet boundary-value problem for the Poisson equation on a finite domain,

$$\nabla^2 u = f \quad \text{on} \quad \Omega \qquad u = g \quad \text{on} \quad \delta\Omega$$

has unique solutions. You cannot have two different solutions u_1 and u_2 to this problem.

Solution ppc-a

2. Assuming that the Dirichlet boundary-value problem for the Laplace equation on a finite domain,

$$\nabla^2 u = 0 \quad \text{on} \quad \Omega \qquad u = f \quad \text{on} \quad \delta\Omega$$

is solvable, show that it depends continuously on the data.
Solution ppc-b

3. Repeat the previous two questions for the Dirichlet initial / boundary value problem for the heat equation,

$$u_t = \kappa \nabla^2 u \quad \text{on} \quad \Omega \qquad u = f \quad \text{on} \quad \delta\Omega \qquad u = g \quad \text{at} \quad t = 0$$

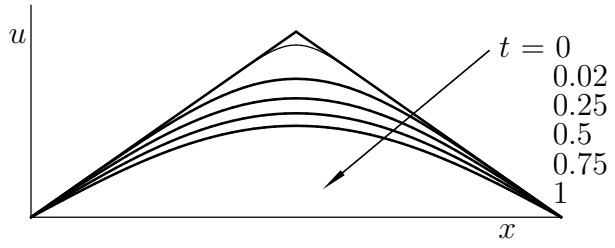
Solution ppc-c

18.3.2 An improperly posed parabolic problem

This subsection will discuss an improperly posed problem involving the heat equation. Recall that the heat equation is an example of a parabolic equation.

Consider first a very standard properly posed problem for the heat equation. The problem is heat conduction in a bar. The unknown is the temperature. The ends of the bar are kept at zero temperature.

The below figure shows some computed temperature profiles in a bar at various times.



At the initial time $t = 0$ the initial condition was assumed to be piecewise linear. There is then a singularity, a kink, at the center of the bar. A kink corresponds to a jump in the derivative. But the heat equation smooths away singularities. At any later time, even as small as 0.02 in the nondimensional units used in this problem, the temperature profile is perfectly smooth, with all derivatives continuous.

This problem was properly posed. Improperly posedness arises for the "backward heat equation". Physically the backward heat equation is the heat equation solved backwards in time. Mathematically, the backward heat equation takes the form

$$u_\tau = -\kappa u_{xx}$$

where the new independent variable τ increases when the time t decreases.

Note that the backward heat equation is equivalent to solving the normal heat equation forwards in time, but with a negative heat conduction coefficient. A negative heat conduction coefficient violates the second law of thermodynamics, so it is not really surprising that you get into trouble with the mathematics.

Suppose that you take the temperature profile u at time $t = 1$ in the figure above as the initial condition for the backward heat equation. Then you compute the solution of the backward heat equation up to $\tau = 1$. That should give you back the singular temperature profile at time zero. And it will, if you manage to do it exactly. The backward heat equation has a unique solution for the chosen initial condition in the interval from $\tau = 0$ to $\tau = 1$.

But now suppose that you use the singular profile u at time zero as the initial condition for the backward heat equation. Then you try again to compute the solution of the backward heat equation up to $\tau = 1$. It will not work. You will not be able to find a solution for any value of τ greater than zero. The reason is easy to understand. Suppose that you did find a solution u at $\tau = 1$. Physically that would be a temperature distribution in the bar at time $t = -1$. But if a temperature distribution at time $t = -1$ existed, even a singular one, then the temperature distribution would be smooth for all times greater than -1. The heat equation smooths away singularities. But the solution at times greater than -1 *is not* nonsingular, because we know it is singular at time zero. So a solution at time -1 can simply not exist.

Now let's return to the problem that *did* have a unique solution. That was when we started the backward heat equation solution from the smooth temperature profile at $t = 1$. There is still a major physical problem with the solution. Physically, (and also in typical numerical solutions), the initial profile will not be exact to infinitely many digits at all locations. There will always some error. Suppose in particular that the actual profile has a very slight kink. It can be so tiny that you cannot see the error in the profile under a microscope if you plot it versus the exact one. But if there is a kink, there is no longer a solution to the backward wave equation. There cannot be a solution at earlier times if there is a singularity, however invisible the kink may be.

In physics your data, here being the temperature profile, are never truly exact. So you have no way of saying which one is the right answer, the unique solution or a complete lack of any solution at all. If the data have any imperfection, it is the latter.

The bottom line is that even though the backward heat equation can have unique solutions for some problems, these solutions are only meaningful if you have a problem that is mathematically exact. So the initial/boundary-value problem for the backward heat equation is improperly posed. Even though it may have unique solutions.

The pure initial-value problem is similarly improperly posed.

18.3.3 An improperly posed elliptic problem

A typical improperly posed problem for the Laplace equation is shown in figure 18.4. Physically, it might correspond to heat conduction in a rectangular plate. For mathematical convenience, the horizontal size of the plate has been rescaled to length π.

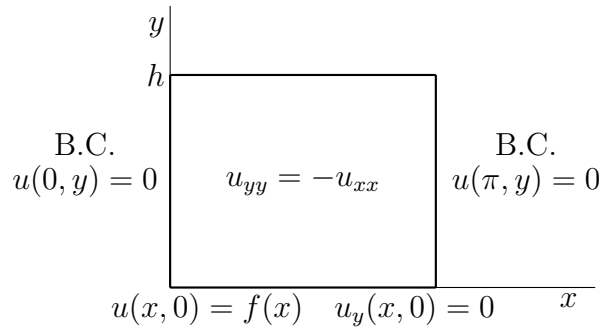


Figure 18.4: An improperly posed Laplace problem.

What is wrong in figure 18.4 is that both the temperature and the heat flow are specified at the lower boundary $y = 0$. That is wrong because the Laplace equation needs exactly one boundary condition at each point of the boundary, not two. It is also wrong that no boundary condition at all is given on the top boundary $y = h$.

Mathematically speaking, you might say that figure 18.4 is an initial / boundary-value problem for the Laplace equation, with y playing the part of time. And initial / boundary-value problems for the Laplace equation are not allowed. Note however that figure 18.4 would be perfectly fine if the partial differential equation was $u_{yy} = u_{xx}$ instead of $u_{yy} = -u_{xx}$. That would be a wave equation with a unit wave velocity, and initial/boundary-value problems are just what you want for wave equations. In short, a single sign in the partial differential equation makes all the difference.

The primary problem with initial/boundary-value problems for the Laplace equation is that they do not meet the third requirement for properly posedness. The effect of small changes in the data on the solution can be much larger than the small changes. For the example figure 18.4, that should be taken to mean that the solution u inside the plate can be much larger than the given value $f(x)$ of u at the lower boundary.

That can be seen as follows. Consider the following type of solution:

$$u(x, y) = \sin(nx) \cosh(ny)$$

where n is a natural number, one of 1, 2, 3, ... For a homework, you can verify that the above is a perfectly valid solution to the problem figure 18.4 when $f(x) = \sin(nx)$. It also happens to be unique.

Now note that the "data", the values of $f(x)$, are no greater than 1 in magnitude. On the other hand, in the interior of the plate, u can reach values up to $\cosh(nh)$. For any given value of n that is a finite number. But there is no *universal* bound to it. You can make $\cosh(nh)$ as large as you want by just taking n large enough. The value of $\cosh(1)$ is only about 1.5, but $\cosh(100)$ is already about 10^{43}. In short, the size of u can exceed the size of f by any arbitrarily large factor.

18.3.3 Review Questions

1. Show that the given solution

$$u(x,y) = \sin(nx)\cosh(ny)$$

for natural n does indeed satisfisfy the Laplace equation

$$u_{yy} + u_{xx} = 0$$

and the boundary conditions

$$u(x,0) = \sin(x) \quad u_y(x,0) = 0 \quad u(0,y) = 0 \quad u(\pi,y) = 0$$

Solution ppe-a

2. For the Laplace equation

$$u_{yy} + u_{xx} = 0$$

with boundary conditions

$$u(x,0) = f(x) \quad u_y(x,0) = 0 \quad u(0,y) = 0 \quad u(\pi,y) = 0$$

the "separation of variables" solution is

$$u(x,y) = \sum_{n=1}^{\infty} f_n \sin(nx)\cosh(ny)$$

Here the "Fourier coefficients" f_n must chosen so that they satisfy

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(nx)$$

Check this solution.

Can you immediately see that this separation of variables solution is probably no good?

Solution ppe-b

3. For the Laplace equation

$$u_{yy} + u_{xx} = 0$$

with boundary conditions

$$u(x,0) = f(x) \quad u_y(x,0) = 0 \quad u(0,y) = 0 \quad u(\pi,y) = 0$$

assume that $f(x)$ is the triangular profile:

$$f(x) = x \quad \text{if} \quad x \leq \tfrac{1}{2}\pi \qquad f(x) = \pi - x \quad \text{if} \quad x \geq \tfrac{1}{2}\pi$$

The "separation of variables" solution for this problem is

$$u(x,y) = \sum_{n=1}^{\infty} f_n \sin(nx)\cosh(ny)$$

where the "Fourier coefficients" f_n must chosen so that they satisfy

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(nx)$$

where $f(x)$ is the triangular profile described above.

Plot this separation of variables solution for $y = 0$ and for a few values greater than zero like $y = 1$, $y = 0.5$, $y = 0.25$. Then comment on whether a solution u exists at $y = 0$ and for $y > 0$.

This example should illustrate that typical improperly posed problems might have solutions if the data are perfectly smooth and their Taylor series have finite radii of convergence. But if there is a singularity, like the kink in the triangular profile, all bets are off.

You might know that if you talk about instability of ordinary differential equations, you wonder about what happens to the solution for infinite time. But in this problem you do not let the "time" coordinate y go to infinity. The problem is not large y, but large "wave number" n. The large wave number problem is really unique to partial differential equations. (If you had a system of infinitely many ordinary differential equations, you might also run into it.)

Include your code, if any.

Solution ppe-c

4. Continuing the previous question, show analytically that for the supposed solution

$$u(x,y) = \sum_{n=1}^{\infty} \frac{4}{\pi n^2} \sin\left(n\tfrac{1}{2}\pi\right) \sin(nx)\cosh(ny)$$

the sum does not converge for any x if $y > 0$.

Also show analytically that at the halfway point $x = \frac{1}{2}\pi$, the values that you get while summing increase monotonically to infinity.

Solution ppe-d

5. Show that the Laplace equation

$$\nabla^2 u = 0 \qquad \text{inside} \qquad \Omega$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial n} = 1 \qquad \text{on} \qquad \delta\Omega$$

has no solution. That makes it an improperly posed problem. To focus your thoughts, you can take an example domain Ω to be the inside of a sphere, and $\delta\Omega$ as its surface.

Explain the lack of solution in physical terms. To do so, consider this a steady heat conduction problem, with u the temperature, and the gradient of u the scaled heat flux.

Generalize the derivation to determine the requirement that

$$\nabla^2 u = f \qquad \text{inside} \qquad \Omega$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial n} = g \qquad \text{on} \qquad \delta\Omega$$

has a solution.

Solution ppe-e

6. Show that if the Poisson equation

$$\nabla^2 u = f \qquad \text{inside} \qquad \Omega$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial n} = g \qquad \text{on} \qquad \delta\Omega$$

has a solution, it is not unique.

Solution ppe-f

18.3.4 Improperly posed hyperbolic problems

A typical improperly posed problem for the wave equation is shown in figure 18.5. Physically, it might correspond to transverse vibrations of a string over a finite time interval. For mathematical convenience, the length of the string has been rescaled to length π. Also, the time has been rescaled to eliminate the wave speed c from the wave equation. The scaled final time has been written as $T\pi$, again for mathematical convenience, but the value of T can be anything.

What is wrong in figure 18.5 is that instead of specifying the initial position and velocity of the string, the initial and final position of the string are given. That is wrong because the wave equation is an evolution equation. It requires initial conditions, not final conditions.

Mathematically speaking, you might say that figure 18.5 is an boundary-value problem for the wave equation, with t playing the part of a spatial coordinate. Boundary-value problems for the wave equation are not allowed. Note

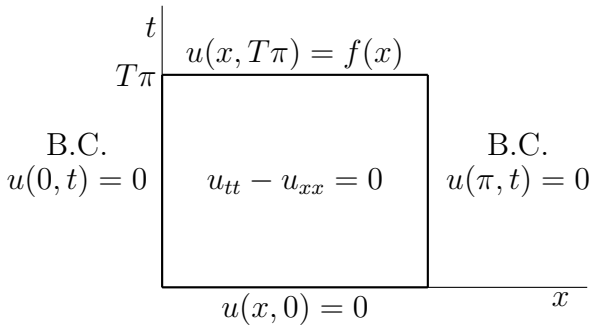


Figure 18.5: An improperly posed wave equation problem.

however that figure 18.5 would be perfectly fine if the partial differential equation was $u_{tt} + u_{xx} = 0$ instead of $u_{tt} - u_{xx} = 0$. That would be the Laplace equation, and boundary-value problems are just what you want for the Laplace equation. In short, a single sign in the partial differential equation makes all the difference.

The given problem has special solutions of the form

$$u = \sin(nx)\sin(nt)$$

where n is an integer. For such a solution

$$f(x) = \sin(nT\pi)\sin(nx)$$

The reason for the fact that figure 18.5 produces an improperly posed problem depends on the value of T. Consider first the possibility that T is a rational number. A rational number is a number that can be written as

$$T = \frac{m_1}{m_2}$$

where m_1 and m_2 are integers.

For such a rational T, the solution to the problem figure 18.5 is not unique. The quickest way to see that is to take the function $f(x)$ in the given problem zero. Then one solution to the problem is obviously $u = 0$. So any nonzero solution means that the solution is not unique. And you get a nonzero solution by taking $n = m_2$ or any whole multiple of m_2 in the special solutions given above.

Since the solution is not unique, the problem violates the second condition for properly posed problems.

Not all numbers are rational numbers, however. In fact, in some sense there are infinitely many more irrational numbers than rational ones. One simple example is $\sqrt{2}$. Irrational numbers can however be approximated to arbitrary

accuracy by rational ones. For example, consider $\sqrt{2}$ to 10 digits accuracy:

$$\sqrt{2} \approx 1.414213562 = \frac{1414213562}{1000000000}$$

The same way, any irrational T value is arbitrarily close to rational ones, and for rational ones the problem figure 18.5 is improperly posed. So surely you would not expect the boundary value problem to be properly posed for an irrational T value. What happens in this case is that the criterion 3 for properly posedness is violated. Consider again the special solutions above. In the interior of the rectangle, the solution $u = \sin(nx)\sin(nt)$ clearly has magnitude 1. (Or something comparable to 1, if you want to use an average magnitude as measure.) The data $f(x) = \sin(nT\pi)\sin(nx)$ however have magnitude $\sin(nT\pi)$. So, if you can find values for n that make $\sin(nT\pi)$ arbitrarily small, you have shown that criterion 3 is violated. The magnitude of u would be much larger than the magnitude of f. Try solving the corresponding homework question only if you are really good in math.

Finally, you might argue that initial/boundary-value problems are not *really* required for the wave equation. Since x appears in the wave equation exactly like t does, surely you should be able to provide two "initial conditions" at $x = 0$ instead of $t = 0$. And provide one "boundary condition" at $t = 0$ and one "boundary condition" at $t = T\pi$. Physically that is not an initial/boundary-value problem; it might be called an initial/final/single-doubled-boundary-value problem.

You have a good point there. The stated problem is indeed properly posed, as you say. However, that trick *only* works in one-spatial dimension. For the wave equation in two spatial dimensions,

$$u_{tt} = c^2 \left(u_{xx} + u_{yy}\right)$$

trying to specify "initial conditions" at $x = 0$ will produce an improperly posed problem. To see why, simply consider solutions that are independent of time. For such solutions, initial conditions at $x = 0$ would produce an initial value problem for the Laplace equation in x and y. That is improperly posed.

18.3.4 Review Questions

1. Show that the given solution

   $$u(x, y) = \sin(nx)\sin(nt)$$

   with $n = m_2$ does indeed satisfisfy the wave equation

   $$u_{tt} = u_{xx}$$

   and the boundary conditions

   $$u(x, 0) = 0 \quad u(0, t) = 0 \quad u(\pi, t) = 0 \quad u(x, \frac{m_1}{m_2}\pi) = 0$$

How about twice that solution? Ten times? How about if $n = 2m_2$? How about if $n = 10m_2$? So how many solutions are there really to this single problem?

Solution pph-a

2. For the brave. Show without peeking at the solution that the problem for irrational T is improperly posed by showing that you can make

$$\sin(nT\pi)$$

arbitrarily small by choosing suitable values of n. Then for these values of n, the solution

$$u = \frac{1}{\sin(nT\pi)} \sin(nx) \sin(nt)$$

becomes arbitrarily large in the interior although is is no larger than 1 on the boundary. So the problem for irrational T is improperly posed too, but not because the solution is not unique, but because small data (f, i.e. u on the top boundary) do not produce correspondingly small solutions in the interior.

Solution pph-b

18.4 Energy methods

Energy methods derive some sort of system "energy" from a partial differential equation. That energy may then be used to derive such things as existence and/or uniqueness of the solution, and whether it depends continuously on the data.

18.4.1 The Poisson equation

Consider the following very general Poisson equation problem:

$$\nabla^2 u = f \quad \text{in} \quad \Omega \qquad Au + B\frac{\partial u}{\partial n} = g \quad \text{on} \quad \delta\Omega$$

This includes the Laplace equation; just take $f = 0$. It includes Dirichlet boundary conditions, (take $B = 0$), and Neumann boundary conditions, (take $A = 0$).

The objective in this subsection is to show that solutions to this problem are unique as long as A and B do not have opposite sign. Assuming that the problem has a solution in the first place, there is only one.

To prove that, it must be proved that any two solutions u_1 and u_2 of the same problem cannot be different. That they must be the same. In other words, it must be proved that the difference $v = u_1 - u_2$ between any two solutions u_1 and u_2 is zero.

The proof starts with deriving the equations satisfied by the difference v. Since by assumption u_1 and u_2 are both solutions of the same problem:

$$\nabla^2 u_2 = f \quad \text{in} \quad \Omega \qquad Au_2 + B\frac{\partial u_2}{\partial n} = g \quad \text{on} \quad \delta\Omega$$

$$\nabla^2 u_1 = f \quad \text{in} \quad \Omega \qquad Au_1 + B\frac{\partial u_1}{\partial n} = g \quad \text{on} \quad \delta\Omega$$

If you subtract these two problems from each other, and replace $u_2 - u_1$ by v, you get

$$\nabla^2 v = 0 \quad \text{in} \quad \Omega \qquad Av + B\frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \delta\Omega$$

Note that both the partial differential equation and the boundary condition are homogeneous. That illustrates an important point:

> *The difference between two solutions of a linear problem always satisfies the homogeneous problem.*

Now we must prove that v is zero. That will mean that the difference between u_1 and u_2 is zero. And that will in turn imply that u_1 and u_2 must be the same; different solutions are not possible.

To prove that v is zero, the trick is to multiply the Laplace equation for v by $-v$ and integrate over the entire domain Ω:

$$-\int_\Omega v\nabla^2 v \, \mathrm{d}V = 0$$

Now note that, (assuming three dimensions, with $(x, y, z) = (x_1, x_2, x_3)$),

$$v\nabla^2 v = v\sum_{i=1}^{3}\frac{\partial^2 v}{\partial x_i^2} = \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(v\frac{\partial v}{\partial x_i}\right) - \sum_{i=1}^{3}\left(\frac{\partial v}{\partial x_i}\right)^2 = \nabla\cdot(v\nabla v) - (\nabla v)^2$$

(If you do not believe the second equals sign, just differentiate out the product immediately to the right of it.) Plugging this into the equation above gives

$$-\int_\Omega \nabla\cdot(v\nabla v)\,\mathrm{d}V + \int_\Omega (\nabla v)^2 \, \mathrm{d}V = 0$$

And now apply the divergence (or Gauss or Ostrogradski) theorem on the first integral to get

$$-\int_{\delta\Omega} v\frac{\partial v}{\partial n}\,\mathrm{d}S + \int_\Omega (\nabla v)^2 \, \mathrm{d}V = 0 \tag{18.4}$$

Note that the second integral is physically a potential energy in some important cases, like for a membrane under tension.

Now consider some special cases, starting with the simplest one:

- *The Dirichlet problem:* For the Dirichlet problem, v is zero on the boundary. So the first integral is zero. What is left is

$$\int_\Omega (\nabla v)^2 \, \mathrm{d}V = 0$$

  This can only be true if $\nabla v = 0$ everywhere. For assume the opposite, that ∇v was nonzero in some vicinity. In that case $(\nabla v)^2$ would be positive in that vicinity. So the vicinity would give a positive contribution to the integral. To get zero for the complete integral, that positive contribution would have to be cancelled by a negative contribution elsewhere. But $(\nabla v)^2$ cannot be negative, so negative contributions to the integral are impossible.

  So ∇v must be zero everywhere, and that means that v must equal some constant C everywhere. That includes that v must be C on the boundary. But $v = 0$ on the boundary, so C must be zero. So v is everywhere zero. So the difference between any two solutions u_1 and u_2 of the original problem must be everywhere zero. That means that the solutions must be equal, different solutions are not possible. So the solution is unique.

- *The Neumann problem:* For the Neumann problem, the first integral in (18.4) must again be zero, now because $\partial v/\partial n$ is zero on the boundary. So just like for the Dirichlet case above, v must everywhere be equal to some constant C. However, this constant does not have to be zero in this case. The Neumann boundary condition is satisfied regardless of what C is.

  So solutions of the Neumann problem are not unique. However, different solutions differ only by a constant, nothing more than that.

- *The mixed problem with A and B of the same sign:* For the mixed problem, you can use the boundary condition to write $\partial v/\partial n$ on the boundary in terms of v. That gives:

$$\frac{A}{B} \int_{\delta\Omega} v^2 \, \mathrm{d}S + \int_\Omega (\nabla v)^2 \, \mathrm{d}V = 0 \tag{18.5}$$

  If A and B are of the same sign, you can use this to show that the solution is unique.

  To do that, note that in that case the final integral must again be zero. The final integral cannot be negative because it is an integral of a square. And the final integral cannot be positive; otherwise the first term in the sum above would have to be negative to make their sum zero. And the first term cannot be negative because it too is an integral of a square.

Since the second integral is zero, v must again be some constant C. And since A is nonzero at at least some point, (otherwise it would be the pure Neumann problem), it follows from the boundary condition that the constant is zero. So like the Dirichlet problem, the solution is unique.

- *Varying boundary conditions:* You could have a problem where, say, on parts of the boundary a Dirichlet condition is satisfied, on other parts a Neumann condition, and on still other parts a mixed condition.

  In that case, the first, surface, integral in (18.4) is a sum over all these parts. As long as you can show that the combined integral is zero or negative, v must be zero and the solution is unique. (Except that in the pure Neumann case, the solution is indeterminate by a constant.)

- *The eigenvalue problem:* The eigenvalue problem for the Laplacian operator is

  $$\nabla^2 u = \lambda u \quad \text{in} \quad \Omega \qquad Au + B\frac{\partial u}{\partial n} = 0$$

  where the solution u must be nonzero. Using the same steps as before for v, you get

  $$-\int_{\delta\Omega} u\frac{\partial u}{\partial n}\,\mathrm{d}S + \int_\Omega (\nabla u)^2\,\mathrm{d}V + \lambda\int_\Omega u^2 = 0 \tag{18.6}$$

  From this it can be seen that for Dirichlet boundary conditions, or mixed boundary conditions with A and B of the same sign, the eigenvalues λ can only be negative. For Neumann boundary conditions, the eigenvalues can only be negative or zero. The zero eigenvalue must correspond to constant u.

18.4.1 Review Questions

1. Show that the Poisson equation

   $$\nabla^2 u = f$$

   with boundary conditions

   $$u_y(x,1) = g_1(x) \qquad u_y(x,0) = g_2(x)$$

   $$u(0,y) = g_3(y) \qquad u(1,y) + u_x(1,y) = g_4(y)$$

   has unique solutions.

   Solution emp-a

2. Using the arguments given in the text, uniqueness can *not* be shown for the Poisson equation

   $$\nabla^2 u = f$$

   with boundary conditions

   $$u_y(x,1) = g_1(x) \qquad u_y(x,0) = g_2(x)$$

$$u(0,y) = g_3(y) \qquad u(1,y) - u_x(1,y) = g_4(y)$$

Of course, just because you cannot *prove* uniqueness does not mean it is not true. But show that this problem never has unique solutions. If it has a solution at all, there are infinitely many different ones.

Solution emp-b

18.4.2 The heat equation

For the heat equation, similar arguments can be made as for the Laplace equation. This subsection briefly indicates the general lines.

Like for the Laplace equation in the previous subsection, the *difference* v between any two solutions of a heat equation problem must satisfy the homogenous problem. That problem is here

$$v_t = \kappa\nabla^2 v \quad \text{in} \quad \Omega \qquad Av + B\frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \delta\Omega \qquad v = 0 \quad \text{at} \quad t = 0$$

Multiply the partial differential equation by v and integrate like for the Laplace equation to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tfrac{1}{2}v^2 \,\mathrm{d}V - \int_{\delta\Omega} \kappa v \frac{\partial v}{\partial n}\,\mathrm{d}S + \int_\Omega \kappa\left(\nabla v\right)^2 \,\mathrm{d}V = 0 \tag{18.7}$$

Now consider conditions like those for the Laplace equation; Dirichlet or Neumann boundary conditions, or mixed boundary boundary conditions where A and B have the same sign. For those the final two terms cannot by negative. So the first term cannot be positive. So the "energy integral"

$$\int_\Omega \tfrac{1}{2}v^2 \,\mathrm{d}V$$

cannot increase in time. Because of the initial condition, it starts at zero. If it cannot grow, it cannot become greater than zero. (And it cannot become less than zero because it is the integral of a positive quantity.) So the energy integral must stay zero for all time. And that is only possible if v is everywhere zero for all time. If v was somewhere nonzero, the energy integral would be positive.

If the difference between two solutions is always zero, different solutions are not possible. Solutions are unique.

Under the same conditions on the boundary conditions, you can see that one condition for properly posedness is satisfied: small changes in the initial conditions produce small changes in the solution. To see this, allow a nonzero initial condition for v in the arguments above. The energy integral for v at a later time is still never larger than the energy integral of the initial condition for v. And the energy integral is a measure for the magnitude of v: if you divide the energy integral by half the volume of the domain and take a square root, it gives the root mean square value of v.

18.4.3 The wave equation

For the wave equation, similar arguments can be made as for the Laplace and heat equations. This subsection briefly indicates the general lines.

Like in the previous two subsections, the *difference* v between any two solutions of a wave equation problem must satisfy the homogenous problem. That problem is here

$$v_{tt} = a^2\nabla^2 v \text{ in } \Omega \qquad Av + B\frac{\partial v}{\partial n} = 0 \text{ on } \delta\Omega \qquad v = v_t = 0 \text{ at } t = 0$$

Multiply by v_t and integrate over Ω as before using the divergence theorem to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tfrac{1}{2}v_t^2 \,\mathrm{d}V - \int_{\delta\Omega} a^2 v_t \frac{\partial v}{\partial n}\,\mathrm{d}S + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tfrac{1}{2}a^2\left(\nabla v\right)^2 \,\mathrm{d}V = 0 \tag{18.8}$$

Now consider conditions like those for the Laplace equation; Dirichlet or Neumann boundary conditions, or mixed boundary boundary conditions where A and B have the same sign. Under such conditions the surface integral term cannot be negative. So the time derivative of the "energy integral"

$$\int_\Omega \tfrac{1}{2}v_t^2 + \tfrac{1}{2}a^2\left(\nabla v\right)^2 \,\mathrm{d}V$$

is never positive; the energy integral cannot increase. So it must stay at its initial value of zero. So the spatial derivatives of v, as well as its time derivative, must be zero for all time. So v must be a constant. And the initial condition says that that constant is zero. So v is zero everywhere for all time.

If the difference between two solutions is always zero, different solutions are not possible. Solutions are unique.

It may be noted that physically, the first term in the energy equation is typically kinetic energy, and the second potential energy. If you generalize the problem for v to still have homogeneous boundary conditions, but inhomogeneous initial conditions, you can derive energy conservation. In particular for homogeneous Dirichlet or Neumann boundary conditions, the total energy is preserved. For mixed boundary conditions where A and B have the same sign, the energy can only decrease. If A and B have opposite sign, the problem is unstable in the sense that the energy will increase. You can also use this to show that small changes in the initial conditions produce small changes in the solution for appropriate boundary conditions.

18.5 Variational methods [None]

18.6 Classification

18.6.1 Introduction

Classification groups partial differential equations with similar properties together.

One set of partial differential equations that has a unambiguous classification are 2D second order quasi-linear equations:

$$au_{xx} + 2bu_{xy} + cu_{yy} = d$$

where $a = a(x, y, u, u_x, u_y)$, $b = b(x, y, u, u_x, u_y)$, $c = c(x, y, u, u_x, u_y)$, and $d = d(x, y, u, u_x, u_y)$.

The classification for these equations is:

- $b^2 - ac > 0$: hyperbolic
- $b^2 - ac = 0$: parabolic
- $b^2 - ac < 0$: elliptic

Example

Question: Classify the equation

$$yu_{xx} - 2u_{xy} + e^x u_{yy} + u = 3$$

Solution:

Identify the coefficients and find the discriminant:

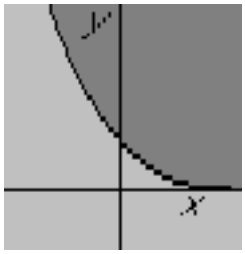
$$a = y, b = -1, c = e^x \qquad \Rightarrow \qquad b^2 - ac = 1 - ye^x$$

So it is parabolic for

$$1 - ye^x = 0 \qquad \Rightarrow \qquad y = e^{-x},$$

It is elliptic if y is greater than the value above and hyperbolic if y is less.

Graphically:



Example

Question: Classify the potential equation for compressible flow for an airfoil:

$$(a^2 - \phi_x^2)\phi_{xx} - -2\phi_x\phi_y\phi_{xy} + (a^2 - \phi_y^2)\phi_{xx} = 0$$

Here a is the speed of sound, not to be confused with the a in the generic partial differential equation. Also ϕ_x and ϕ_y are the velocity components u and v in the x and y directions. Discuss the associated physics.

Solution:

The discriminant is

$$(\phi_x\phi_y)^2 - (a^2 - \phi_x^2)(a^2 - \phi_y^2)$$

or in terms of velocity and speed of sound:

$$u^2v^2 - (a^2 - u^2)(a^2 - v^2)$$

Multiplying out gives

$$a^2[a^2 - (u^2 + v^2)]$$

Note that the term within square brackets is positive if the magnitude of the speed of sound is greater than the magnitude of the velocity. That is subsonic flow, with a Mach number

$$M = \frac{\sqrt{u^2 + v^2}}{a}$$

less than 1. The equation is then elliptic. Conversely, if the flow velocity is greater than the speed of sound, supersonic flow, then the equation is hyperbolic.

Consider a picture of transonic flow around an airfoil:

The subsonic region does indeed behave elliptic. There are smooth solutions and an unlimited region of dependence. Numerically, this region must be solved through simultaneous global solution.

The supersonic region behaves hyperbolic. There are singularities like expansion fans and shocks. The propagation of Mach lines is given by the so-called Mach-angle. The solution can numerically be found station by station using a marching scheme.

18.6.2 Scalar second order equations

The general n-dimensional second order quasi-linear second order equation is:

$$\begin{array}{ccccccc} a_{11}u_{x_1x_1} & + & a_{12}u_{x_1x_2} & + & a_{13}u_{x_1x_3} & + & \dots \\ a_{21}u_{x_2x_1} & + & a_{22}u_{x_2x_2} & + & a_{23}u_{x_2x_3} & + & \dots \\ a_{31}u_{x_3x_1} & + & a_{32}u_{x_3x_2} & + & a_{33}u_{x_3x_3} & + & \dots \\ \vdots & & \vdots & & \vdots & & \ddots \end{array} = d$$

The coefficients are not quite unique: since the order of differentiation can be reversed, a coefficient like a_{12} is equivalent to a_{21}. The coefficients will be made unique by requiring that they form a symmetric matrix A. That means, for example, that $a_{12} = a_{21}$.

In index notation, the n-dimensional equation can then be written as:

$$\boxed{\sum_i \sum_j a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = d} \tag{18.9}$$

where the coefficients $a_{ij} = a_{ij}(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_n})$ form a symmetric matrix A and $d = d(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_n})$.

Example

Question: Find matrix A for the generic two-dimensional equation

$$au_{xx} + 2bu_{xy} + cu_{yy} = d$$

Solution: Writing the equation in n-dimensional form gives:

$$a_{11}u_{xx} + a_{12}u_{xy} + a_{21}u_{xy} + a_{22}u_{yy} = d$$

Comparing with the equation above, $a_{11} = a$, $a_{12} = a_{21} = b$, and $a_{22} = c$. So the matrix A is here:

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Classification is based on the eigenvalues of A:

- parabolic if any eigenvalues are zero; otherwise:
- elliptic if all eigenvalues are the same sign;
- hyperbolic if all eigenvalues except one are of the same sign;
- ultrahyperbolic, otherwise.

The reason for this classification will be explained later, in section 18.7.4.

Example

Question: Figure out why that is consistent with what we defined for the two-dimensional case,

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \text{which should be} \quad \begin{cases} \text{hyperbolic if } b^2 - ac > 0 \\ \text{parabolic if } b^2 - ac = 0 \\ \text{elliptic if } b^2 - ac < 0 \end{cases}$$

Solution: From linear algebra it is known that the determinant of a matrix equals the product of its eigenvalues. In this case there are two eigenvalues. Their product is the determinant:

$$\lambda_1\lambda_2 = \begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac - b^2$$

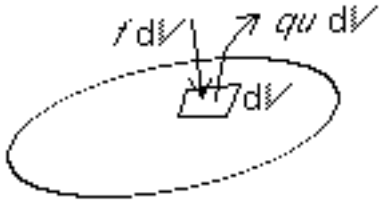
According to the n-dimensional classification scheme, the equation is parabolic if an eigenvalue is zero. But then the product $ac - b^2$ of the eigenvalues will be zero. So $b^2 - ac$ is zero too, and that also makes the equation parabolic according to the two-dimensional classification scheme.

According to the n-dimensional classification scheme, the equation is elliptic when the two eigenvalues are of the same sign. But then the product of the eigenvalues is positive, $ac - b^2 > 0$. That makes $b^2 - ac$ negative in agreement with the two-dimensional classification.

The only remaining possibility is that the eigenvalues are of opposite sign. That makes their product $ac - b^2$ negative, again in agreement with the two-dimensional classification.

Example

Question:



The equation

$$u_{tt} - \nabla \cdot (p\nabla u) + qu + ru_t = f$$

is a generic wave equation, with u the displacement from equilibrium. The first term is the substance acceleration at a point. The second term represents net force per unit volume due to elastic distortion. In it, p could be the Young's modulus. The third term would be a distributed force that wants to keep the substance in the equilibrium location. The forth term would be a viscous damping. The right hand side represents a distributed force per unit volume. Classify this equation.

Solution:

Write out the equation using the definition of nabla:

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Then

$$\nabla u \equiv \text{grad } u = \hat{\imath}u_x + \hat{\jmath}u_y + \hat{k}u_z$$

while for any vector $\vec{v}$,

$$\nabla \cdot \vec{v} \equiv \text{div } \vec{v} = v_{1x} + v_{2y} + v_{3z}$$

The equation is therefore

$$u_{tt} - (pu_x)_x - (pu_y)_y - (pu_z)_z + qu + ru_t = f$$

Now identify the highest derivatives of u:

$$u_{tt} - pu_{xx} - pu_{yy} - pu_{zz} + \ldots = f$$

Find the coefficient matrix A

$$A = \begin{pmatrix} -p & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = -p$ and $\lambda_4 = 1$. Since all eigenvalues except one are of the same sign, it is hyperbolic.

18.6.2 Review Questions

1. The equation

$$u_t - \nabla \cdot (p \nabla u) + qu = f$$

is a generic unsteady heat conduction equation, with u the temperature relative to the surroundings. The first term is the rate of temperature change at a point. The second term represents heat accumulation at the point due to conduction of heat. In it, p is the heat conduction coefficient. The third term would in be an approximation to the heat radiated away to the surroundings, either in two-dimensions or for a transparant medium. The right hand side represents heat that is explicitly added from other sources. Classify this equation. Also classify the steady version, i.e. the equation without the u_t term.

Solution clasnd-a

18.7 Changes of Coordinates

Changes of coordinates are a primary way to understand, simplify, and sometimes even solve, partial differential equations.

18.7.1 Introduction

It is possible to simplify many partial differential equation problems by using coordinate systems that are special to the problem:

- In unsteady pipe flows, use the lines along which sound waves propagate (characteristic lines) as coordinate lines to simplify the partial differential equation.

- In steady supersonic flows, use the Mach lines along which disturbances propagate (characteristic lines) as coordinate lines to simplify the partial differential equation.
- In problems with anisotropic properties, rotate your coordinate system along the principal or physical directions.
- In problems with spherical symmetry, spherical coordinates are usually easier than Cartesian ones.
- ...

18.7.2 The formulae for coordinate transformations

Assume the purpose is to address a problem in an n-dimensional space. The coordinates in this space form a vector

$$\vec{x} = (x_1, x_2, \ldots, x_n)$$

For example, we may have a problem in three-dimensional Cartesian coordinates x, y, and z. Then $x_1 = x$, $x_2 = y$, and $x_3 = z$, and $\vec{x}$ is the position vector $\vec{r}$. Or we might have a problem in four-dimensional space-time, in which case $x_1 = x$, $x_2 = y$, $x_3 = z$, and $x_4 = t$.

The idea is now to switch to some new set of independent coordinates

$$\vec{\xi} = \xi_1, \xi_2, \ldots, \xi_n$$

that simplify the problem. Of course, these new coordinates will have to be some sort of functions of the old ones,

$$\xi_1 = \xi_1(x_1, x_2, \ldots, x_n) \quad \xi_2 = \xi_2(x_1, x_2, \ldots, x_n) \quad \ldots \quad \xi_n = \xi_n(x_1, x_2, \ldots, x_n) \tag{18.10}$$

and vice-versa.

The change of coordinates is characterized by Jacobian matrices

$$\boxed{\mathcal{J} \equiv \frac{\partial \vec{x}}{\partial \vec{\xi}} \qquad \mathcal{J}^{-1} \equiv \frac{\partial \vec{\xi}}{\partial \vec{x}}} \tag{18.11}$$

These matrices are inverses of each other, as the above notation indicates. The determinant $|\mathcal{J}|$ is the Jacobian J of the transformation from $\vec{x}$ to $\vec{\xi}$. It is used in converting volume integrals. In particular

$$\mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n = J \mathrm{d}\xi_1 \mathrm{d}\xi_2 \ldots \mathrm{d}\xi_n$$

The complete Jacobian matrices describes how a small change in $\vec{\xi}$ relates to the corresponding change in $\vec{x}$, and vice versa:

$$\boxed{\mathrm{d}\vec{x} = \mathcal{J} \mathrm{d}\vec{\xi} \qquad \mathrm{d}\vec{\xi} = \mathcal{J}^{-1} \mathrm{d}\vec{x}} \tag{18.12}$$

Note that while $\mathcal{J}$ is called the transformation matrix *from* $\vec{x}$ *to* $\vec{\xi}$, it really allows you to compute $\mathrm{d}\vec{x}$ from $\mathrm{d}\vec{\xi}$; the opposite of what you would expect.

In index notation, the components of the Jacobian matrices are

$$\boxed{\left(\mathcal{J}\right)_{ik} = \left(\frac{\partial \vec{x}}{\partial \vec{\xi}}\right)_{ik} \equiv \frac{\partial x_i}{\partial \xi_k} \qquad \left(\mathcal{J}^{-1}\right)_{ki} = \left(\frac{\partial \vec{\xi}}{\partial \vec{x}}\right)_{ki} \equiv \frac{\partial \xi_k}{\partial x_i}} \tag{18.13}$$

For any values of i and k between 1 and n. Note that the first index is the one of the top vector, and the second index the one of the bottom vector.

The purpose is now to simplify second order quasi-linear partial differential equations using coordinate transforms. As noted in the previous section, second order quasi-linear equations are of the form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = d$$

The set of independent coordinates $\vec{x} = (x_1, x_2, \ldots)$ is to be replaced by a cleverly chosen different set $\vec{\xi} = (\xi_1, \xi_2, \ldots)$ to simplify the equation.

Of course, before you can do anything clever like that, you have to first know what happens to the partial differential equation when the coordinates are changed. It turns out that the *form* of the equation remains the same in the new coordinates:

$$\sum_{k=1}^{n} \sum_{l=1}^{n} a'_{ij} \frac{\partial^2 u}{\partial \xi_k \partial \xi_l} = d'$$

The coefficients a_{kl} do again form a symmetric matrix, call it A'. However, the matrix A' is different from the matrix A in the original coordinates. Also, the right hand side d' is different from d.

The expression for the new matrix A' can be written in either matrix notation or index notation:

$$\boxed{A' = \mathcal{J}^{-1} A \mathcal{J}^{-\mathrm{T}} \qquad a'_{kl} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \xi_k}{\partial x_i} a_{ij} \frac{\partial \xi_l}{\partial x_j}} \tag{18.14}$$

Here $-\mathrm{T}$ means the transpose of $\mathcal{J}^{-1}$. The transpose matrix has the columns of $\mathcal{J}^{-1}$ as its rows.

The expression for the new right hand side d' is best written in index notation:

$$\boxed{d' = d - \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{\partial^2 \xi_k}{\partial x_i \partial x_j} \right) \frac{\partial u}{\partial \xi_k}} \tag{18.15}$$

Using equations (18.14) and (18.15) above, you can figure out what the new matrix and right hand side are. However, that may not yet be enough to fully

transform the problem to the new coordinates. Recall that the coefficients a_{ij} and d might involve first order derivatives with respect to $x_1, x_2, \ldots$. These derivatives must be converted to derivatives with respect to $\xi_1, \xi_2, \ldots$. To do that, use:

$$\boxed{\frac{\partial u}{\partial \vec{x}} = \frac{\partial u}{\partial \vec{\xi}} \frac{\partial \vec{\xi}}{\partial \vec{x}} \qquad \frac{\partial u}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial u}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_i}} \tag{18.16}$$

in vector and index notation respectively.

That may still not be enough, because the resulting equation will probably still contain $x_1, x_2, \ldots$ themselves. You will also need to express these in terms of $\xi_1, \xi_2, \ldots$ to get the final partial differential equation completely in terms of the new coordinates.

But that is it. You are now done. At least with the partial differential equation. There might also be boundary and/or initial conditions to invert. That can be done in a similar way, but we will skip it here.

One additional point should be made. If you follow the procedure as outlined above exactly, you will have to express $\xi_1, \xi_2, \ldots$ in terms of $x_1, x_2, \ldots$, and differentiate these expressions. You will also need to express $x_1, x_2, \ldots$ in terms of $\xi_1, \xi_2, \ldots$ to get rid of $x_1, x_2, \ldots$ in the equations. That is a lot of work. Also, if you are, say, switching from Cartesian to sperical coordinates, the expressions for the spherical coordinates in terms of the Cartesian ones are awkward. You would much rather just deal with the expressions of the Cartesian coordinates in terms of the spherical ones.

Now differentiating the $x_1, x_2, \ldots$ with respect to the $\xi_1, \xi_2, \ldots$ will give you matrix $\mathcal{J}$ instead of $\mathcal{J}^{-1}$. But you can invert the matrix relatively easily using the method of minors. While that is a bit of work, you also save a lot of work because you no longer have to convert $x_1, x_2, \ldots$ in the results to $\xi_1, \xi_2, \ldots$ and clean up the mess.

To convert d into d', as described above, you will need to evaluate the second order derivatives of $\xi_1, \xi_2, \ldots$ in it. Do that as

$$\frac{\partial^2 \xi_k}{\partial x_i \partial x_j} = \sum_{l=1}^{n} \frac{\partial}{\partial \xi_l} \left(\frac{\partial \xi_k}{\partial x_i} \right) \frac{\partial \xi_l}{\partial x_j}$$

Take the two first order derivatives at the end of this expression from the inverse matrix that you already computed.

Derivation {D.4} gives the derivation of the various formulae above.

18.7.3 Rotation of coordinates

The purpose of this section is to simplify second order partial differential equations by rotating the coordinate system to a different orientation. This allows you to simplify the matrix A of the partial differential equation considerably.

In particular, in this way you can bring the new matrix A' into the form of a "diagonal" matrix:

$$\boxed{A' = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ 0 & 0 & \lambda_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \equiv \Lambda} \tag{18.17}$$

So the partial differential equation simplifies in the new coordinates to:

$$\boxed{\lambda_1 u_{\xi_1\xi_1} + \lambda_2 u_{\xi_2\xi_2} + \ldots + \lambda_n u_{\xi_n\xi_n} = d'} \tag{18.18}$$

There are no longer any mixed derivatives. And the remaining coefficients of the PDE are the eigenvalues of the original matrix A.

One limitation to the procedure in this section should be stated right away. It concerns the case that the matrix A is not constant, but varies from point to point. For such a partial differential equation, you can select a point, any point you like, and bring the equation in the above diagonal form *at that one selected point*. At other points there will then still be mixed derivatives in the transformed equation.

To figure out *how* to convert a partial differential equation to the above diagonal form, first a brief review of linear algebra is needed. First recall that in three dimensions, you can define "basis vectors" $\hat{\imath} = (1,0,0)$, $\hat{\jmath} = (0,1,0)$, and $\hat{k} = (0,0,1)$. And you can write any other three dimensional vector in terms of these three basis vectors, like for example $3\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$. Similarly in n dimensions you can define n basis vectors $\hat{\imath}_1 = (1,0,0,\ldots)$, $\hat{\imath}_2 = (0,1,0,\ldots)$, $\ldots \hat{\imath}_n = (\ldots,0,0,1)$.

Next, a simple linear transformation of coordinates (which leaves the origin unchanged) takes the form, by definition,

$$\boxed{\vec{x} = P\vec{\xi} \qquad \vec{\xi} = P^{-1}\vec{x}} \tag{18.19}$$

Here P is a matrix that is called the transformation matrix from $\vec{x}$ to $\vec{\xi}$. (Although it really computes $\vec{x}$ from $\vec{\xi}$.)

Matrix P consists of the "basis vectors" of the new coordinate system, viewed from the old coordinate system. So for the special case that the transformation is a simple rotation of the coordinate system, matrix P consists of the rotated basis vectors $\hat{\imath}_1, \hat{\imath}_2, \ldots, \hat{\imath}_n$, call them $\hat{\imath}'_1, \hat{\imath}'_2, \ldots, \hat{\imath}'_n$, viewed from the original coordinate system. (Conversely, P^{-1} consists of the original basis vectors $\hat{\imath}_1, \hat{\imath}_2, \ldots, \hat{\imath}_n$ when viewed from the new coordinate system.) The important thing to remember is that for the special case of coordinate rotation, the inverse of P is just its transpose:

$$\boxed{\text{Coordinate system rotation:} \quad P^{-1} = P^{\mathrm{T}}} \tag{18.20}$$

Note further that for linear transformations, the Jacobian matrices are just

$$\mathcal{J} = \frac{\partial \vec{x}}{\partial \vec{\xi}} = P \qquad \mathcal{J}^{-1} = \frac{\partial \vec{\xi}}{\partial \vec{x}} = P^{-1}$$

So the expression of the previous subsection for the new matrix A' becomes in terms of P:

$$A' = \mathcal{J}^{-1} A \mathcal{J}^{-\mathrm{T}} = P^{-1} A P^{-\mathrm{T}}$$

Now it is known from linear algebra that this becomes the diagonal matrix Λ given at the start of the subsection if you take $P^{-\mathrm{T}}$ as the matrix E of eigenvectors of A'. (If A varies from point to point, that means more specifically the eigenvectors of the selected point.) But for a rotation of coordinates, $P^{-\mathrm{T}} = P^{\mathrm{TT}}$ is just P. So the needed coordinate transform is

$$\boxed{\vec{x} = E\vec{\xi} \qquad \vec{\xi} = E^{\mathrm{T}}\vec{x}} \tag{18.21}$$

where the eigenvectors of A are the columns of matrix E. The eigenvalues in diagonal matrix $A' = \Lambda$ will be in the same order as the corresponding eigenvectors in E.

Warning: you must normalize the eigenvectors (divide them by their length) because the basis vectors $\hat{\imath}_1, \hat{\imath}_2, \ldots, \hat{\imath}_n$ are all of length one. And first you must make sure every one is orthogonal to all the others. Fortunately, that is normally automatic. However, if you have a double eigenvalue, any two corresponding eigenvectors are not necessarily orthogonal; you must explicitly make them so. Similarly, for a triple eigenvalue, you will need to create three corresponding orthogonal eigenvectors. And then divide each by its length. (To check whether vectors are orthogonal, check that their dot product is zero.)

Some books, like [3], do not bother to normalize the eigenvectors to length one. In that case the coordinate transformation is not just a rotation, but also a stretching of the coordinate system. The matrix A' is still diagonal, but the values on the main diagonal are no longer the eigenvalues of A. Also, it becomes messier to find the old coordinates in terms of the new ones. You would have to find the inverse of P^{-1} using minors. Using orthonormal rather than just orthogonal eigenvectors is recommended.

You might wonder, if A varies from point to point, why can we not simply set

$$\mathcal{J} \equiv \frac{\partial \vec{x}}{\partial \vec{\xi}} = E$$

at every point, where matrix E consists of the eigenvectors of A at that point. That would make A' diagonal at every point, instead of just a selected point. Unfortunately however, this does not work, because it is equivalent to n^2 scalar differential equations for the n scalar components of $\vec{x}$. If the number of equations is larger than the number of unknowns, there is normally no solution.

Do recall that you will also have to transform the right hand side d to the new coordinates. However, the second formula in (18.21) implies that the right hand side d is the same in the transformed equation as in the original one: $x'_1, x'_2, \ldots$ are linear in $x_1, x_2, \ldots$, so their second order derivatives in (18.15) are zero.

You will need the first formula in (18.21), in terms of its components, to get rid of any coordinates $x_1, x_2, \ldots$ in the right hand side d in favor of $x'_1, x'_2, \ldots$. Also, if d contains derivatives with respect to the unknowns $x_1, x_2, \ldots$, you will need to convert those using (18.16) of the previous subsection. To get the derivatives $x'_1, x'_2, \ldots$ with respect to $x_1, x_2, \ldots$ while doing so, write out the second formula in (18.21) in terms of its components.

Example

Question: Classify the equation

$$3u_{xx} - 2u_{xy} + 2u_{yy} - 2u_{yz} + 3u_{zz} + 12u_y - 8u_z = 0$$

and put it in canonical form.

Solution:

$$3u_{xx} - 2u_{xy} + 2u_{yy} - 2u_{yz} + 3u_{zz} + 12u_y - 8u_z = 0$$

Identify the matrix:

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

To find the new coordinates (transformation matrix), find the eigenvalues and eigenvectors of A:

The eigenvalues are the roots of $|A - \lambda I| = 0$:

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2(2-\lambda) - (3-\lambda) - (3-\lambda)$$

Hence $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 4$.

The eigenvectors are solutions of $(A - \lambda I)\vec{v} = 0$ that are normalized to length one. For $\lambda_1 = 1$, writing matrix $A - \lambda_1 I$ and applying Gaussian elimination on it produces

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

which gives the normalized eigenvector

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} / \sqrt{6}$$

For $\lambda_2 = 3$,

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which gives the normalized eigenvector

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} / \sqrt{2}$$

For $\lambda_3 = 4$,

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

which gives the normalized eigenvector

$$\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} / \sqrt{3}$$

The new equation is:

$$u_{\xi\xi} + 3u_{\eta\eta} + 4u_{\theta\theta} + 12u_y - 8u_z = 0$$

However, that still contains the old coordinates in the first order terms. Use the transformation formulae and total differentials to convert the first order derivatives:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \theta \end{pmatrix}$$

and its inverse

$$\begin{pmatrix} \xi \\ \eta \\ \theta \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The partial derivatives of $(\xi\eta\theta)$ with respect to (x, y, z) can be read off from the final matrix. So

$$u_y = u_\xi \frac{2}{\sqrt{6}} - u_\theta \frac{1}{\sqrt{3}}$$

$$u_z = u_\xi \frac{1}{\sqrt{6}} - u_\eta \frac{1}{\sqrt{2}} + u_\theta \frac{1}{\sqrt{3}}$$

Hence in the rotated coordinate system, the partial differential equation is:

$$u_{\xi\xi} + 3u_{\eta\eta} + 4u_{\theta\theta} + \frac{16}{\sqrt{6}}u_\xi + \frac{8}{\sqrt{2}}u_\eta - \frac{20}{\sqrt{3}}u_\theta = 0$$

18.7.3 Review Questions

1. Simplify the partial differential equation

$$10u_{xx} + 6u_{xy} + 2u_{yy} = u_x + x + 1$$

by rotating the coordinate system. Classify the equation. Draw the original and rotated coordinate system and identify the angle of rotation.

Solution rotcoor-a

18.7.4 Explanation of the classification

The previous subsection showed how partial differential equations can be simplified by rotating the coordinate system. Using this procedure it is possible to understand why second order partial differential equations are classified as described in section 18.6.2.

From the above, it already starts to become clearer why the classification of second order partial differential equations is in terms of the eigenvalues A. If two different second order partial differential equations have the same eigenvalues of their matrix A, then you can simply rotate the coordinate system to make their matrices A equal. And the highest order derivatives make the biggest difference for the physical behavior of the system. For short-scale effects, which include singularities, the highest order derivatives dominate. For them, the right hand side d is relatively unimportant. And since the highest order terms are now equal for the two partial differential equations, they must behave very similarly. So they should be classified as being in the same group.

Rotation of the coordinate system reduces a partial differential equation of the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} = d$$

to

$$\lambda_1 u_{x'_1 x'_1} + \lambda_2 u_{x'_2 x'_2} + \ldots + \lambda_n u_{x'_n x'_n} = d$$

where $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of the matrix A that has coefficients a_{ij}.

That immediately explains why only the eigenvalues of matrix A are of importance for the classification. Rotating the mathematical coordinate system obviously does not make any difference for the physical nature of the solutions. And in the rotated coordinates, all that is left of matrix A are its eigenvalues.

The next question is why the classification only uses the signs of the eigenvalues, not their magnitudes. The reason is that the magnitude can be scaled away by stretching the coordinates. That is demonstrated in the next example.

Example

Question: The previous example reduced the elliptic partial differential equation

$$3u_{xx} - 2u_{xy} + 2u_{yy} - 2u_{yz} + 3u_{zz} + 12u_y - 8u_z = 0$$

to the form

$$u_{\xi\xi} + 3u_{\eta\eta} + 4u_{\theta\theta} + \frac{16}{\sqrt{6}}u_\xi + \frac{8}{\sqrt{2}}u_\eta - \frac{20}{\sqrt{3}}u_\theta = 0$$

Reduce this equation further until it becomes as closely equal to the Laplace equation as possible.

Solution:

The first step is to make the coefficients of the second order derivatives equal in magnitude. That can be done by stretching the coordinates. If

$$\xi = \bar{\xi} \quad \eta = \sqrt{3}\bar{\eta} \quad \theta = 2\bar{\theta}$$

then

$$u_{\bar{\xi}\bar{\xi}} + u_{\bar{\eta}\bar{\eta}} + u_{\bar{\theta}\bar{\theta}} + \frac{16}{\sqrt{6}}u_{\bar{\xi}} + \frac{8}{\sqrt{6}}u_{\bar{\eta}} - \frac{10}{\sqrt{3}}u_{\bar{\theta}} = 0$$

Note that all that is left in the second order derivative terms is the *sign* of the eigenvalues.

You can get rid of the first order derivatives by changing to a new independent variable v. To do so, set $u = v\, e^{a\bar{\xi}+b\bar{\eta}+c\bar{\theta}}$. Plug this into the differential equation above and differentiate out the product. Then choose a, b, and c so that the first derivatives drop out. You will find that you need:

$$a = -\frac{8}{\sqrt{6}} \quad b = -\frac{4}{\sqrt{6}} \quad c = \frac{5}{\sqrt{3}}$$

Then the remaining equation turns out to be:

$$v_{\bar{\xi}\bar{\xi}} + v_{\bar{\eta}\bar{\eta}} + v_{\bar{\theta}\bar{\theta}} - \frac{65}{3}v = 0$$

It is not exactly the Laplace equation because of the final term. But the final term does not even involve a first order derivative. It makes very little difference for short-scale phenomena. And short scale phenomena (such as singularities) are the most important for the qualitative behavior of the partial differential equation.

As this example shows, the values of the nonzero eigenvalues can be normalized to 1 by stretching coordinates. However, the *sign* of the eigenvalues cannot be changed. And neither can you change a zero eigenvalue into a nonzero one, or vice-versa, by stretching coordinates.

You might wonder why all this also applies to partial differential equations that have variable coefficients a_{ij} and d. Actually, what d is does not make much of a difference. But generally speaking, rotation of the coordinate system only works if the coefficients a_{ij} are constant. If they depend on position, the eigenvectors $\hat{\imath}'_1, \hat{\imath}'_2 \ldots$ at every point can still be found. So it might seem logical to try to find the new coordinates $x'_1, x'_2, \ldots$ from solving $\partial\vec{x}'/\partial\vec{x} = (\hat{\imath}'_1, \hat{\imath}'_2, \ldots)^T$. But the problem is that that are n^2 equations for only n unknown coordinates.

If the unit vectors are not constant, these equations normally mutually conflict and cannot be solved.

The best that can normally be done for arbitrary A is to select a single point that you are interested in. Then rotate the coordinate system to diagonalize the partial differential equation at that one point. In that case, A is diagonal near the considered point. And that is enough to classify the equation at that point. For, the most important feature that the classification scheme tries to capture is what happens to short scale phenomena. Short scale phenomona will "see" the locally diagonal equation. So the classification scheme continues to work.

18.7.4 Review Questions

1. Convert the equation

$$11u_{x'x'} + u_{y'y'} = \frac{3}{\sqrt{10}}u_{x'} - \frac{1}{\sqrt{10}}u_{y'} + \frac{3}{\sqrt{10}}x' - \frac{1}{\sqrt{10}}y' + 1$$

to be as close as possible to the Laplace equation.

Solution expclass-a

18.8 Two-Dimensional Coordinate Transforms

More powerful simplifications by changing coordinates are possible in 2D.

Assume that in terms of coordinates x and y, we have a partial differential equation:

$$au_{xx} + 2bu_{xy} + cu_{yy} = d$$

Then, if we transform to new coordinates, call them ξ and η, we will get a new partial differential equation of the form: ξ, η

$$a'u_{\xi\xi} + 2b'u_{\xi\eta} + c'u_{\eta\eta} = d'$$

The idea is again to choose the new coordinates ξ and η so that the new partial differential equation is as simple as possible.

For example, for a hyperbolic equation, you may like coordinates ξ and η such that a' and c' are zero. To find out for what coordinates ξ and η that is the case, expressions for the new coefficients a', b', c' and d' in terms of the new coordinates are needed. These can be found by writing out the general transformation formulae from section 18.7.2 for the special case of two dimensions. You get, {D.5}:

$$\boxed{\begin{aligned} a' &= a\left(\xi_x\right)^2 + 2b\left(\xi_x\right)\left(\xi_y\right) + c\left(\xi_y\right)^2 \\ b' &= a\left(\xi_x\right)\left(\eta_x\right) + b\left(\xi_x\right)\left(\eta_y\right) + b\left(\xi_y\right)\left(\eta_x\right) + c\left(\xi_y\right)\left(\eta_y\right) \\ c' &= a\left(\eta_x\right)^2 + 2b\left(\eta_x\right)\left(\eta_y\right) + c\left(\eta_y\right)^2 \\ d' &= d - \left(a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy}\right)u_\xi - \left(a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy}\right)u_\eta \end{aligned}} \tag{18.22}$$

18.8.1 Characteristic Coordinates

Characteristic coordinates are coordinates so that the the $u_{\xi\xi}$ and $u_{\eta\eta}$ derivatives are eliminated. That leaves only the $u_{\xi\eta}$ derivative, greatly simplifying the partial differential equation. It reduces to the two-dimensional canonical form:

$$\boxed{2b' u_{\xi\eta} = d'} \tag{18.23}$$

The first thing is to find out how this may be achieved. In terms of the coefficients of the transformed equation as discussed above, a' and c' must vanish. The condition $a' = 0$ requires, according to the given formulae:

$$a\left(\xi_x\right)^2 + 2b\left(\xi_x\right)\left(\xi_y\right) + c\left(\xi_y\right)^2 = 0$$

That can be considered to be a partial differential equation for ξ. A nonlinear first order equation, to be sure. Similarly for c' to vanish,

$$a\left(\eta_x\right)^2 + 2b\left(\eta_x\right)\left(\eta_y\right) + c\left(\eta_y\right)^2 = 0$$

Note that ξ and η must satisfy the exact same equation, but they must be different solutions. Otherwise they are not valid independent coordinates.

To solve the equation for ξ (η goes the same way), divide by $\left(\xi_y\right)^2$:

$$a\left(-\frac{\xi_x}{\xi_y}\right)^2 - 2b\left(-\frac{\xi_x}{\xi_y}\right) + c = 0$$

and note that, from your calculus or thermo,

$$-\frac{\xi_x}{\xi_y} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{\xi \text{ is constant}}$$

So the lines of constant ξ should satisfy the ordinary differential equation

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a}}$$

We can achieve this by taking ξ to be the integration constant in the solution of this ordinary differential equation! Integration constants are, like the word says, constant for solutions.

By taking the other sign for the square root, you can get a second independent coordinate η.

Bottom line, to get characteristic coordinates, solve the plus and minus sign ordinary differential equations above, and equate the integration constants to ξ and η.

A couple of notes:

1. Since integration constants are not unique, the characteristic coordinates are not. But the lines of constant ξ and η are unique, and are called *characteristic lines* or *characteristics*.

2. Elliptic equations do not have characteristics, because the square root in the ordinary differential equation would be imaginary. The coordinates ξ and η must be real; you do not want to deal with partial differential equations in complex coordinates.

3. Parabolic equations have only one family of characteristic lines. That is because the square root is zero, so taking the other root does not make a difference.

Example

Question: Use characteristic coordinates to reduce the wave equation in multi-dimensional canonical form

$$u_{tt} - a^2 u_{xx} = 0$$

to its equivalent two-dimensional canonical form. Then solve it.

Solution:

First find the characteristics by solving the ordinary differential equation given above:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \pm a$$

Note that the final a is the wave propagation speed, not the coefficient a in the generic second order equation.

The solution is simple:

$$x = at + \xi \qquad x = -at + \eta$$

where ξ and η are the integration constants (as well as the characteristic coordinates). So the lines $x - at =$ constant are one set of characteristic lines, and the lines $x + at =$ constant are the other set.

Now find the coefficient d'. The coefficient d was zero, and the second order derivatives of ξ and η in the formula for d' are also zero, so d' is zero too. So the wave equation in characteristic coordinates is

$$\boxed{u_{\xi\eta} = 0} \tag{18.24}$$

Note that b' could be divided out, so there is no need to figure out what it is.

The wave equation can now easily be solved. Integration with respect to η gives

$$u_\xi = f(\xi)$$

where the integration constant f can be any arbitrary function of ξ. Integrating with respect to ξ gives the final solution:

$$u = f_1(\xi) + f_2(\eta)$$

Here f_1 is an antiderivative of f, so it is arbitrary just like f. The additional integration constant f_2 is an arbitrary function of η.

However, you would surely want the solution in terms of the physical coordinates x and t, rather than the mathematical characteristic coordinates. So substitute for ξ and η using the obtained equations for the characteristics. That gives the final solution:

$$\boxed{u = f_1(x - at) + f_2(x + at)} \tag{18.25}$$

That is the general solution of the wave equation. In order to solve a particular problem, you will still need to figure out what f_1 and f_2 are using whatever the initial and boundary conditions are. One special case, in which the x-range is doubly infinite, will be solved in detail later.

Example

Question: Find and sketch the characteristics of the equation

$$u_{xx} + yu_{yy} = 0$$

Solution:

Figure out the coefficients in the the characteristic equation by looking at the partial differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \pm\sqrt{-y}$$

Note that there are only characteristics for negative y. For positive y the equation is elliptic. And for zero y there will only be one direction for the characteristics, horizontal.

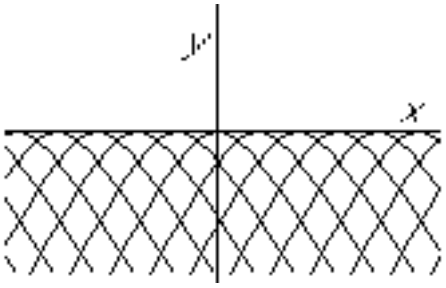
Use separation of variables to solve. In other words, take the y factors to one side and the x-factors to the other side:

$$\frac{\mathrm{d} - y}{\sqrt{-y}} = \pm\mathrm{d}x \qquad \Rightarrow \qquad 2\sqrt{-y} = \pm(x - C)$$

Squaring both sides to get rid of the square root gives

$$y = -\tfrac{1}{4}(x - C)^2$$

These are parabolae.



Example

Question: Reduce the equation

$$e^y u_{xx} + 2e^x u_{xy} - e^{2x-y} u_{yy} = 0$$

to two-dimensional canonical form.

Solution:

Two-dimensional canonical form means characteristic form. Find the ordinary differential equation for the characteristics:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a} = (1 \pm \sqrt{2}) e^{x-y}$$

Solve it using separation of variables:

$$e^y \mathrm{d}y = (1 \pm \sqrt{2}) e^x \mathrm{d}x \qquad \Rightarrow \qquad e^y = (1 \pm \sqrt{2}) e^x + C$$

The integration constants are the new coordinates:

$$\xi = (1 + \sqrt{2}) e^x - e^y \qquad \eta = (1 - \sqrt{2}) e^x - e^y$$

Work out the partial differential equation in these coordinates using the formulae given at the start of this section:

$$b' = a\left(\xi_x\right)\left(\eta_x\right) + b\left(\xi_x\right)\left(\eta_y\right) + b\left(\xi_y\right)\left(\eta_x\right) + c\left(\xi_y\right)\left(\eta_y\right) = -4e^{2x+y}$$

$$d' = d - \left(a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy}\right) u_\xi - \left(a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy}\right) u_\eta$$

so

$$d' = -\left[(1 + \sqrt{2}) e^{x+y} + e^{2x}\right] u_\xi - \left[(1 - \sqrt{2}) e^{x+y} + e^{2x}\right] u_\eta$$

The partial differential equation becomes

$$8e^{x+y} u_{\xi\eta} = \left[(1 + \sqrt{2}) e^y + e^x\right] u_\xi \left[(1 - \sqrt{2}) e^y + e^x\right] u_\eta$$

Get rid of x and y completely using the equations for the characteristics:

$$e^x = \frac{1}{2\sqrt{2}}(\xi - \eta) \quad e^y = -\frac{1 - \sqrt{2}}{2\sqrt{2}}\xi - \frac{1 + \sqrt{2}}{2\sqrt{2}}\eta$$

The resulting partial differential equation is

$$(\xi - \eta)[(1 - \sqrt{2})\xi + (1 + \sqrt{2})\eta] u_{\xi\eta} = (1 + \sqrt{2})\eta u_\xi + (1 - \sqrt{2})\xi u_\eta$$

It does not look easily solvable.

Example

Question: Find the characteristic coordinates of the equation

$$\sin^2(x) u_{xx} + 2\cos(x) u_{xy} - u_{yy} = 0$$

Solution:

Find the ordinary differential equation for the characteristics:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \frac{\cos(x) \pm 1}{\sin^2(x)}$$

Solve it:

$$y = -\frac{1}{\sin(x)} \pm \operatorname{cotg}(x) + C$$

The characteristic coordinates are the integration constants:

$$\xi = y + \frac{1}{\sin(x)} + \operatorname{cotg}(x) \quad \eta = y + \frac{1}{\sin(x)} - \operatorname{cotg}(x)$$

It does not look like the partial differential is going to be very simple.

18.8.2 Parabolic equations in two dimensions

In the parabolic case there is only one equation for the characteristics because the discriminant $b^2 - ac$ is zero:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a}$$

So you can only find one characteristic coordinates, call it η.

You will need to take the other coordinate something else, say $\xi = x$. You want to take something simple, but it should be independent of the other coordinate.

The partial differential equations then simplifies to the two-dimensional canonical form

$$\boxed{a' u_{\xi\xi} = d'} \tag{18.26}$$

You may be surprised by that. In choosing η, all we did was make the coefficient c' zero. We did not explicitly make b' zero. But b' is zero automatically. The reason is that the physical properties of partial differential equations do not change just because you use different coordinates. A parabolic equation should stay parabolic; there are fundamental differences between the physical behaviors of parabolic, elliptic, and hyperbolic equations. And the equation above would not be parabolic if b' was nonzero.

Example

Question: Reduce the equation

$$x u_{xx} + 2\sqrt{xy} u_{xy} + y u_{yy} - u_y = 0$$

to two-dimensional canonical form.

Solution:

Write the equation for the characteristics

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \sqrt{\frac{y}{x}}$$

The square root is zero, so the equation is parabolic.

Solve the equation and call the integration constant η:

$$\frac{\mathrm{d}y}{\sqrt{y}} = \frac{\mathrm{d}x}{\sqrt{x}} \qquad \Rightarrow \qquad \sqrt{y} = \sqrt{x} + \eta$$

So take the new coordinates as

$$\xi = x \qquad \eta = \sqrt{y} - \sqrt{x}$$

The final partial differential equation then becomes

$$-4\xi u_{\xi\xi} = \left[\frac{1}{\sqrt{\xi}} + \frac{3}{\eta + \sqrt{\xi}}\right] u_\eta$$

18.8.3 Elliptic equations in two dimensions

Characteristic lines are solutions to the ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a}$$

Elliptic equations have no real characteristics, because the square root is imaginary. However, elliptic equations can still be simplified, assuming that the above ordinary differential equation can be solved analytically.

Take either sign of the square root. Solve the equation and call the integration constant, say, $\tilde{\xi}$. Then write this integration constant in the form

$$\boxed{\tilde{\xi} = \xi + \mathrm{i}\eta} \tag{18.27}$$

where ξ and η are real and $\mathrm{i} = \sqrt{-1}$. In other words, take $\xi = \Re(\tilde{\xi})$ and $\eta = \Im(\tilde{\xi})$.

Using ξ and η as the new coordinates, it turns out that the partial differential equation takes the two-dimensional canonical form:

$$\boxed{a' u_{\xi\xi} + a' u_{\eta\eta} = d'} \tag{18.28}$$

You may note that this is quite similar to what you can get from rotating the coordinate system, as in the previous section. However, the above procedure works even if the coefficients a, b, and c of the original partial differential equation are not constants.

There are significant limitations on this procedure, however, {D.6}

Example

Question: Reduce the equation

$$u_{xx} + (1+y)^2 u_{yy} = 0$$

to two-dimensional canonical form.

Solution:

Write the equation for the characteristics

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \pm \mathrm{i}(1+y)$$

This is complex, so the equation is elliptic.

Solve using separation of variables

$$\frac{\mathrm{d}y}{1+y} = \mathrm{i}\,\mathrm{d}x \qquad \Rightarrow \qquad \ln|1+y| - \mathrm{i}x = \tilde{\xi}$$

The new coordinates can therefore be chosen as

$$\xi = \ln|1+y| \qquad \eta = -x$$

In terms of these coordinates, the equation becomes

$$a' u_{\xi\xi} + a' u_{\eta\eta} = d'$$

The new partial differential equation becomes

$$u_{\xi\xi} + u_{\eta\eta} = -u_\xi$$

18.8.3 Review Questions

1. Convert the equation

$$10u_{xx} + 6u_{xy} + 2u_{yy} = u_x + x + 1$$

to two-dimensional canonical form.

Using rotation and stretching of the coordinates you would get

$$u_{\xi\xi} + u_{\eta\eta} = \frac{3}{\sqrt{110}} u_\xi - \frac{1}{\sqrt{10}} u_\eta + \frac{3\sqrt{11}}{\sqrt{10}} \xi - \frac{1}{\sqrt{10}} \eta + 1$$

Do you get the same equation? Should you? Comment.

Solution 2dcanel-a

Chapter 19

Green's Functions

The purpose of this chapter is to find solutions of linear partial differential equations in integral form. That is done using so-called Green's functions. Green's functions are solutions to the partial differential equations under forcing by spikes. By integrating such solutions together, arbitrary forcing can be handled.

19.1 Introduction

The purpose of this section is to introduce the Green's function ideas.

19.1.1 The one-dimensional Poisson equation

This subsection will consider a very simple problem, the Poisson equation in one-dimensional infinite space. The solution will be obtained using a Green's function approach.

In general, the Poisson equation reads

$$\nabla^2 u = f$$

where f is a given function and u the unknown to be found. In one-dimensional infinite space that becomes

$$u_{xx} = f(x) \qquad -\infty < x \infty$$

Of course, this equation is trivial to solve. That makes it such a good example to understand the Green's function approach.

It may be noted that the solution is not quite unique; adding $A + Bx$ to any solution, with A and B constants, produces another solution. Therefore, solving the problem will simply be taken to be finding a solution, whichever one.

Figure 19.1 shows a sketch of an arbitrary given function $f(x)$.

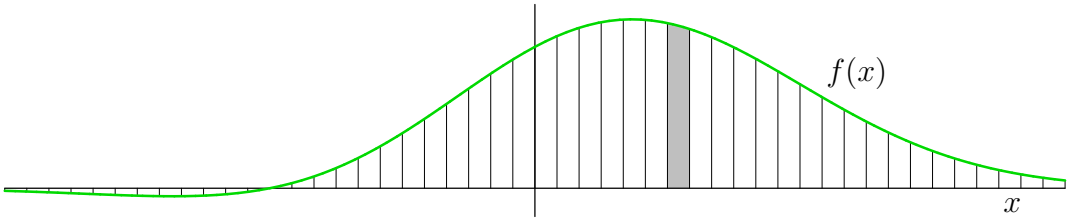


Figure 19.1: Chopping a one-dimensional function up into spikes.

The basic idea of a Green's function approach is to chop the function into narrow spikes and solve for each spike separately.

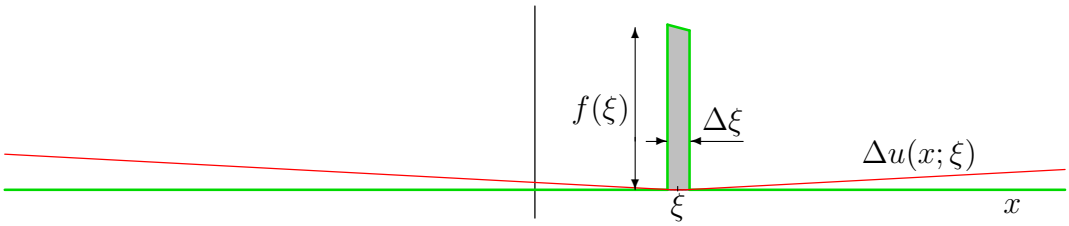


Figure 19.2: Contribution of one spike to the solution.

Consider an arbitrary example spike, shown in grey in figure 19.1. Figure 19.2 shows this one spike separately. The solution due to this one spike, call it Δu, is shown in red. The total solution can be obtained by summing the solutions for all the spikes together:

$$u = \sum_{\text{all spikes}} \Delta u$$

To be sure, solving the problems for the spikes *exactly* is just as difficult as solving the original problem. But if the spikes are narrow, approximations can be made. Before doing so however, consider the exact solution Δu in figure 19.2 more closely. Note that the solution is linear everywhere except in the narrow region of the spike. That is because it satisfies the Poisson equation

$$\Delta u_{xx} = \Delta f$$

Here Δf is the single-spike function, shown in green in figure 19.2. It is zero everywhere outside the single spike, because the other spikes are emitted. So outside the single spike, the second derivative of Δu is zero. And if its second derivative is zero, then Δu is linear.

The slope does change from one side of the spike region to the other. In fact, integration of the equation above produces

$$\Delta u_{x,\text{after}} - \Delta u_{x,\text{before}} = \int_{\text{spike}} f(\xi')\,\mathrm{d}\xi'$$

Note that the integral is the spike area. So the slope changes by an amount equal to the spike area. Since the spike area is small for small $\Delta\xi$, so is change in slope. Figure 19.2 shows a sketch how the solution Δu looks. It took the slopes equal and opposite at both sides. That keeps the maximum slope as small as possible.

If we approximate the spike area as $f(\xi)\,\Delta\xi$, where ξ is the center point of the spike, we get

$$\Delta u_{x,\text{after}} - \Delta u_{x,\text{before}} \approx f(\xi)\,\Delta\xi$$

Now first consider an *idealized* spike problem. In this idealized spike problem, the spike is given a unit area. Then the limit is taken that the width of the spike becomes zero. (In that limit, the height of the spike must go to infinity to keep the area constant.) The limiting infinitely narrow, infinitely high, spike is called the "Dirac delta function" $\delta(x-\xi)$.

The corresponding solution Δu is called the "Green's function" $G(x;\xi)$. It is equal to

$$\boxed{G(x;\xi) = \tfrac{1}{2}|x-\xi|} \tag{19.1}$$

That is easily checked. Indeed, it is linear at both sides of point ξ. And for $x > \xi$, the absolute signs do nothing, so the slope is $\frac{1}{2}$. For $x < \xi$, the absolute signs produce a minus sign so the slope is $-\frac{1}{2}$. That makes the total change in slope 1, the area of the delta function, as it should.

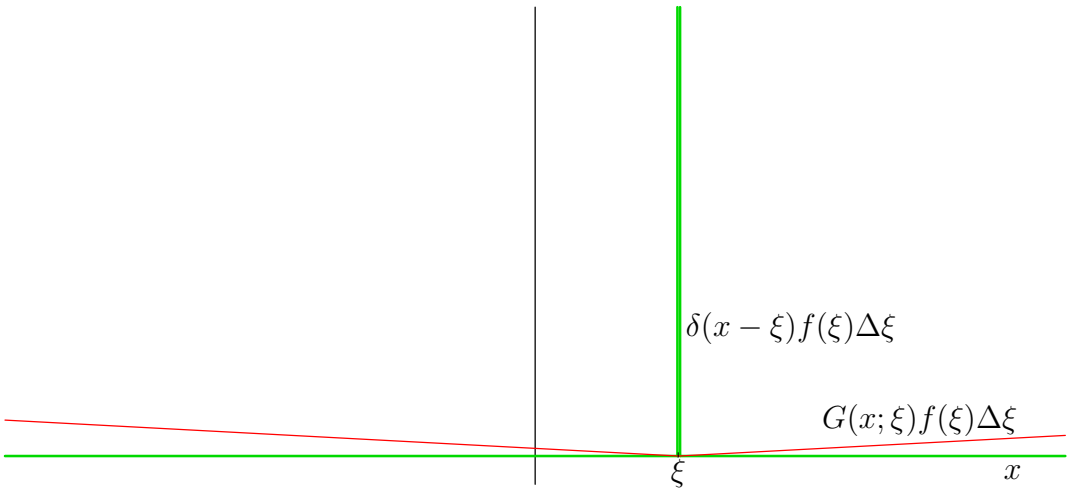


Figure 19.3: Approximation of the spike by an infinitely narrow one.

Back to the original spike of small area $f(\xi)\,\Delta\xi$. We can approximate its solution in terms of the Green's function above as

$$\Delta u \approx G(x;\xi) f(\xi)\,\Delta\xi$$

Since the Green's function has a unit change in slope, multiplying by the spike area gives the correct change in slope. The idea is shown in figure 19.3. We replace the narrow spike by an infinitely narrow one, but still with the same area. That gives essentially the same solution as in figure 19.2. (There will be small deviations. In particular, the solution for the infinite thin spike will be linear right up to the point ξ while the original was somewhat rounded. But these differences can be neglected.)

The total solution u to the original problem is obtained by summing the contribution of all the spikes:

$$u = \sum_{\text{all spikes}} \Delta u \approx \sum_{\text{all spikes}} G(x,\xi) f(\xi)\,\Delta\xi$$

To make this exact, we take the limit that the width $\Delta\xi$ of the spikes becomes zero. In that limit, the summation becomes integration. The exact solution of the Poisson equation is therefore:

$$\boxed{u_{xx} = f(x) \quad -\infty < x < \infty \quad \Longrightarrow \quad u(x) = \int_{\xi=-\infty}^{\infty} G(x;\xi) f(\xi)\,\mathrm{d}\xi} \tag{19.2}$$

where $G(x;\xi) = \frac{1}{2}|x-\xi|$. You can verify this solution by splitting the integral into two and integrating by parts. It is somewhat messy to do so, however.

Example

Question: Find Green's function approximations to the solution u of the Poisson problem

$$u_{xx} = -2xe^{-x^2} \qquad -\infty < x < \infty$$

Use various spike widths $\Delta\xi$. Verify that you do seem to get the exact solution when $\Delta\xi \to 0$ as claimed above.

Solution:

Figure 19.4 shows some results obtained using matlab. First of all, this problem has an exact solution

$$\frac{\sqrt{\pi}}{2}\mathrm{erf}(x)$$

where erf is the so-called error function. This exact solution is indicated by the blue dots in figure 19.4.

The question is now, how good is a Green's function approximation for this problem?

The right hand side in the Poisson problem is negligibly small outside the range $-3 < x < 3$, so no spikes are needed outside that range. In the top left graph,

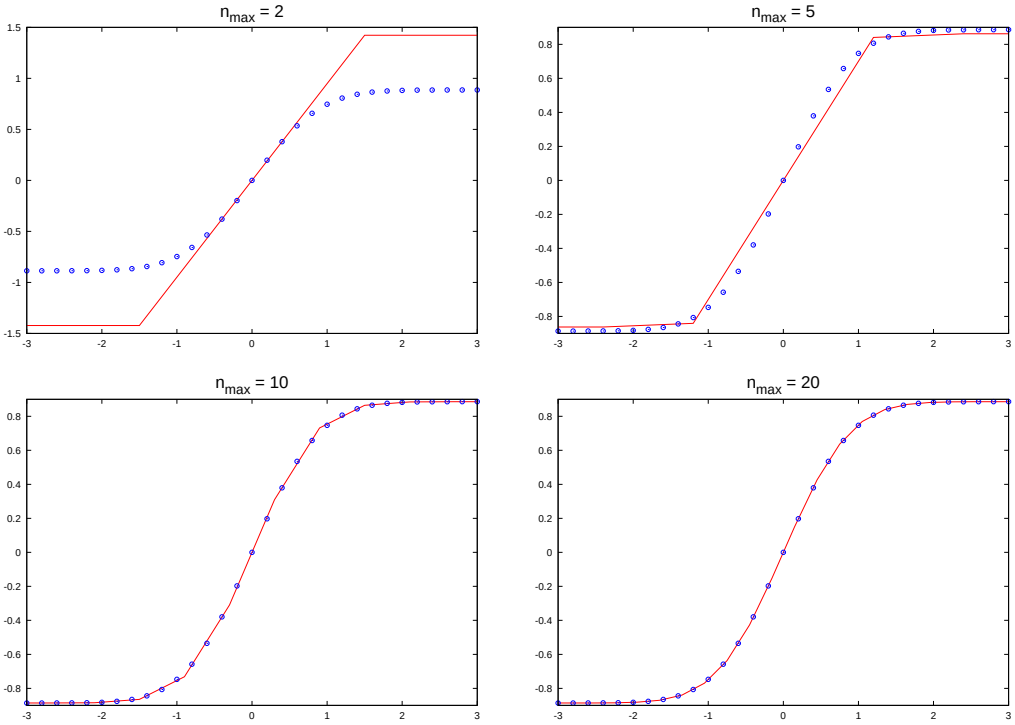


Figure 19.4: Green's function solution of an example one-dimensional Poisson equation.

the interval $-3 < x < 3$ was chopped up into two "spikes." Each spike was approximated by a delta function spike at its center as described above. Then the two Green's function solutions of these spikes were added to give the red solid line. You can see the locations of the delta functions from the kinks in this solution. Obviously, this Green's function solution is not accurate.

It gets better if the interval $-3 < x < 3$ is divided up into 5 narrower spikes, and each one is approximated by a delta function spike. The solution for that is shown in the top right graph of figure 19.4. The next graph shows that for 10 spikes, the Green's function solution is quite close to the exact solution. However, there are still visible kinks at the locations of the delta function spikes. At 20 spikes, the kinks are virtually invisible.

19.1.1 Review Questions

1. Solve the Poisson equation

$$u_{xx} = -2\frac{\sinh x}{\cosh^3 x}$$

numerically using Green's functions. Experiment with numerical parameters and show convergence.

Include your code.

Solution gf1d-a

2. Show that

$$\tilde{u}(x) = \int_{\xi=-\infty}^{\infty} \tfrac{1}{2}|x-\xi| f(\xi)\,\mathrm{d}\xi$$

is a solution to

$$u_{xx} = f(x) \quad -\infty < x < \infty$$

You can assume that function $f(\xi)$ becomes zero rapidly at large ξ. (If you want, you can assume it is zero beyond some value $\xi_{\max}$ of $|\xi|$.) Find out what function $\tilde{u}$ is relative to some given second anti-derivative u_0 of f.

Solution gf1d-b

19.1.2 More on delta and Green's functions

Figure 19.5 shows the definition of the one-dimensional delta function. Note that the function value of $\delta(x-\xi)$ is zero at all points except at the single point ξ. At that single point however, the function value is infinite.

Of course, infinite function values are invalid mathematics. The delta function is not a properly defined function. The best way to deal with that as an engineer is to mentally not make the delta function infinitely narrow. Instead think of a delta function as an extremely narrow, extremely high spike that integrates to 1. Mathematicians have better but more complicated ways of dealing with the problem, {A.1}.

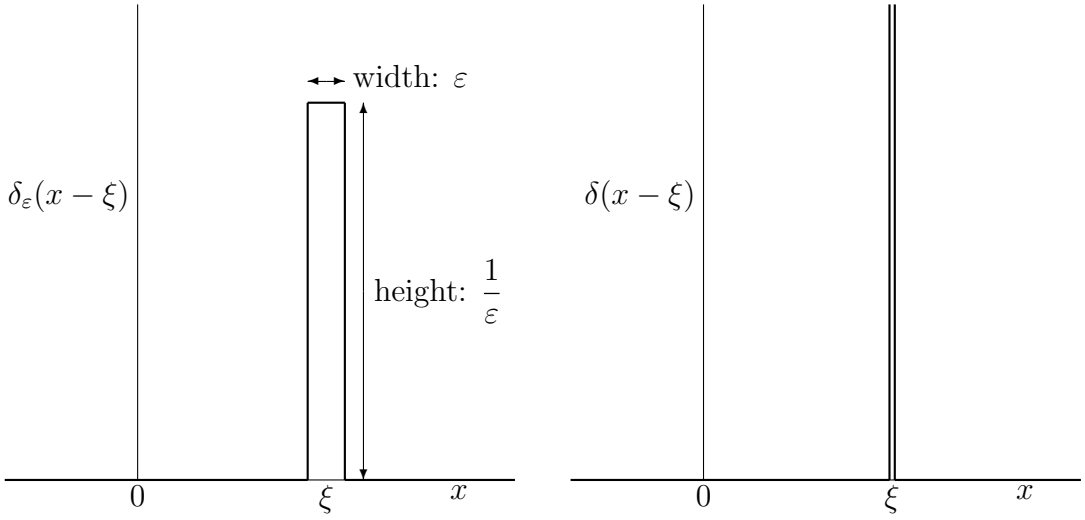


Figure 19.5: Approximate Dirac delta function $\delta_\varepsilon(x-\xi)$ is shown left. The true delta function $\delta(x-\xi)$ is the limit when ε becomes zero, and is an infinitely high, infinitely thin spike, whose bottom is shown right.

Usually delta functions are used as inhomogeneous terms in differential equation problems. The solutions to these problems are called Green's functions. Fortunately, it turns out that while the delta function is not well defined, the Green's function typically is. In the limit that the width of the delta function becomes zero, the Green's function stays a perfectly good function.

For example, the Green's function of the Poisson equation in one dimension formally satisfies

$$\frac{\partial^2 G(x;\xi)}{\partial x^2} = \delta(x-\xi) \qquad -\infty < x < \infty$$

You can make this equation meaningful by replacing the delta function by the approximate delta function of figure 19.5 and then taking the limit that the width ε becomes zero. In that limit, the right hand side becomes the poorly defined delta function. However, as discussed in the previous section, $G(x;\xi)$ becomes $\frac{1}{2}|x-\xi|$. That is a mathematically completely legal function.

Another thing to note is that Green's functions in infinite domains are usually not unique. The most general Green's function for the Poisson equation in one dimension is

$$G(x;\xi) = \tfrac{1}{2}|x-\xi| + A + Bx$$

where A and B are arbitrary constants. The final two terms are a solution of the homogeneous equation.

You would typically like the Green's function to be zero at large distances. But for the one dimensional Green's function above, (as well as for the two-dimensional equivalent, for that matter), there is no way to do it. There is no

way to choose A and B so that G is zero at both $\pm\infty$. The best you can do is make the derivatives at $\pm\infty$ as small as possible. If B is nonzero, the derivative at either $-\infty$ or ∞ is greater than $\frac{1}{2}$ in magnitude. So you take B zero so that neither derivative exceeds $\frac{1}{2}$ in magnitude. There is nothing defensible that you can take for the constant A, so you take it also zero.

It may be noted that in wave propagation problems, trying to make the wave function as small as possible typically does not work. Instead you take the Green's function so that at large distances it describes waves that move away to infinity. Green's functions that describe waves that come in from infinity are physically undesirable.

19.2 The Poisson equation in infinite space

(Book, section 8.2)

19.2.1 Overview

This section works out the Green's function idea for the Poisson equation

$$\nabla^2 u = f$$

in infinite space. In two dimensions, this can be understood to be heat conduction in an infinite plate, with u the temperature and $-f$ the heat added to the plate per unit surface area. In three dimensions, the heat could be generated internally, due to chemical or nuclear reactions, say, or be due to absorbed radiation passing through the body. Or f can be understood to be charge density in electro-statics, with u being the potential. In the fluid dynamics of viscous unidirectional flows, f would be the pressure gradient. In various numerical schemes for viscous incompressible flows, u could be pressure and f velocity terms, or in two dimensions u could be streamfunction and f vorticity.

In this section, the Green's function in infinite two-dimensional space will be derived. For discussion purposes, the problem will be assumed to be heat conduction in a plate, but the mathematical solution does not depend on what the physical meaning is. In the homework you will derive the Green's function for the Poisson equation in infinite three-dimensional space; the analysis is similar but the result will be quite different.

First of all, a Green's function G for the above problem is by definition a solution when function f is a delta function. A delta function is an infinitely narrow spike that integrates to one. We will write $\vec{x}$ for the point at which the temperature is desired. Further $\vec{\xi}$ is the position of the delta function.

In two dimensions $\vec{x} = (x, y)$ and $\vec{\xi} = (\xi, \eta)$. Also, the two-dimensional delta function can be written in terms of one-dimensional ones as

$$\delta^2(\vec{x} - \vec{\xi}) = \delta(x - \xi)\delta(y - \eta)$$

In any number of dimensions, the Green's function solution of the Poisson equation with a delta function as right hand side will be written as

$$G(\vec{x};\vec{\xi})$$

(Since it is a constant coefficient problem, the Green's function will simplify to $G(\vec{x}-\vec{\xi})$, and even further to $G(|\vec{x}-\vec{\xi}|)$ because the problem is rotationally symmetric.)

In terms of the Green's function, the solution u to the Poisson equation with an arbitrary right hand side f can be written as

$$u(\vec{x}) = \int_{\text{all } \xi} G(\vec{x};\vec{\xi}) f(\vec{\xi})\, \mathrm{d}V_{\vec{\xi}} \tag{19.3}$$

Here $\mathrm{d}V_{\vec{\xi}}$ stands for a "volume" integral over all components of vector $\vec{\xi}$. In two dimensions that becomes

$$u(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y;\xi,\eta) f(\xi,\eta)\, \mathrm{d}\xi\mathrm{d}\eta \tag{19.4}$$

The "volume" is here really an area.

It will be found that the Green's function for the two-dimensional infinite-domain Poisson problem is:

$$\boxed{G(\vec{x};\vec{\xi}) = \frac{1}{2\pi}\ln d \qquad \text{where} \qquad d = \left|\vec{x}-\vec{\xi}\right| = \sqrt{(x-\xi)^2+(y-\eta)^2}} \tag{19.5}$$

The physical meaning of d is the distance between the point $\vec{\xi} = (\xi,\eta)$ where the heat is added and the point $\vec{x} = (x,y)$ at which the temperature is desired.

Similarly, you will find in the homework that the Green's function for the three-dimensional infinite-domain Poisson problem is:

$$\boxed{G(\vec{x};\vec{\xi}) = \frac{-1}{4\pi d} \qquad \text{where} \qquad d = \left|\vec{x}-\vec{\xi}\right| = \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\theta)^2}} \tag{19.6}$$

Therefore the two-dimensional temperature distribution $u(x,y)$ corresponding to a *general* distribution of added heat $f(\xi,\eta)$ is:

$$\boxed{u(x,y) = \iint \frac{f(\xi,\eta)}{2\pi}\ln\sqrt{(x-\xi)^2+(y-\eta)^2}\,\mathrm{d}\xi\mathrm{d}\eta} \tag{19.7}$$

while the three-dimensional temperature distribution is

$$\boxed{u(x,y,z) = \iiint \frac{-f(\xi,\eta)}{4\pi\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\theta)^2}}\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\theta} \tag{19.8}$$

Given an added-heat distribution f, you can find the temperature distribution u by doing the corresponding integral. Especially if you only want the temperature at a few points, this can be quite effective. (If you want the temperature at essentially all points, a "multigrid" numerical method that directly solves the partial differential equation is far more efficient. However, there are so-called "fast-summation" methods, like the one by Van Dommelen and Ründensteiner, that can do the integrals very fast too, especially if the region of heat addition is limited.)

The physical meaning of the Green's function varies with setting. In heat transfer, it is the solution for a point heat source, in electrostatics a point charge, in gravitation a point mass, in potential flows a point source of fluid, in two-dimensional vortex flows a point vortex, etcetera.

19.2.2 Loose derivation

To verify the two-dimensional Green's function G given in the previous section, the solution to the Poisson equation must be found in which f is a delta function spike at some point $\vec{\xi}$.

However, dealing with infinite functions like delta functions is a very abstract and fishy problem. Therefore an approach like in the first section will be used. It will assumed that we are really trying to solve the Poisson equation for an arbitrary function f. (And so we are, really.) We then mentally cut up this function f into spikes. That idea is sketched in two-dimensions in figure fig:2dspike.

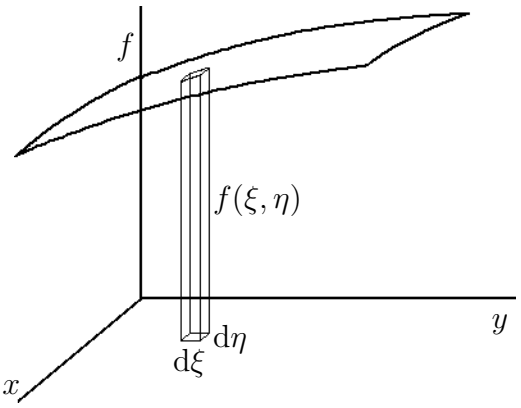


Figure 19.6: One of the spikes of which an arbitrary two-dimensional function f consists is shown in outline.

The problem for such a narrow, but finite spike can be solved with some physical intuition. The solution will again be called Δu. The total solution will then be the sum of the solutions Δu for all the spikes. Of course, each

solution will not just depend on the position (x, y) at which you want to know the temperature. It will also depends on where the spike is, as indicated by its center point (ξ, η).

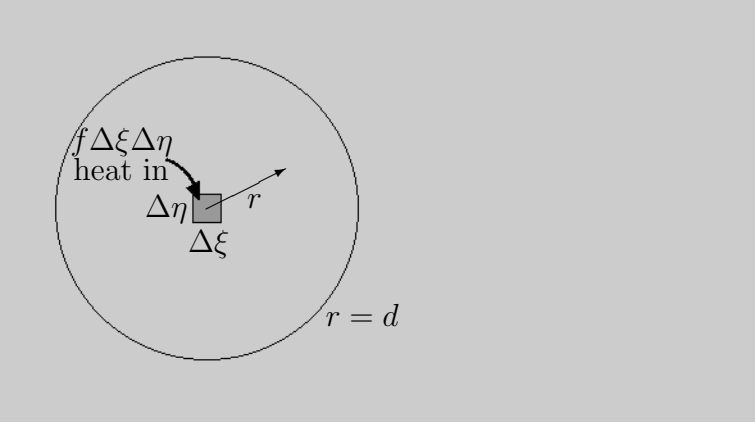


Figure 19.7: Sketch of the problem to be solved: heat is added only to the small dark rectangle around a point (ξ, η).

Figure 19.7 shows a two-dimensional top view equivalent to figure 19.6. In other words, it shows just the plate, not the function f. The dimensions of the little rectangle to which the heat is added by the considered spike will be indicated by $\Delta\xi\Delta\eta$. The amount of heat added is $f(\xi, \eta)\Delta\xi\Delta\eta$, since variations in f over the small rectangle can be ignored. Since the Green's function G is the solution for *unit* added heat flux, the final Green's function will be obtained by dividing the solution Δu by the amount of heat $f(\xi, \eta)\Delta\xi\Delta\eta$. (Formally speaking, you would then still need to take the limit $\Delta\xi, \Delta\eta \to 0, 0$, but that becomes trivial under the approximations to be made.)

The mathematical problem being solved is:

$$\nabla^2 \Delta u = \begin{cases} f(\xi, \eta) \text{ within the vicinity } \Delta\xi\Delta\eta \text{ of point } (\xi, \eta) \\ 0 \text{ everywhere else} \end{cases}$$

For convenience, for now use a polar coordinate system r, ϑ centered around the point (ξ, η) of heat addition, as indicated in figure 19.7. Further, since the rectangle $\Delta\xi\Delta\eta$ is assumed to be very small, almost a single point, you can reasonably assume that the temperature distribution Δu depends only on the distance r from the point where the heat is added, not on ϑ.

Under those assumptions, it is easiest to simply integrate the mathematical problem above over the inside of a circle of radius d:

$$\int_{r=0}^{d} \int_{\vartheta=0}^{2\pi} \nabla^2 \Delta u \, r \, \mathrm{d}r\mathrm{d}\vartheta = f(\xi, \eta)\Delta\xi\Delta\eta$$

But $\nabla^2 \Delta u = \operatorname{div}(\operatorname{grad} \Delta u)$, so you can apply the divergence theorem here:

$$\int_S \vec{n} \cdot \nabla \Delta u \, \mathrm{d}S = f(\xi, \eta) \Delta\xi \Delta\eta.$$

In this two-dimensional problem the "surface" S is the perimeter $2\pi d$ of the circle. And $\vec{n}$ is the radial polar unit vector $\hat{\imath}_r$, which makes the total derivative $\vec{n} \cdot \nabla \Delta u$ equal to the derivative with respect to radius, $\mathrm{d}\Delta u/\mathrm{d}d$. So you have:

$$\frac{\mathrm{d}\Delta u}{\mathrm{d}d} 2\pi d = f(\xi, \eta) \Delta\xi \Delta\eta$$

Take the $2\pi d$ to the other side and integrate to get Δu:

$$\Delta u = \frac{f(\xi, \eta) \Delta\xi \Delta\eta}{2\pi} \ln d$$

The Green's function is the solution for unit heat added:

$$G = \frac{1}{2\pi} \ln d$$

This derivation used a polar coordinate system centered around the heat addition point (ξ, η). To get the expression for whatever the origin of the coordinate system is, substitute

$$d = |\vec{x} - \vec{\xi}| = \sqrt{(x - \xi)^2 + (y - \eta)^2}.$$

That gives the Green's function as stated in the overview.

19.2.2 Review Questions

1. Do an analysis similar to either this subsection, or the next one, to derive the Green's function of the Poisson equation in three dimensional infinite space.

   Solution pninfl-a

19.2.3 Rigorous derivation

For students who do not like the above derivation with infinitesimal regions, and the assumption that their temperature distribution only depends on distance, here is a mathematically solid derivation.

It will be assumed that a suitable heat distribution $f(x, y)$ is given with at least continuous low-order derivatives. Also that it disappears sufficiently quickly at large distances that you do not have to worry about that region. Then it is to be shown that if you do the Green's function integration

$$u(x, y) = \iint G(x, y; \xi, \eta) f(\xi, \eta) \, \mathrm{d}\xi \mathrm{d}\eta \qquad G = \frac{1}{2\pi} \ln |\vec{x} - \vec{\xi}|$$

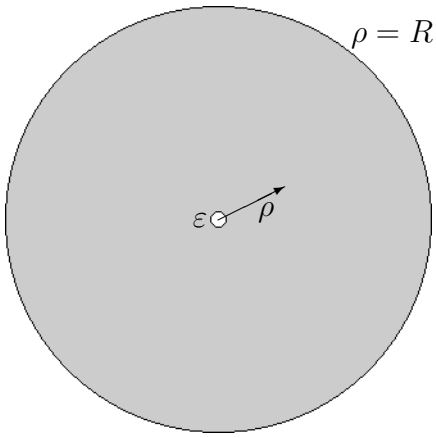


Figure 19.8: Region of integration for the Green's function integral. Excluded regions are left blank. The point (x, y) at which the temperature u is to be found is in the center of the excluded small circle.

you get a function $u(x, y)$ satisfying the Poisson equation:

$$\nabla^2 u(x, y) = f(x, y)$$

The first thing to check is that you do at least get *some* function $u(x, y)$ by doing the integration. That is not automatic, since the integrand is infinite when $(\xi, \eta) = (x, y)$. And integration over an infinite region is not proper either. What you must do is exclude the inside of a very small circle around (x, y) and the outside of a very large circle from the integration. Then you define $u(x, y)$ to be the limit of the integral when the radius of the small circle ε becomes zero, and the radius R of the the large circle becomes infinite. (Assuming that those limits exist.) See figure 19.8.

A local polar coordinate system ρ, φ will be used centered at the point (x, y) at which the temperature is desired. Then ρ is the distance that the heat addition point (ξ, η) is away from the point (x, y) at which the temperature is desired. Note that the variables in the integration are ξ and η; (x, y) is just a fixed point in this entire story. The two-dimensional Green's function is

$$G = \frac{1}{2\pi} \ln(\rho)$$

in these terms.

The effect of the excluded area outside the large circle may be taken to be vanishingly small if the radius of the large circle, call it R, is large, since it was assumed that function f vanishes sufficiently quickly at large distances. The effect of the excluded area inside the small circle is can be estimated as

$$\left| \iint \frac{1}{2\pi} \ln(\rho) f(\xi, \eta) \, \mathrm{d}\xi \mathrm{d}\eta \right| \leq |f_{\max}| \int_{\rho=0}^{\varepsilon} \frac{1}{2\pi} \ln(\rho) \, 2\pi\rho \, \mathrm{d}\rho$$

where $|f_{\max}|$ is the maximum value of f within the circle and ε is the radius of the small circle. Now $|f_{\max}|$ is finite since f is continuous, and the integral becomes vanishingly small when $\epsilon \to 0$. So the effect of the small circle too can be ignored if it is small enough. In particular, the limit when the radius ϵ of the small circle tends to zero and the radius of the large circle R tends to infinity exists. So the temperature $u(x, y)$ exists.

But does it satisfy the Poisson equation $\nabla^2 u = f$? Now what you *cannot* do here is simply differentiate the Green's function (the logarithm) within the integral

$$\begin{aligned} u(x,y) &= \iint G(x,y;\xi,\eta) f(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta \\ &= \iint \frac{f(\xi,\eta)}{2\pi} \ln\sqrt{(x-\xi)^2+(y-\eta)^2}\,\mathrm{d}\xi\mathrm{d}\eta \end{aligned}$$

a couple of times with respect to x and y. If you differentiate the logarithm, it becomes a more singular function, and you get into trouble.

Instead you need to go back to the basic definition of the partial derivatives. As an example, take

$$\frac{\partial u}{\partial x} \equiv \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}.$$

The integral for $u(x + \Delta x, y)$;

$$u(x+\Delta x, y) = \iint \frac{f(\xi,\eta)}{2\pi} \ln\sqrt{(x+\Delta x-\xi)^2+(y-\eta)^2}\,\mathrm{d}\xi\mathrm{d}\eta,$$

can be manipulated by defining $\bar{\xi} = \xi - \Delta x$ to become

$$u(x+\Delta x, y) = \iint \frac{f(\bar{\xi}+\Delta x,\eta)}{2\pi} \ln\sqrt{(x-\bar{\xi})^2+(y-\eta)^2}\,\mathrm{d}\bar{\xi}\mathrm{d}\eta,$$

You can now again drop the bar on $\bar{\xi}$ since it is just an integration variable whose name makes no difference.

Plug this, and the expression for $u(x, y, z)$ itself, into the limit above to get:

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \iint \frac{[f(\xi+\Delta x,\eta) - f(\xi,\eta)]/\Delta x}{2\pi} \ln\sqrt{(x-\xi)^2+(y-\eta)^2}\,\mathrm{d}\xi\mathrm{d}\eta,$$

In the limit $\Delta x \to 0$, the term in the numerator becomes the partial derivative $\partial f/\partial \xi$! So x-derivatives transform to ξ-derivatives on f inside the integral. And the equivalent thing happens to y-derivatives. Since the derivatives of f are assumed to be continuous just like f, there are no problems with these integrals.

The net result is that

$$\nabla^2 u = \iint \frac{\nabla^2_{\vec{\xi}} f}{2\pi} \ln \sqrt{(x-\xi)^2 + (y-\eta)^2}\, \mathrm{d}\xi \mathrm{d}\eta,$$

using the notations

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \text{and} \qquad \nabla^2_{\vec{\xi}} = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}.$$

Now you need to show that the integral in the right hand side is equal to $f(x, y)$ to finish the proof that $\nabla^2 u = f$. To shorten the notations, the Green's function will again be written as G, and the question is whether

$$\iint G \nabla^2_{\vec{\xi}} f \, \mathrm{d}\xi \mathrm{d}\eta \text{ equals } f?$$

To show this, exclude again the inside of a small circle of radius ε and the outside of a large circle of radius R around (x, y) from the integration, as in figure 19.8. In that case, you do not have to worry about infinite quantities, and you can again take the limit $\varepsilon \to 0$ and $R \to \infty$ later to get the final result.

If you add a second term,

$$\iint G \nabla^2_{\vec{\xi}} f \, \mathrm{d}\xi \mathrm{d}\eta - \iint f \nabla^2_{\vec{\xi}} G \, \mathrm{d}\xi \mathrm{d}\eta$$

you can use Green's second integral identity from section 1.2 in the book. The added term is zero, since $\nabla^2_{\vec{\xi}} G = 0$ away from the singular point $\vec{x} = \vec{\xi}$. (Remember, G is the solution of the Poisson problem where the forcing is a delta function, zero everywhere except at the singular point. So $\nabla^2 G = 0$ away from the point, and since the components of $\vec{\xi}$ appear in G in exactly the same way as those of $\vec{x}$, if $\nabla^2 G = 0$, then so is $\nabla^2_{\vec{\xi}} G$.)

Green's second identity now says that the expression that should equal f is the "surface" integral

$$\int G \frac{\partial f}{\partial n_{\vec{\xi}}} \, \mathrm{d}S_{\vec{\xi}} - \int f \frac{\partial G}{\partial n_{\vec{\xi}}} \, \mathrm{d}S_{\vec{\xi}}$$

where $\partial / \partial n_{\vec{\xi}} = \vec{n} \cdot \nabla_{\vec{\xi}}$ is the derivative normal to the surface with respect to the components of $\vec{\xi}$. The "surface" $S_{\vec{\xi}}$ is in this two-dimensional case the perimeter of the region of integration. Part of it is the large circle $\rho = R$, but you can ignore that since it is assumed that f vanishes sufficiently rapidly at large distances. The other part is the perimeter of the little circle $\rho = \varepsilon$ that was excluded from the integration. The integral over this little circle cannot be ignored. On it, the normal derivative $\partial / \partial n_{\vec{\xi}}$ is minus the radial derivative

$\partial/\partial\rho$, (minus since the normal vector must stick out of the region of integration, which here is into the excluded little circle.) So you have on the little circle

$$G = \frac{1}{2\pi}\ln(\rho) = \frac{1}{2\pi}\ln(\varepsilon) \qquad \frac{\partial G}{\partial n_{\vec{\xi}}} = -\frac{1}{2\pi\varepsilon}$$

The perimeter $S_{\vec{\xi}}$ of the little circle is $2\pi\varepsilon$. So you can estimate the two integrals above as

$$-\varepsilon\ln(\varepsilon)\oint_{\rho=\varepsilon}\frac{\partial f}{\partial n_{\vec{\xi}}}\frac{\mathrm{d}S_{\vec{\xi}}}{S_{\vec{\xi}}} + \oint_{\rho=\varepsilon} f\frac{\mathrm{d}S_{\vec{\xi}}}{S_{\vec{\xi}}}$$

The first term is certainly no larger than $\varepsilon\ln(\varepsilon)$ times the maximum value of the gradient of f on the spherical surface and disappears when you take the limit $\varepsilon \to 0$. The second term however is the average value of f on the perimeter, and that becomes simply f when the circle becomes so small that the variations in f can be ignored.

So finally, you conclude that $\nabla^2 u$ is indeed f.

19.3 The Poisson or Laplace equation in a finite region

19.3.1 Overview

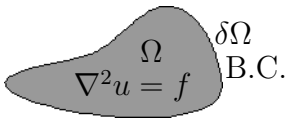


Figure 19.9: Example finite domain Ω in which the Poisson or Laplace equation is to be solved.

This section will derive the solution of the Poisson equation in a finite region as sketched in figure 19.9. The region will be denoted as Ω, and its boundary by $\delta\Omega$. It will again be assumed that the region is two-dimensional, leaving the three-dimensional case to the homework. As shown in figure 19.9, inside the region the Poisson equation applies. In case $f = 0$, this becomes the Laplace equation. On the boundary there is some boundary condition that will for now be left arbitrary.

19.3.2 Intro to the solution procedure

The big idea is to relate the finite domain solution to the infinite domain solution derived earlier.

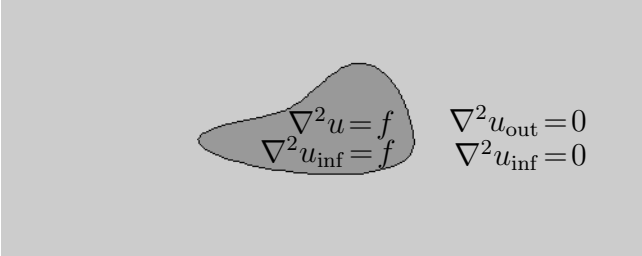


Figure 19.10: Temperature distributions involved in the solution process: (i) u is the desired temperature distribution satisfying the given boundary conditions; (ii) $u_{\rm out}$ is an external temperature distribution whose boundary conditions can be cleverly chosen to achieve various objectives; (iii) $u_{\rm inf}$ is the infinite-domain solution that satisfies no particular boundary conditions on $\delta\Omega$.

For the given heat addition f, you can still do the integral $\int Gf \, \mathrm{d}V_{\vec{\xi}}$ over the domain Ω to get the infinite domain solution. That solution will now be called $u_{\rm inf}$, as indicated in figure 19.10.

The infinite domain solution satisfies the Poisson equation, but it does not satisfy the boundary conditions. It will turn out that some surface integrals must be added to it to get the boundary conditions right.

The precise form of these integrals can vary. The different versions all give the same, correct, solution u inside the domain Ω. However, they give different answers for the continuation of this solution to *outside* the domain Ω. So a meaningful discussion of the various possibilities requires consideration of the solution outside the domain. Even though the solution outside domain Ω is not actually a part of the problem.

The solution outside Ω will be indicated as $u_{\rm out}$, so the picture becomes as shown in figure 19.10. Since you surely do not want to just make up an arbitrary function f outside Ω, it will be assumed that $f = 0$ outside. So $u_{\rm out}$ satisfies the homogeneous Poisson equation, the Laplace equation. And so does the infinite space solution $u_{\rm inf}$ outside Ω, for that matter. Only integrating f over the domain Ω is the same as setting f to zero outside the domain.

19.3.3 Derivation of the integral solution

(book, example 8.2)

The desired integral solution u for the finite-region Poisson solution is a generalization of the infinite domain solution

$$u_{\rm inf}(\vec{x}) = \int G(\vec{x};\vec{\xi}) f(\vec{\xi}) \, \mathrm{d}V_{\vec{\xi}},$$

where G is the infinite domain Green's function derived in the previous section. In the two-dimensional case discussed here:

$$G = \frac{1}{2\pi} \ln \left| \vec{x} - \vec{\xi} \right| .$$

Since $f = \nabla^2 u$, the expression above represents an important relationship between the infinite-domain solution u_{inf} and the actual, finite-domain, solution u:

$$u_{\text{inf}}(\vec{x}) = \int G(\vec{x}; \vec{\xi}) \nabla^2_{\vec{\xi}} u(\vec{\xi}) \, \mathrm{d}V_{\vec{\xi}}$$

where

$$\nabla^2_{\vec{\xi}} = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}.$$

is the Laplacian with respect to ξ and η. George Green discovered that the integral in the right hand side could be simplified into surface integrals using the divergence theorem, and that doing so directly relates u to u_{inf}.

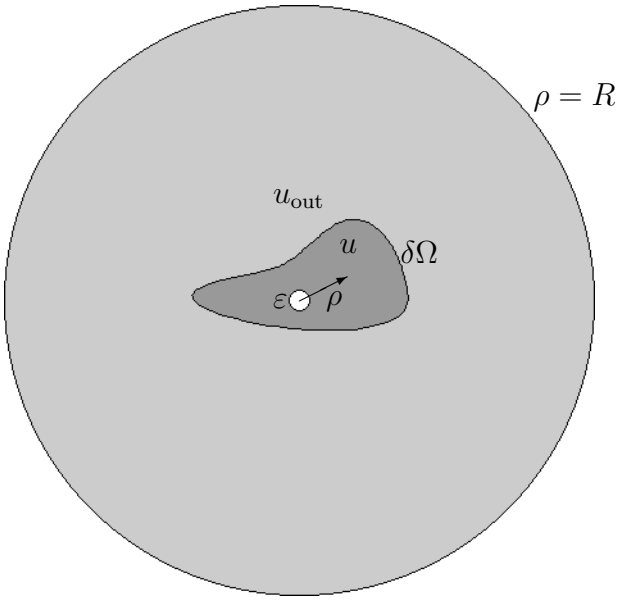


Figure 19.11: Region of integration of the integral for the infinite-space solution. Note that $\delta\Omega$ is a bounding surface of both dark grey domain Ω and of the light grey exterior region.

Some caution is needed, however. The Green's function is infinite when $\vec{\xi} = \vec{x}$, and integrals of infinite functions are not proper. And neither are integrals over infinite regions. You must exclude a very small circle around the point $\vec{x}$ at which u_{inf} is desired from the integration, and also the outside of a very large circle, as indicated in figure 19.11. The correct value for u_{inf} can then be

obtained as the limit in which the radius ε of the small circle becomes zero and the radius R of the large circle becomes infinite. Also, you must think of the integral as really consisting of two separate integrations; one over the dark grey region Ω and one over the light grey exterior of Ω. The reason is that u and its derivatives are not normally continuous on the surface $\delta\Omega$, and the divergence theorem can only be used for fairly smooth functions.

To simplify the remaining discussion, the origin of the coordinate system will be shifted towards the point $\vec{x}$ at which $u_{\rm inf}$ is desired. The integration coordinate $\vec{\xi}$ can then be described by polar coordinates ρ and φ centered around this point. That simplifies the expression to be evaluated to

$$u_{\rm inf} = \int G \nabla^2_{\vec{\xi}} u \, {\rm d}V_{\vec{\xi}} \qquad \text{with} \qquad G = \frac{1}{2\pi} \ln \rho$$

To get a divergence integral, move a $\nabla_{\vec{\xi}}$ out in front, adding a correction term to make up for the error in doing so:

$$u_{\rm inf} = \int \nabla_{\vec{\xi}} \Big(G \nabla_{\vec{\xi}} u \Big) \, {\rm d}V_{\vec{\xi}} - \int \Big(\nabla_{\vec{\xi}} G \Big) \Big(\nabla_{\vec{\xi}} u \Big) \, {\rm d}V_{\vec{\xi}}.$$

Move a $\nabla_{\vec{\xi}}$ out in front in the second integral to create another divergence integral, adding another correction term:

$$u_{\rm inf} = \int \nabla_{\vec{\xi}} \Big(G \nabla_{\vec{\xi}} u \Big) \, {\rm d}V_{\vec{\xi}} - \int \nabla_{\vec{\xi}} \Big(\Big(\nabla_{\vec{\xi}} G \Big) u \Big) \, {\rm d}V_{\vec{\xi}} + \int \Big(\nabla^2_{\vec{\xi}} G \Big) u \, {\rm d}V_{\vec{\xi}}.$$

This final correction term, however, is zero. To see why, remember that the Green's function $G(\vec{x}, \vec{\xi})$ is the temperature distribution due to a spike of heat at point $\vec{\xi}$. So $\nabla^2 G = 0$ everywhere except at the singular point $\vec{x} = \vec{\xi}$. And since $\vec{\xi}$ appears exactly the same way in the Green's function as $\vec{x}$, then so is $\nabla^2_{\vec{\xi}} G$ zero.

The remaining two terms become "surface" (actually, contour in 2D,) integrals using the divergence theorem. In particular:

$$\begin{aligned} u_{\rm inf} \;=\; & - \oint_{\rho=\varepsilon} G \frac{\partial u}{\partial \rho} \, {\rm d}S_{\vec{\xi}} + \oint_{\rho=\varepsilon} u \frac{\partial G}{\partial \rho} \, {\rm d}S_{\vec{\xi}} \\ & + \oint_{\rho=R} G \frac{\partial u_{\rm out}}{\partial \rho} \, {\rm d}S_{\vec{\xi}} - \oint_{\rho=R} u_{\rm out} \frac{\partial G}{\partial \rho} \, {\rm d}S_{\vec{\xi}} \\ & + \oint_{\delta\Omega} G \left(\frac{\partial u}{\partial n_{\vec{\xi}}} - \frac{\partial u_{\rm out}}{\partial n_{\vec{\xi}}} \right) {\rm d}S_{\vec{\xi}} - \oint_{\delta\Omega} (u - u_{\rm out}) \frac{\partial G}{\partial n_{\vec{\xi}}} \, {\rm d}S_{\vec{\xi}} \end{aligned} \tag{19.9}$$

To verify this expression, note that the "surfaces" include the small and big circles, and that $\delta\Omega$ counts as both part of the "surface" of the dark grey region in figure 19.11 as well as part of the "surface" of the light grey region. The

normal vector $\vec{n}$ on $\delta\Omega$ was taken to stick out of region Ω, which accounts for the additional minus sign in the corresponding u_{out} terms. Also $\vec{n} \cdot \nabla_{\vec{\xi}}$ is according to the total differential of calculus the derivative $\partial/\partial n_{\vec{\xi}}$ in the direction normal to the surface. On the big circle, that is the same as $\partial/\partial\rho$, and on the small circle it is $-\partial/\partial\rho$ since there the outward normal points towards the origin.

The second integral over the small circle is particularly interesting: since $G = \ln(\rho)/2\pi$, its ρ derivative is $1/2\pi\rho$, which is the inverse of the "surface" (perimeter) of the circle. So you get

$$\oint_{\rho=\varepsilon} u \frac{\partial G}{\partial \rho} \, \mathrm{d}S_{\vec{\xi}} = \oint_{\rho=\varepsilon} u \frac{\mathrm{d}S_{\vec{\xi}}}{S_{\vec{\xi}}}.$$

That is just the average of u on the small circle, and it becomes u at the point $\vec{x}$, (used here as origin,) in the limit that the radius of the small circle becomes zero. So, since this integral simplifies to u, all the other integrals in equation (19.9) merely describe the difference between the true solution u and the infinite-domain solution u_{inf}.

The first integral over the small circle in (19.9) is vanishingly small and can be ignored. To see why, note that it is no larger than the maximum value of the gradient of u on the small circle times

$$\oint_{\rho=\varepsilon} G \, \mathrm{d}S_{\vec{\xi}} = \oint_{\rho=\varepsilon} \frac{1}{2\pi} \ln(\rho) \, \rho \, \mathrm{d}\varphi = \ln(\varepsilon) \, \varepsilon$$

and that becomes zero in the limit $\varepsilon \to 0$.

There is little that can be done about the integrals over "surface" $\delta\Omega$. However, the integrals over the big circle in (19.9) still must be evaluated. To do so, you must know something about the behavior of the solution u_{out} for large values of ρ. In general, it is described by

$$u_{\text{out}} \sim C_0 \ln \rho + C_1 + \frac{C_2}{\rho} + \ldots$$

In three or more dimensions, the constant C_0 is zero. The two integrals over the big circle become, noting that $\mathrm{d}S_{\vec{\xi}} = \rho \, \mathrm{d}\varphi$,

$$\oint_{\rho=R} \frac{1}{2\pi} \ln(\rho) \left(\frac{C_0}{\rho} - \frac{C_2}{\rho^2} + \ldots \right) \rho \, \mathrm{d}\varphi - \oint_{\rho=R} \left(C_0 \ln(\rho) + C_1 + \frac{C_2}{\rho} \right) \frac{1}{2\pi\rho} \, \rho \, \mathrm{d}\varphi$$

which becomes $-C_1$ in the limit $R \to \infty$.

Collecting the results together, the solution for the temperature u at any point (x, y) is:

$$\boxed{u(\vec{x}) = \int_\Omega G f \, \mathrm{d}V_{\vec{\xi}} + \oint_{\delta\Omega} (u - u_{\text{out}}) \frac{\partial G}{\partial n_{\vec{\xi}}} \, \mathrm{d}S_{\vec{\xi}} - \oint_{\delta\Omega} G \left(\frac{\partial u}{\partial n_{\vec{\xi}}} - \frac{\partial u_{\text{out}}}{\partial n_{\vec{\xi}}} \right) \mathrm{d}S_{\vec{\xi}} + C_1} \tag{19.10}$$

where G is the *infinite domain* Green's function, $(2\pi)^{-1} \ln |\vec{x} - \vec{\xi}|$, with $|\vec{x} - \vec{\xi}|$ the distance between the point of integration $\vec{\xi} = (\xi, \eta)$ and the point $\vec{x} = (x, y)$ at which the temperature u is desired. The first integral is therefor the infinite domain solution u_{inf}, which has the right values for the added heat f, but does not satisfy the correct boundary condition on $\delta\Omega$

19.3.3 Review Questions

1. Perform the equivalent analysis in the three dimensional case.
   Solution pnfd-a

19.3.4 Boundary integral (panel) methods

The previous subsection derived the solution to the Poisson equation in a finite domain. It was given by equation (19.10). This subsection will examine how this solution may be evaluated.

Except for $G(\vec{x}; \vec{\xi})$, all other quantities in the right hand side of equation (19.10) are evaluated at the point of integration $\vec{\xi}$. For example, $\partial u/\partial n_{\vec{\xi}}$ stands for the normal derivative $\partial u/\partial n$ evaluated at the boundary point ξ of integration. That means that if you merely know u and the normal derivative $\partial u/\partial n$ on the boundary, you can find u in the interior by taking u_{out} to be zero and doing the integrals above. Unfortunately, a priori at most only one of u (Dirichlet boundary condition) or $\partial u/\partial n$ (Neumann boundary condition) will be known on the boundary.

Various solutions for this problem are possible. A panel method might decide to compute the particular solution where u_{out} is not zero, but has the same values as u on the boundary. The big advantage is then that the second integral in (19.10) drops out, leaving only the last integral as a problem.

A simple panel method will now discretize the boundary in a large number of densely spaced points, and then put a Green's function at each point. Since each Green's function corresponds to the addition of a spike of heat at that point, this is called a surface "source" distribution. The problem remains that the strengths

$$-\left(\partial u/\partial n_{\vec{\xi}} - \partial u_{\text{out}}/\partial n_{\vec{\xi}}\right) \, \mathrm{d}S_{\vec{\xi}}$$

of these sources are not known, since even if $\partial u/\partial n$ is given on the boundary, $\partial u_{\text{out}}/\partial n$ is not. So the strength of each source is an unknown, and an equally large number of equations is needed. These equations can be found from requiring that at that many points, the error in the boundary condition as computed from (19.10) is zero. Put all these equations on a computer and solve. And with the source strength now known, u can then be evaluated at any arbitrary point.

Alternatively, a panel method might decide to compute the solution for the case that u and u_{out} have the same normal derivatives on the boundary. That

kills off the source integral, leaving the second integral in (19.10). The quantity $\partial G/\partial n_{\vec{\xi}}$ in this integral is called a "dipole." The reason for that name can be understood by writing the definition of the derivative:

$$\frac{\partial G}{\partial n_{\vec{\xi}}} \equiv \lim_{\Delta n \to 0} \left(\frac{1}{\Delta n} G(\vec{x}; \vec{\xi} + \vec{n}\Delta n) - \frac{1}{\Delta n} G(\vec{x}; \vec{\xi}) \right). \tag{19.11}$$

This shows that a dipole corresponds to an infinitely large source of heat and an infinitely large sink of heat infinitely close together.

19.3.5 Poisson's integral formulae

The previous subsection showed that the Poisson equation can be solved by using suitable source and/or dipole distributions on the boundary of the domain. However, the strengths of these distributions are not usually known, since they involve both u and its normal derivative on the boundary, and there is only one boundary condition. And if an exterior solution u_{out} is chosen to eliminate one of them, that has the effect of introducing the unknown values of u_{out} or its derivative into the problem. So at least one distribution strength must be found using brute numerical force. Or by brute analytical force, maybe, if the domain is simple.

There is an exception, however, and it occurs for the Dirichlet problem inside a ball (a circle in two dimensions, a sphere in three-dimensions, etcetera.) In that case, suitable distribution strengths can be found by simple means.

The following discussion will restrict itself to the Laplace equation, since the Poisson equation can always be turned into the Laplace equation by subtracting the unbounded space solution u_{inf}. This only produces an unimportant change in the inhomogeneous term of the boundary condition. The problem to be solved is then:

$$\nabla^2 u = 0 \text{ for } |\vec{x}| \le R, \qquad u = g \text{ on } |\vec{x}| = R,$$

where g is a given function, physically the temperature on the boundary in heat conduction problems, and R is the radius of the ball.

In two dimensions, using polar coordinates, the solution is

$$\boxed{u(r,\vartheta) = \frac{R^2 - r^2}{2\pi} \oint \frac{g(\bar{\vartheta})\,\mathrm{d}\bar{\vartheta}}{R^2 - 2Rr\cos(\bar{\vartheta} - \vartheta) + r^2}} \tag{19.12}$$

and in three dimensions, using spherical coordinates, the solution is

$$\boxed{\begin{aligned} u(r,\vartheta,\varphi) &= \frac{R^2 - r^2}{4\pi} R \\ &\times \oiint \frac{g(\bar{\vartheta},\bar{\varphi}) \sin\bar{\vartheta}\,\mathrm{d}\bar{\vartheta}\mathrm{d}\bar{\varphi}}{\left\{R^2 - 2Rr[\cos\bar{\vartheta}\cos\vartheta + \sin\bar{\vartheta}\sin\vartheta\cos(\bar{\varphi} - \varphi)] + r^2\right\}^{3/2}} \end{aligned}} \tag{19.13}$$

These results are known as "Poisson's integral formula" in two, respectively three dimensions.

19.3.6 Derivation

This subsection will derive the two-dimensional formula above, leaving the three-dimensional one for the homework. For simplicity, from now on it will be assumed that the ball (i.e. circle in two-dimensions) has unit radius,

$$R = 1$$

It is a simple matter of rescaling r to get back to the formulae for a ball of arbitrary radius.

The integral formula can be derived by a clever selection for the solution $u_{\rm out}$ outside the circle in the integral solution (19.10). In particular, the trick is to take

$$u_{\rm out}(r,\vartheta) = Au(\bar{r},\vartheta) \qquad \text{where} \qquad \bar{r} = \frac{1}{r} \tag{19.14}$$

Here A is a constant still to be selected. Note that if $\bar{r} \leqslant 1$ then $r \geqslant 1$: these rules turn solutions inside the ball into solutions outside the ball. The transformation $\bar{r} = 1/r$ is called an inversion with respect to the surface of the unit ball.

The first thing to show is that $u_{\rm out}$ satisfies the Laplace equation. The integral solution (19.10) does not apply otherwise. The Laplacian,

$$\nabla^2 u_{\rm out} = \frac{\partial^2 u_{\rm out}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{\rm out}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_{\rm out}}{\partial \theta^2}$$

must be zero.

To show that this is so, first differentiate (19.14) once, using the chain rule to convert the $\bar{r}$ derivatives of u to r derivatives:

$$\frac{\partial u_{\rm out}}{\partial r} = A\frac{\partial u}{\partial \bar{r}}\frac{\partial \bar{r}}{\partial r} = -A\frac{\partial u}{\partial \bar{r}}\frac{1}{r^2}$$

Differentiate this once more to get the second derivative. Note that you now have to use the product rule of differentiation to differentiate the factors. And you need again the chain rule for differentiating the first factor. You get

$$\frac{\partial^2 u_{\rm out}}{\partial r^2} = A\frac{\partial^2 u}{\partial \bar{r}^2}\frac{1}{r^2}\frac{1}{r^2} + A\frac{\partial u}{\partial \bar{r}}\frac{2}{r^3}$$

Also,

$$\frac{\partial^2 u_{\rm out}}{\partial \theta^2} = A\frac{\partial^2 u}{\partial \theta^2}$$

If you plug these derivatives into the Laplacian given above, you get

$$\nabla^2 u_{\rm out} = A\frac{1}{r^4}\left[\frac{\partial^2 u}{\partial \bar{r}^2} + r\frac{\partial u}{\partial \bar{r}} + r^2\frac{\partial^2 u}{\partial \theta^2}\right]$$

Since $r = 1/\bar{r}$, you recognize the Laplacian of u inside the square brackets. That is zero because u satisfies the Laplace equation. Then you see that so does $u_{\rm out}$.

Now the idea is to try to choose the constant A so that the integral solution (19.10) only involves the given values of u on the boundary. In particular, the normal derivative of $u - u_{\text{out}}$ must be eliminated. Now for a spherical boundary, the normal derivative is the radial derivative. And on the surface of the ball, $r = \bar{r} = 1$. So on the boundary, using the expressions above,

$$u_{\text{out}}(1,\theta) = Au(1,\theta) \qquad \frac{\partial u_{\text{out}}}{\partial r}(1,\theta) = -A\frac{\partial u_{\text{out}}}{\partial r}(1,\theta)$$

Note that in the final term, the first independent variable in u has been renamed simply r. It does not make a difference what you call the independent variable of a function; we just used a bar on it when we were treating u at one location to define u_{out} at another location. The bar was merely to keep the two locations apart.

For $\partial(u - u_{\text{out}})/\partial r$ to vanish on the surface of the sphere. according to the above equations you need to take $A = -1$. In that case, $u - u_{\text{out}}$ on the sphere equals $2u$, and u is the given function g on the surface of the sphere. So the integral solution (19.10) becomes

$$u(\vec{x}) = \oint_{|\vec{\xi}|=1} 2g\frac{\partial G}{\partial n_{\vec{\xi}}}\,\mathrm{d}S_{\vec{\xi}} + C_1 \qquad \text{with} \qquad G = \frac{1}{2\pi}\ln\left|\vec{x} - \vec{\xi}\right|. \tag{19.15}$$

The above solution u is completely in terms of the *given* function g. So the Dirichlet problem has been solved.

But of course you want to clean it up. You would like the solution of a problem in a circle to be in terms of polar coordinates. So set

$$\vec{x} = r\hat{\imath}_r \text{ with } \hat{\imath}_r = \begin{pmatrix} \cos\vartheta \\ \sin\vartheta \end{pmatrix} \qquad \vec{\xi} = \rho\hat{\imath}_\rho \text{ with } \hat{\imath}_\rho = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

for the point $\vec{x}$ at which the temperature u is desired and the point of integration $\vec{\xi}$ respectively. Then the "surface" element $\mathrm{d}S_{\vec{\xi}}$ in the integral over the circle perimeter is $\rho\,\mathrm{d}\varphi$, and $\rho = 1$ on the circle.

Also, the derivative $\partial G/\partial n_{\vec{\xi}}$ normal to the circle is simply $\partial G/\partial\rho$. G is a function of the distance $d = |\vec{x} - \vec{\xi}|$ between the points $\vec{x}$ and $\vec{\xi}$; in particular $G = \ln(d)/2\pi$ in two dimensions. You can write

$$d^2 = (r\hat{\imath}_r - \rho\hat{\imath}_\rho)\cdot(r\hat{\imath}_r - \rho\hat{\imath}_\rho) = r^2 - 2r\rho\hat{\imath}_r\cdot\hat{\imath}_\rho + \rho^2, \tag{19.16}$$

whose derivative with respect to ρ equals

$$\frac{\mathrm{d}d^2}{\mathrm{d}\rho} = \frac{-2r\rho\hat{\imath}_r\cdot\hat{\imath}_\rho + 2\rho^2}{\rho}$$

or getting rid of the ugly dot product term using the expression (19.16) for d^2,

$$\frac{\mathrm{d}d^2}{\mathrm{d}\rho} = \frac{\rho^2 - r^2 + d^2}{\rho}$$

So you can write using the chain rule that

$$\frac{\partial G}{\partial n_{\vec{\xi}}} = \frac{\mathrm{d}G}{\mathrm{d}d}\frac{\mathrm{d}d}{\mathrm{d}d^2}\frac{\partial d^2}{\partial \rho} = \frac{\mathrm{d}G}{\mathrm{d}d}\frac{\rho^2 - r^2 + d^2}{2d\rho}$$

Plug in the expression $G = \ln(d)/2\pi$ for the two-dimensional Green's function, and note that $\rho = 1$ on the circle to get:

$$\frac{\partial G}{\partial n_{\vec{\xi}}} = \frac{1 - r^2}{4\pi d^2} + \frac{1}{4\pi}$$

Plug that into the integral expression (19.15) for $u(\vec{x})$, taking d^2 from (19.16) with $\hat{\imath}_r \cdot \hat{\imath}_\rho$ equal to $\cos(\varphi - \vartheta)$, to get

$$u(r,\vartheta) = \frac{1 - r^2}{2\pi} \oint \frac{g(\varphi)\,\mathrm{d}\varphi}{1 - 2r\cos(\varphi - \vartheta) + r^2} + \frac{1}{2\pi} \oint g(\varphi)\,\mathrm{d}\varphi + C_1.$$

The two final terms are just constants, and they cancel each other. The reason is that

$$C_1 = u_{\text{out}}(\infty, \vartheta) = -u(0, \vartheta)$$

The mean value theorem, proved in {D.3}, says that $u(0,\vartheta)$ equals the average of $g = u$ on the circle.

Also, to allow for the case that the radial coordinate r is not normalized with the circle radius R, you want to replace r in the above result with r/R. That produces the Poisson integral as stated in the previous subsection.

19.3.6 Review Questions

1. Find a suitable solution u_{out} outside the sphere in three dimensions. Show that it satisfies the Laplace equation.
   Solution pnifd-a
2. Derive the Poisson integral formula in three dimensions as given in the previous subsection.
   Solution pnifd-b

19.3.7 The integral formula for the Neumann problem

The Neumann problem in two dimensions is:

$$\nabla^2 u = 0 \text{ for } r \le R, \qquad \frac{\partial u}{\partial r} = g \text{ on } r = R,$$

This corresponds physically to a problem where the heat flux instead of the temperature is described on the boundary. The solution is

$$\boxed{u(r,\vartheta) = \frac{-1}{2\pi} \oint g(\varphi) \ln \frac{R^2 - 2Rr\cos(\varphi - \vartheta) + r^2}{R^2}\,\mathrm{d}\varphi + C_1} \tag{19.17}$$

Note that there is only a proper solution for u if

$$\oint g(\varphi)\,\mathrm{d}\varphi = 0$$

If you put in an invalid g, you will get a u, but it will not have heat flux g through the boundary. In particular, putting in a nonzero constant for g will produce $u = 0$ and no heat flux. It can be seen from the above expression that the undetermined constant C_1 is the temperature at the center of the circle.

The derivation of the formula above is similar to the one in the previous subsection. You will be disappointed to learn that you must miss doing it in the homework. The same story does not work in three dimensions since you cannot get rid of the unknown surface values of u in both the source and dipole distributions. In two dimensions, however, if you take $u_{\text{out}}(r,\vartheta) = u(1/r,\vartheta)$, the dipole strength is zero and only the source integral remains:

$$u(\vec{x}) = -\oint_{|\vec{\xi}|=1} 2gG\,\mathrm{d}S_{\vec{\xi}} + C_1 \text{ with } G = \frac{1}{2\pi}\ln d$$

and use of the expression (19.16) for d gives the stated result.

19.3.8 Smoothness of the solution

One important qualitative conclusion that can be drawn from the various results of the previous subsections is that the solution of a Laplace equation problem is infinitely smooth in the interior of the region Ω in which it applies.

For example, consider the derived expression for u if the exterior solution is zero:

$$u(\vec{x}) = \oint_{\delta\Omega} u\frac{\partial G}{\partial n_{\vec{\xi}}}\,\mathrm{d}S_{\vec{\xi}} - \oint_{\delta\Omega} G\frac{\partial u}{\partial n_{\vec{\xi}}}\,\mathrm{d}S_{\vec{\xi}} \tag{19.18}$$

If you take derivatives of u with respect to the components of $\vec{x}$, you will be differentiating $G(\vec{x};\vec{\xi})$ inside the integral. And G has infinitely many finite derivatives away from the singular point $\vec{\xi} = \vec{x}$, in other words, away from the boundary.

So, if u and $\partial u/\partial n$ are merely integrable on the boundary, which still allows them to be quite singular, the solution at every point in the interior will have infinitely many continuous derivatives.

It is somewhat different for the Poisson equation, since if the forcing f has a singularity at some point, then so will the solution u. But still the solution for u will be less singular than f is. For example, in two-dimensions a delta function in f, whose square is not integrable, produces a logarithmic Green's function, for which every power is integrable over the singular point. In general, it can be seen from Fourier solution of the Poisson problem that u will in general have two more square integrable derivatives than f. (Assuming that lack of decay of u at large distances is not a factor or subtracted out first.)

Chapter 20

First Order Equations

The book's treatment of first order equations is not very intuitive, and the artificial parameter r may not be a convenient one. Also, you would like to write a solution of a partial differential equation without applying a boundary condition. This chapter gives an alternate approach that defines no artificial quantities.

20.1 Classification and characteristics

The general quasi-linear first order equation in two dimensions takes the form

$$\boxed{au_x + bu_y = c} \tag{20.1}$$

where a, b, and c may depend on x, y, and u. All scalar first order equation are classified as hyperbolic equations.

The characteristics are defined by

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a}} \tag{20.2}$$

They will form a single family of lines in the x, y-plane. (In contrast, the characteristics of scalar second order hyperbolic differential equations form two intersecting families of lines.)

In general, the variation of a function u of two variables along a line is given by the total differential of calculus,

$$\frac{\mathrm{d}u}{\mathrm{d}x} = u_x + u_y \frac{\mathrm{d}y}{\mathrm{d}x}$$

but along the characteristic lines that becomes

$$\frac{\mathrm{d}u}{\mathrm{d}x} = u_x + u_y \frac{b}{a}$$

and comparing with the partial differential equation, it is seen that the right hand side is c/a. So the variation of the solution u along the characteristic lines is given by

$$\boxed{\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{c}{a}} \tag{20.3}$$

Note that frequently, you may have to solve the ordinary differential equations in a different form or order. For example, if b/a depends on u, you will not be able to solve $\mathrm{d}y/\mathrm{d}x = b/a$ to find y as a function of x since u in b/a is still an unknown function of x. But maybe, say, c/b is not a function of x, in which case you could solve $\mathrm{d}u/\mathrm{d}y = c/b$; then you could plug that solution for u as a function of y into $\mathrm{d}x/\mathrm{d}y = a/b$ to get an equation for x that no longer involves u. The bottom line is that it is really best to write the characteristic equations as

$$\boxed{\mathrm{d}x : \mathrm{d}y : \mathrm{d}u = a : b : c} \tag{20.4}$$

and pick from those proportionalities the ratio that is easiest to solve first.

In all the unsolved problems in the book, there is at least one solvable ratio. But if there is none, you may be forced to try to change variables, e.g. to polar, or eliminate one variable by, say, differentiating a ratio, hopefully producing a second order ordinary differential equation with one variable eliminated.

Example

Question: (5.30) Solve

$$yu_x + xu_y = cu$$

Solution:

This example wants to solve the partial differential equation

$$yu_x + xu_y = cu$$

For this equation, a ratio like $\mathrm{d}u/\mathrm{d}x = cu/y$ is not immediately solvable for u, since y besides u would be an unknown function of x. The only solvable ratio is in fact that between $\mathrm{d}x$ and $\mathrm{d}y$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} \qquad \Rightarrow \qquad y\,\mathrm{d}y = x\,\mathrm{d}x \qquad \Rightarrow \qquad y^2 = x^2 + C_1$$

where C_1 is the integration constant. These characteristic lines are hyperbola; they are sketched in figure 20.1.

Now that y is a known function of x, specifically $y = \sqrt{x^2 + C_1}$ assuming it is positive, the ordinary differential equation for u can be solved

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{cu}{y} = \frac{cu}{\sqrt{x^2 + C_1}}$$

to give

$$\frac{\mathrm{d}u}{u} = c\frac{\mathrm{d}x}{\sqrt{x^2 + C_1}} \qquad \Rightarrow \qquad \ln|u| = c\ln\left(x + \sqrt{x^2 + C_1}\right) + \ln|C_2|$$

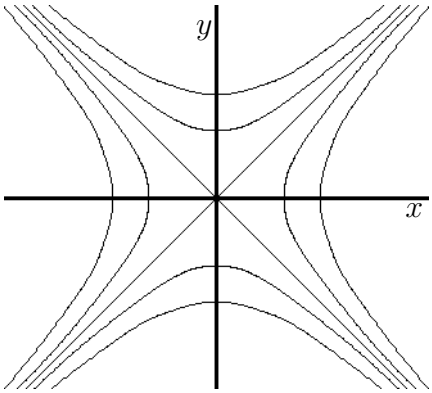


Figure 20.1: Characteristics of the partial differential equation of problem 5.30.

Taking exponentials and noting that the square root equals y, this simplifies to

$$u = C_2(x+y)^c \qquad \text{along a characteristic} \qquad y^2 = x^2 + C_1$$

For example, if it is given that $u = 1$ at the point $x = y = 1$ shown as a fat dot in figure 20.2, then it follows from the above general expressions that $C_1 = 0$, so the characteristic line is the line $y = x$ shown in grey, and that $C_2 = 1/2^c$, so you would get $u(x, x) = x^c$ for u on the grey line.

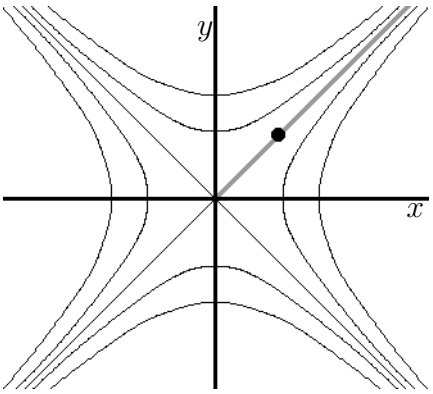


Figure 20.2: Given the value of u at a single point on a characteristic line, u can be found at every point on that line.

Example

Question: (5.6) Solve the nasty example

$$xu_x + yuu_y = -xy.$$

Solution:

In this case, none of the ratios

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{uy}{x} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -y \qquad \frac{\mathrm{d}u}{\mathrm{d}y} = -\frac{x}{u}$$

is a solvable ordinary differential equation; each involves three variables. You might try to take a derivative of an equation, like

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = -\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{uy}{x} = \frac{u}{x}\frac{\mathrm{d}u}{\mathrm{d}x} \qquad \Longrightarrow \qquad \frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{u}{x}\frac{\mathrm{d}u}{\mathrm{d}x}$$

which is indeed an ordinary differential equation for $u(x)$ not involving the unknown y. But it is an awkward second order nonlinear equation.

The trick is to guess that the combination xy can be found as a function of u:

$$y\frac{\mathrm{d}x}{\mathrm{d}u} + x\frac{\mathrm{d}y}{\mathrm{d}u} = -1 - u \qquad \Longrightarrow \qquad \frac{\mathrm{d}xy}{\mathrm{d}u} = -1 - u$$

which produces

$$xy = -u - \tfrac{1}{2}u^2 + C_1.$$

This can then be plugged into

$$\frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{y} = -\frac{x}{xy} = \frac{x}{u + \frac{1}{2}u^2 - C_1}$$

to get a separable equation giving x as a function of u, with another integration constant C_2. However, that becomes a mess, involving either an arctan or logarithm, depending on the value of C_1.

20.2 Numerical solution

It is certainly straightforward to numerically solve the two ordinary differential equations of the previous subsection along a characteristic line using say a Runge-Kutta method. You would need to start from some point at which an initial or boundary condition is given.

If you find the solution along each of a densely spaced set of characteristic lines, you have essentially found u everywhere.

Of course, if a is zero somewhere in the region of interest, it may be a better idea to find u and x as functions of y instead of u and y as functions of x, by taking suitable ratios from (20.4). Or you could just find all three variables as function of the arc length s along the characteristic lines, by solving

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{a}{\sqrt{a^2+b^2}} \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = \frac{b}{\sqrt{a^2+b^2}} \qquad \frac{\mathrm{d}u}{\mathrm{d}s} = \frac{c}{\sqrt{a^2+b^2}}$$

This allows either a or b to be zero; it only fails if both are zero at the same point, and that is a true physical problem rather than a mathematical one.

20.3 Analytical solution

Often, you want an analytical expression for the solution u of a first order equation in terms of x and y. Solving the differential equations gives expressions valid along characteristic lines, which is not the same thing. These expressions involve two integration "constants", call them C_1 and C_2, that themselves are unknown functions of x and y: if you move from one characteristic line to another, the values of C_1 and C_2 will normally change. They are only constants along the characteristic lines.

To get a relationship for u as a function of x and y, the trick is to recognize that there is a functional dependence between the two integration constants involved. You can use, say, C_1 as a label for what characteristic curve you are on: different values of C_1 correspond to different characteristic lines. And C_2 *only* depends on what characteristic line you are on, not on the position on the line. So C_2 *only* depends on what C_1 is; C_2 is some function $C_2(C_1)$ of C_1. What function that is remains unknown; that depends on the relevant initial or boundary condition, but it is *some* function.

The procedure to find u as a function of x and y, or at least, to find the most general and precise expression between these three quantities, is therefor:

1. In one of the two ordinary differential equation solutions you obtained, say the one involving the integration constant you called C_2, replace C_2 by the more precise $C_2(C_1)$ to indicate that it is not really a constant, but still depends on what C_1 is.

2. Substitute for C_1 from the other ordinary differential equation solution.

Note that in some special cases, it makes a difference in which of the two ordinary differential equation solutions you take the integration constant to be a function of the other one: sometimes $C_2(C_1)$ is not a well-defined function, but $C_2(C_1)$ is. (An example is in subsubsection 20.5.5.)

Example

Question: (5.30 continued) Solve

$$yu_x + xu_y = cu$$

Solution:

Previously, it was found that the characteristics of this example were given by

$$u = C_2(x+y)^c \qquad \text{along a characteristic} \qquad y^2 = x^2 + C_1$$

To get the general expression for $u(x, y)$, first note that more precisely,

$$u = C_2(C_1)(x+y)^c,$$

then plug in the expression for C_1 from the other equation to get

$$u(x, y) = C_2(y^2 - x^2)(x + y)^c.$$

This is the most general solution of the partial differential equation. Function $C_2()$ remains undetermined; the above expression is a solution of the partial differential equation regardless what one-argument function you take for $C_2()$.

In fact, you need an undetermined one-argument function in the solution, because you must still match the function used to specify the relevant initial or boundary condition, also a one-argument function.

20.4 Using the boundary or initial condition

After you have found the general solution of the partial differential equation as described in the previous sections, you probably want to find the remaining undetermined function that it involves by applying a given boundary or initial condition. To do so:

1. Write the boundary condition in terms of a single parameter, call it s.
2. Plug the general solution to the partial differential equation into it.
3. Call the argument of the unknown function, say α, and express everything else in terms of α instead of s by solving for s in terms of α.
4. That will produce the expression for the unknown function for any value of its argument. Plug that back into the general solution which is now fully determined.

Example

Question: (5.30 continued) Solve

$$yu_x + xu_y = cu$$

with initial condition

$$u = x^2 + y \qquad \text{on} \qquad x + y = 1,$$

Solution:

The general solution to the partial differential equation for this example is given by

$$u(x, y) = C_2(y^2 - x^2)(x + y)^c.$$

Plug that into the initial condition on the line $x + y = 1$ in order to figure out what function C_2 must be:

$$C_2(y^2 - x^2)(x + y)^c = x^2 + y \qquad \text{on} \qquad x + y = 1,$$

Don't try to deduce one-parameter function C_2 directly from an expression involving two different parameters. Instead convert to a single parameter by expressing one parameter in terms of the other. In this case, you can use $x + y = 1$ to express y in terms of x as $y = 1 - x$. That gives

$$C_2(1 - 2x + x^2 - x^2)(x + 1 - x)^c = x^2 + 1 - x \qquad \text{on} \qquad y = 1 - x,$$

or cleaned up

$$C_2(1 - 2x) = x^2 - x + 1$$

Now you have an expression for function C_2 in terms of a single parameter. To get the function itself, give some name to its parameter that is not already used. Call the argument, say, α. So $C_2 = C_2(\alpha)$. According to the initial condition above

$$\alpha \equiv 1 - 2x.$$

Solve this for x in terms of α,

$$x = \tfrac{1}{2}(1 - \alpha)$$

and plug that into the expression for C_2 to get an expression for function $C_2(\alpha)$ in terms of α only:

$$C_2(\alpha) = \Big(\tfrac{1}{2}(1 - \alpha)\Big)^2 - \tfrac{1}{2}(1 - \alpha) + 1$$

or worked out

$$C_2(\alpha) = \tfrac{1}{4}\alpha^2 + \tfrac{3}{4}.$$

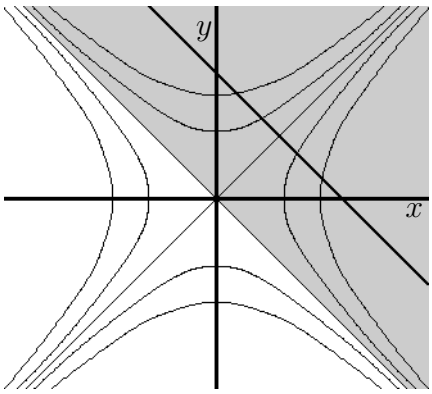


Figure 20.3: Region where u is determined by an initial condition given on the line $x + y = 1$.

Now that function C_2 has been identified, plug it into the general solution, valid everywhere,

$$u(x, y) = C_2(y^2 - x^2)(x + y)^c.$$

to get the final solution

$$u(x, y) = \Big(\tfrac{1}{4}(y^2 - x^2)^2 + \tfrac{3}{4}\Big)(x + y)^c.$$

Note that this solution is only valid in the grey region of figure 20.3; the characteristics in the white region never intersect the line $x + y = 1$. To find the solution there, you would need an initial condition on, say, the line $x + y = -1$.

You see how important it is to graph the characteristics.

20.5 The inviscid Burgers' equation

The inviscid Burgers' equation is a model for nonlinear wave propagation, especially in fluid mechanics. It takes the form

$$\boxed{u_t + uu_x = 0} \tag{20.5}$$

The characteristic equations are, according to (20.4),

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = 0.$$

The second of these shows that u is constant along the characteristics of the Burgers' equation, and then the first equation shows that the characteristic lines are straight lines in the x, t-plane.

The solution of the two characteristic ordinary differential equations above is simple:

$$x = ut + C_1 \qquad u = C_2$$

The general solution of the partial differential equation may be found in terms of x and t by noting that C_2 must be a function of C_1, $C_2 = C_2(C_1)$, and then substituting $x - ut$ for C_1:

$$u = C_2(x - ut).$$

Some special cases are singular in those terms; they require that C_1 is written in terms of $C_2 = u$:

$$x = ut + C_1(u).$$

Normally, either expression may be taken to be the general solution of the ordinary differential equation. One-parameter function C_2, respectively C_1 remains to be identified from whatever initial or boundary conditions there are.

20.5.1 Wave steepening

The given solution of the inviscid Burgers' equation shows that the characteristics are straight lines. This is troubling, since straight lines are likely to intersect. In particular, since the point on a given characteristic lines propagates with speed u, faster points behind less fast ones will eventually overtake them.

As an example, consider the following problem:

$$u_t + uu_x = 0 \quad u(x,0) = 1 - \cos(x)$$

This problem is self-evidently periodic of period 2π. Figure 20.4 shows how the characteristics intersect starting from time $t = 1$.

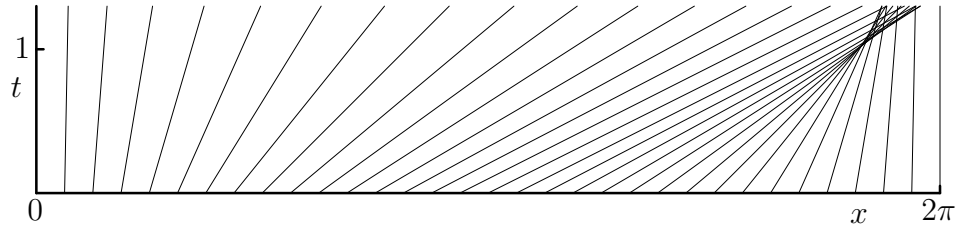


Figure 20.4: Characteristics of Burgers' equation for an example initial condition intersect for times greater than $t = 1$.

Figure 20.5 shows profiles u versus x at various times. Note that for times greater than one, u becomes a multiple-valued function. Physically, this is normally not acceptable: you can not have three different pressures or flow velocities at the same point.

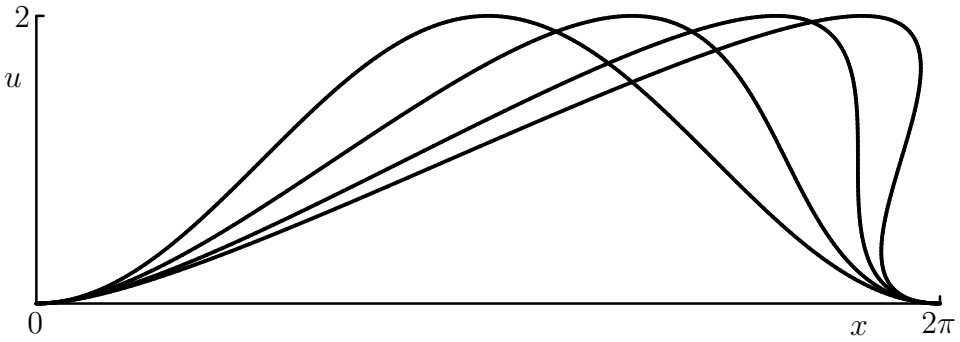


Figure 20.5: Profiles at times $t = 0$, .5, 1, and 1.3 show wave steepening leading to a multiple-valued solution for times greater than $t = 1$.

20.5.2 Shocks

The previous subsection noted that solutions of hyperbolic equations with intersecting characteristics are usually not physically acceptable. In fact, the desired solution for the inviscid Burgers' equation is usually taken to be the solution of the viscous Burgers' equation:

$$u_t + uu_x = \varepsilon u_{xx}$$

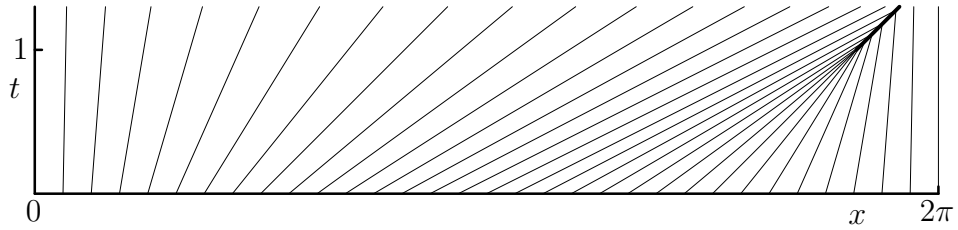


Figure 20.6: Correct solution of Burgers' equation for the same initial condition as the previous subsection.

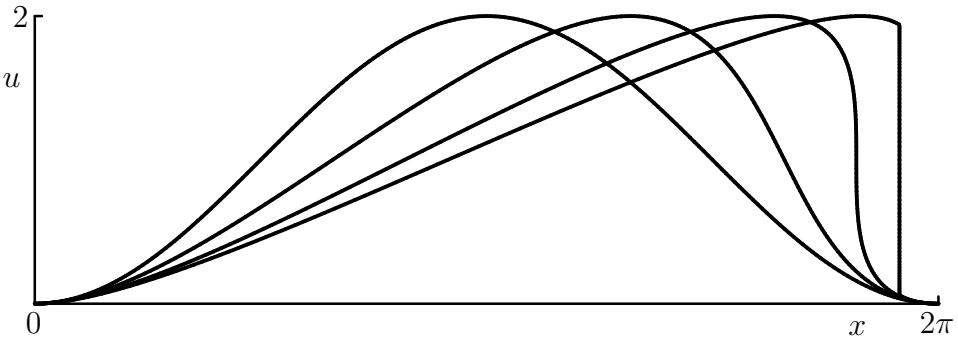


Figure 20.7: Correct profiles for Burgers' equation for the same initial condition as the previous subsection.

in the limit that the coefficient of viscosity ε becomes zero.

The viscous Burgers' equation, too, is analytically solvable, though the solution will be skipped here. The bottom line is that it *does not* have multiple valued solutions. So what does the solution of the viscous Burgers' equation look like in the limit that the viscosity becomes zero? Like figures 20.6 and 20.7. A jump discontinuity called a "shock" develops in u. The characteristics run into this shock and disappear.

The question now is of course, what determines the precise location of the shock? Clearly, it should be somewhere in the region of intersecting characteristics, but that still leaves a considerable uncertainty. Equations for the shock are needed. They usually follow from the requirement that certain quantities remain conserved in the solution. This is addressed in the next subsections.

20.5.3 Conservation laws

Often, partial differential equations express conservation of some physical quantity. For example, the continuity equation for the density of a fluid expresses conservation of mass of the fluid: the mass of a region of fluid is found by integrating the density over the volume of the region, and the continuity equation implies that mass is preserved in time.

The viscous Burgers' equation, too, preserves some quantity. To see what, integrate the equation over an interval from some position $x = a$ to some position $x = b$:

$$\int_a^b u_t \,\mathrm{d}x + \int_a^b u u_x \,\mathrm{d}x = \varepsilon \int_a^b u_{xx} \,\mathrm{d}x$$

The last two integrals can be integrated after noting that $uu_x = \frac{1}{2}(u^2)_x$, to give

$$\int_a^b u_t \,\mathrm{d}x + \tfrac{1}{2}u_b^2 - \tfrac{1}{2}u_a^2 = \varepsilon u_{x,b} - \varepsilon u_{x,a}$$

First consider the case that the problem is periodic and the integral is over a full period. Then the quantities at a and b are the same because of periodicity and drop away against each other. This shows that

$$\int_a^b u_t \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \int_a^b u \,\mathrm{d}x = 0$$

so that $\int u \,\mathrm{d}x$ over a period is a conserved quantity, unchanging in time. The unknown u itself can then be identified as the amount of conserved quantity per unit length.

Next consider the case that the region of integration is not a period. In that case, the Leibniz rule for differentiating integrals says that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b u_t \,\mathrm{d}x = \int_a^b u_t \,\mathrm{d}x + u_b \frac{\mathrm{d}b}{\mathrm{d}t} - u_a \frac{\mathrm{d}a}{\mathrm{d}t}$$

and plugging that into the integrated equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u_t\,\mathrm{d}x = \left(u_b\frac{\mathrm{d}b}{\mathrm{d}t} - \tfrac{1}{2}u_b^2 + \varepsilon u_{x,b}\right) - \left(u_a\frac{\mathrm{d}a}{\mathrm{d}t} - \tfrac{1}{2}u_a^2 + \varepsilon u_{x,a}\right).$$

Now think of interval a, b as being preceded by a similar interval $a'b'$, with $b' = a$. It is evident from the above expression that the reduction in the value of $\int_a^b u\,\mathrm{d}x$ caused by the term

$$\left(u_a\frac{\mathrm{d}a}{\mathrm{d}t} - \tfrac{1}{2}u_a^2 + \varepsilon u_{x,a}\right)$$

is fully compensated for by a corresponding increase in $\int_{a'}^{b'} u\,\mathrm{d}x$, because the same term shows up there as

$$\left(u_{b'}\frac{\mathrm{d}b'}{\mathrm{d}t} - \tfrac{1}{2}u_{b'}^2 + \varepsilon u_{x,b'}\right)$$

with a plus sign. So whatever goes out of interval ab at a goes into interval $a'b'$. The same way, whatever comes in at b comes out of the region $x > b$. It follows that $\int u\,\mathrm{d}x$ is still preserved.

It may be noted that in

$$\left(u_b\frac{\mathrm{d}b}{\mathrm{d}t} - \tfrac{1}{2}u_b^2 + \varepsilon u_{x,b}\right)$$

the first term represents the amount of conserved quantity being swept into the interval by the motion of its end point b. Typically, the second term physically corresponds to the amount of conserved quantity being convected out by motion of the substance, and the final term to the amount diffusing in by random molecular motion.

20.5.4 Shock relation

If the solution of the inviscid Burgers' equation is indeed supposed to approximate the solution of the viscous equation when the coefficient of viscosity ϵ becomes zero, it puts a condition on how the shocks must move. The shock is vanishingly thin and can only hold a negligible amount of conserved material. So, whatever goes into the shock at one side must come out at the other side.

The amounts going in and out of a region were derived in the previous section for an interval ab. Taking point a just before the shock and b just behind the shock, so that to practical purposes $a = x_s = b$ with x_s the shock velocity, equality of the amounts going in and out requires

$$u_b\frac{\mathrm{d}x_s}{\mathrm{d}t} - \tfrac{1}{2}u_b^2 = u_a\frac{\mathrm{d}x_s}{\mathrm{d}t} - \tfrac{1}{2}u_a^2$$

Solving for the shock velocity $\mathrm{d}x_s/\mathrm{d}t$, you get

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \tfrac{1}{2}(u_b + u_a)$$

It follows that the shock must move with the average of the characteristic velocities u_a and u_a just before and after the shock. Figures 20.6 and 20.7 were obtained by finding the shock position from that relationship.

Shock relations, like this one for Burgers' equation, are known as Rankine-Hugoniot relations in fluid mechanics. When deriving shock relations, make sure that the unknown variables are the conserved quantities per unit volume. If you multiply the inviscid Burgers' equation by $2u$, you get

$$(u^2)_t + 2u^2 u_x = 0$$

from which it can be seen that as far as the *inviscid* Burgers' equation is concerned, $\int u^2 \,\mathrm{d}x$ is also a conserved quantity. But the shocks you would compute using the corresponding conservation law are going to be different, and wrong if the true conserved quantity across shocks is the $\int u \,\mathrm{d}x$ of the viscous Burgers' equation.

20.5.5 The entropy condition

Consider now Burgers' equation for a unit "pulse" initial condition:

$$u_t + uu_x = 0 \qquad u(x,0) = \begin{cases} 1 \text{ for } 0 \le x \le 10 \\ \text{everywhere else} \end{cases}$$

This problem has a simple solution that is also quite wrong. It is shown in figure 20.8. It implies that the pulse moves with velocity $\frac{1}{2}$ towards the right. Note that both shocks satisfy the shock condition of the previous section; $u = 0$ at one side of each shock and $u = 1$ at the other side average in each case to $\mathrm{d}x_s/\mathrm{d}t = \frac{1}{2}$.

The problem is with the left shock. Characteristics should run into the shock for increasing time like for the right shock, not emerge out of it as happens for the left one. In fluid mechanics, the left shock is what is called an "expansion" shock. It produces an adiabatic decrease in entropy over the shock, something the second law of thermodynamics does not allow. For that reason, the condition that characteristics must run into the shock is called the "entropy condition."

The correct solution is shown in figure 20.9. The left jump in the initial condition spreads out into what is called an "expansion far." Unlike the shock, the expansion fan is a perfectly good nonsingular solution of the Burgers" equation, though you must use the solution form $x = ut + C_1(u)$ with $C_1 = 0$. The solution form $u = C_2(x - ut)$ does not work since $x - ut$ is the same, zero, on all characteristics, and u must be different on different characteristics. Conversely, in the other three regions, you must use the solution form $u = C_2(x - ut)$ with C_2

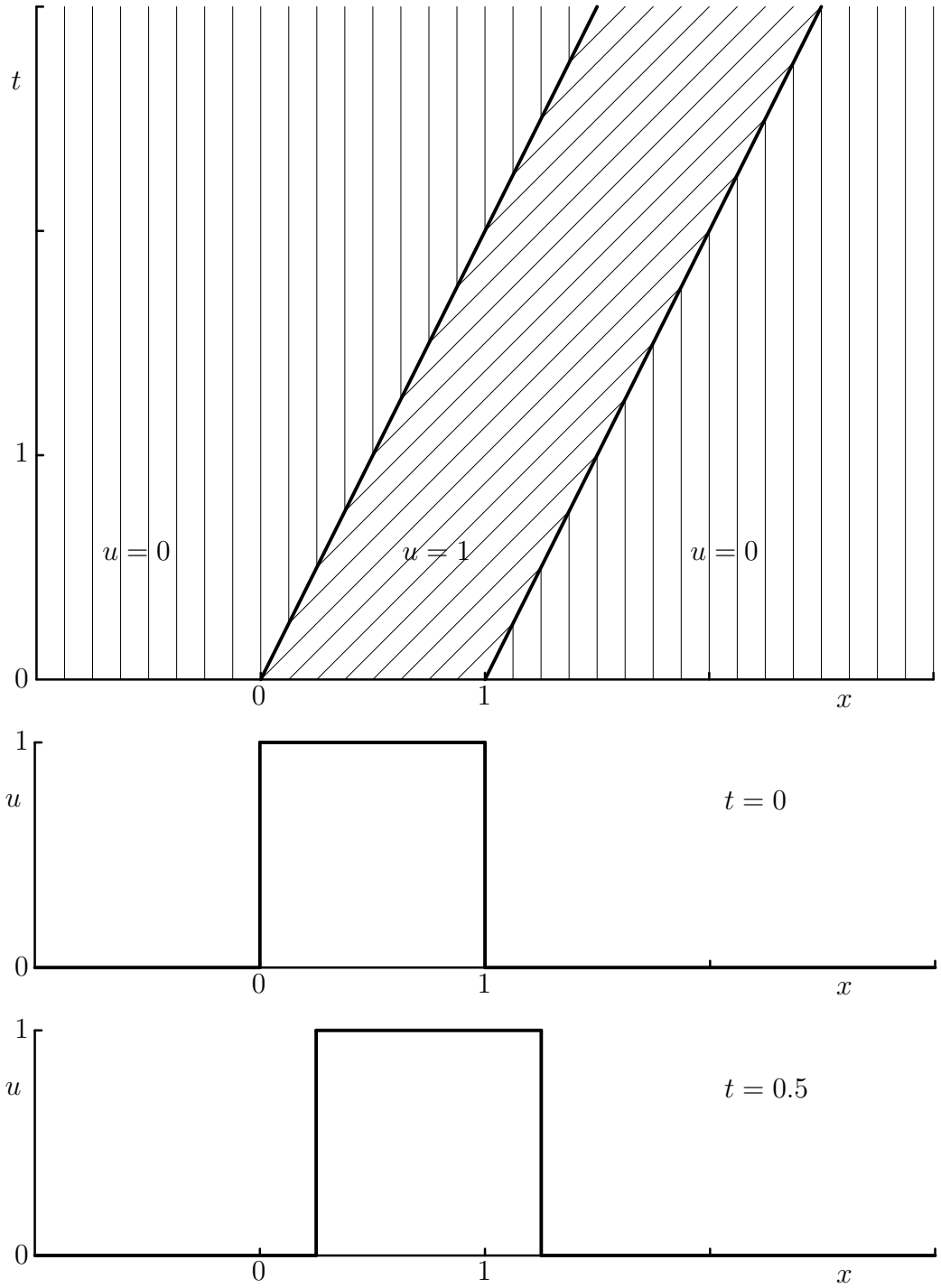


Figure 20.8: Incorrect solution to Burgers' equation for the initial pulse profile shown in the center graphic. The left shock violates the entropy condition.

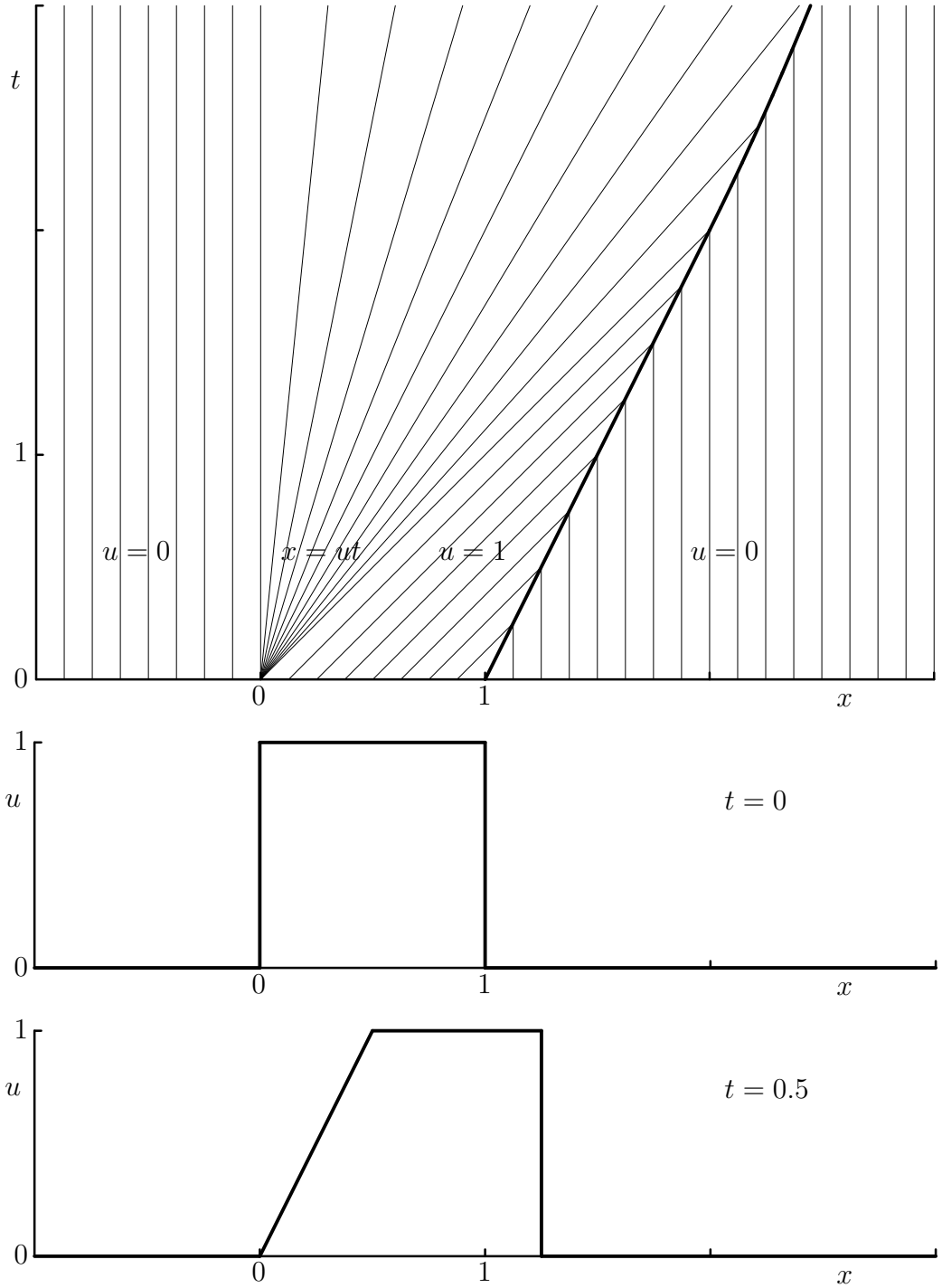


Figure 20.9: Correct solution to Burgers' equation for the initial pulse profile shown in the center graphic. The left shock has been replaced by an expansion fan.

either uniformly zero or uniformly one. There the solution form $x = ut + C_1(u)$ does not work since u is the same for all characteristics and $x - ut$ is not.

It may also be observed that the entropy condition is necessary to get a unique solution; both figures 20.8 and 20.9 satisfy the Burgers" equation at all continuous points and the shock conditions at all discontinuities.

20.6 First order equations in more dimensions

The procedures of the previous subsections extend in a logical way to more dimensions. If the independent variables are $x_1, x_2, \ldots, x_n$, the first order quasi-linear partial differential equation takes the form

$$\boxed{a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_n u_{x_n} = d} \tag{20.6}$$

where the a_n and d may depend on $x_1, x_2, \ldots, x_n$ and u.

The characteristic equations can now be found from the ratios

$$\boxed{\mathrm{d}x_1 : \mathrm{d}x_2 : \ldots : \mathrm{d}x_n : \mathrm{d}u = a_1 : a_2 : \ldots : a_n : d} \tag{20.7}$$

After solving n different ordinary differential equations from this set, the integration constant of one of them, call it C_n can be taken to be a general $n-1$-parameter function of the others,

$$C_n = C_n(C_1, C_2, \ldots, C_{n-1})$$

and then substituting for $C_1, C_2, \ldots, C_{n-1}$ from the other ordinary differential equation, an expression for u results involving one still undetermined, $n-1$ parameter function C_n.

To find this remaining undetermined function, plug in whatever initial condition is given, renotate the parameters of C_n to $\alpha_1, \alpha_2, \ldots$ and express everything in terms of them to find function $C_n(\alpha_1, \alpha_2, \ldots)$.

20.7 Systems of First Order Equations (None)

TBA

Chapter 21

D'Alembert Solution of the Wave equation

This chapter discusses a way to solve the wave equation in one spatial dimension in a relatively easy way.

21.1 Introduction

The wave equation in two dimensions,

$$u_{tt} = a^2 u_{xx}$$

has the general solution

$$u(x,t) = f_1(x - at) + f_2(x + at)$$

Here $f_1(x-at)$ is a function that moves to the right with speed a; a 'right-going wave'. And $f_2(x - at)$ is a function that moves to the left with speed a; a 'left-going wave'.

This solution was derived earlier. The functions f_1 and f_2 must be found from whatever initial and boundary conditions are given. One special case of that is the D'Alembert solution, which is the subject of this brief chapter.

In its simplest form, the D'Alembert solution assumes that there are no boundaries. That means that the x-range is doubly infinite:

$$-\infty < x < \infty$$

Therefore, only initial conditions are needed at the starting time $t = 0$:

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x)$$

Here f and g are assumed to be *given* functions.

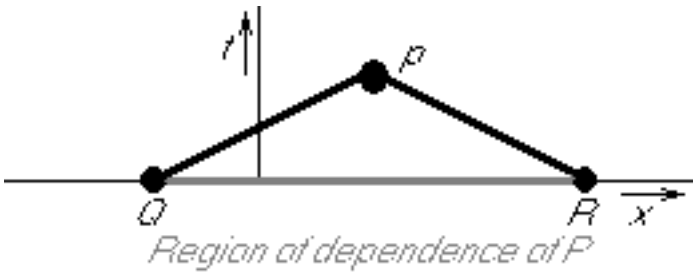
Note that since the wave equation is second order in time, it needs two initial conditions. Physically, the wave equation might describe the vibrations of a string. The two initial conditions are then that both the initial position and the initial velocity of the string must be given at each point.

The two given initial conditions allow the unknown functions f_1 and f_2 in the solution of the wave equation to be determined. That gives the solution directly in terms of the given functions f and g:

$$\boxed{u(x,t) = \frac{f(x-at)+f(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)\,\mathrm{d}\xi} \tag{21.1}$$

This is derived in example 4.10 in the book.

The solution can be understood more physically from looking at the x,t-plane:



To get the solution u at a position and time P, we need to average the f, (i.e. u), values at points Q and R. To that we need to add an integral of g, (i.e. the velocity u_t), between points Q and R:

$$u_P = \frac{u_Q + u_R}{2} + \frac{1}{2a}\int_Q^R u_t\,\mathrm{d}\xi$$

Note that the solution at point P depends on the initial conditions in the interval QR. In other words, the "region of dependence" of point P is the triangle QPR. Whatever is outside that triangle does not affect the solution at P at all.

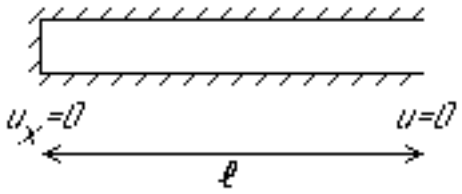
Conversely, point P is inside the region of influence of all points inside the triangle. A change in the initial conditions at any initial point inside the triangle will influence the solution at P.

21.2 Extension to finite regions

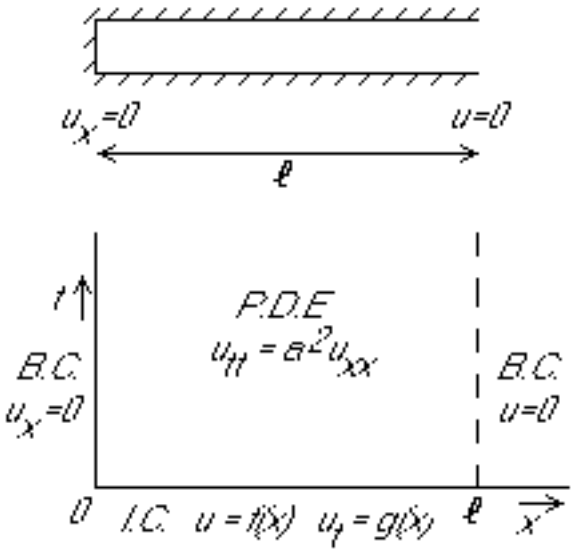
If x is restricted by finite boundaries, the D'Alembert solution does not really apply. To use it anyway, we must somehow extend the problem to a doubly infinite x-range without boundaries. But our solution without boundaries should still *satisfy* the boundary conditions for the finite range. That is often possible by clever use of symmetry. An example can clarify that.

21.2.1 The physical problem

The problem is to find the pressure for sound wave propagation in a tube with one end closed and one end open:



21.2.2 The mathematical problem



- There is a finite domain $\bar{\Omega}$ given by $0 \leqslant x \leqslant \ell$
- There is an unknown pressure $u = u(x,t)$ to be found from the wave equation

$$u_{tt} = a^2 u_{xx}$$

- The constant a is the speed of sound a. The equation is only valid for normal acoustics in which the gas velocities are much less than the speed of sound.
- This equation is second order in time so it needs two initial conditions. They are:

$$u(x,0) = f(x) \quad u_t(x,0) = g(x) \qquad \text{for} 0 \leq x \leq \ell$$

- This equation is second order in space so it needs two boundary conditions. Given are one homogeneous Neumann boundary condition at $x = 0$ and one homogeneous Dirichlet condition at $x = \ell$.

21.2.3 Dealing with the boundary conditions

The D'Alembert solution applies to an infinite domain $-\infty < x < \infty$. So to use the D'Alembert solution, the given initial conditions, that are valid for $0 < x < \ell$ must be extended to all x. In other words, functions $f(x)$ and $g(x)$ must be converted into functions $\bar{f}(x)$ and $\bar{g}(x)$ that have values for all x. Of course, in the interval $0 < x < \ell$, they must stay the same as $f(x)$ and $g(x)$. Assume now for example that $f(x)$ looks as sketched below:

You might think that you could now simply take $\bar{f}(x)$ to be zero for all x outside the range of the pipe. The corresponding D'Alembert solution will satisfy the wave equation everywhere, including inside the pipe $0 < x < \ell$. That is good, because the wave equation must indeed be satisfied. Unfortunately, the solution you get that way will not satisfy the boundary conditions at $x = 0$ and $x = \ell$. So it will still be wrong.

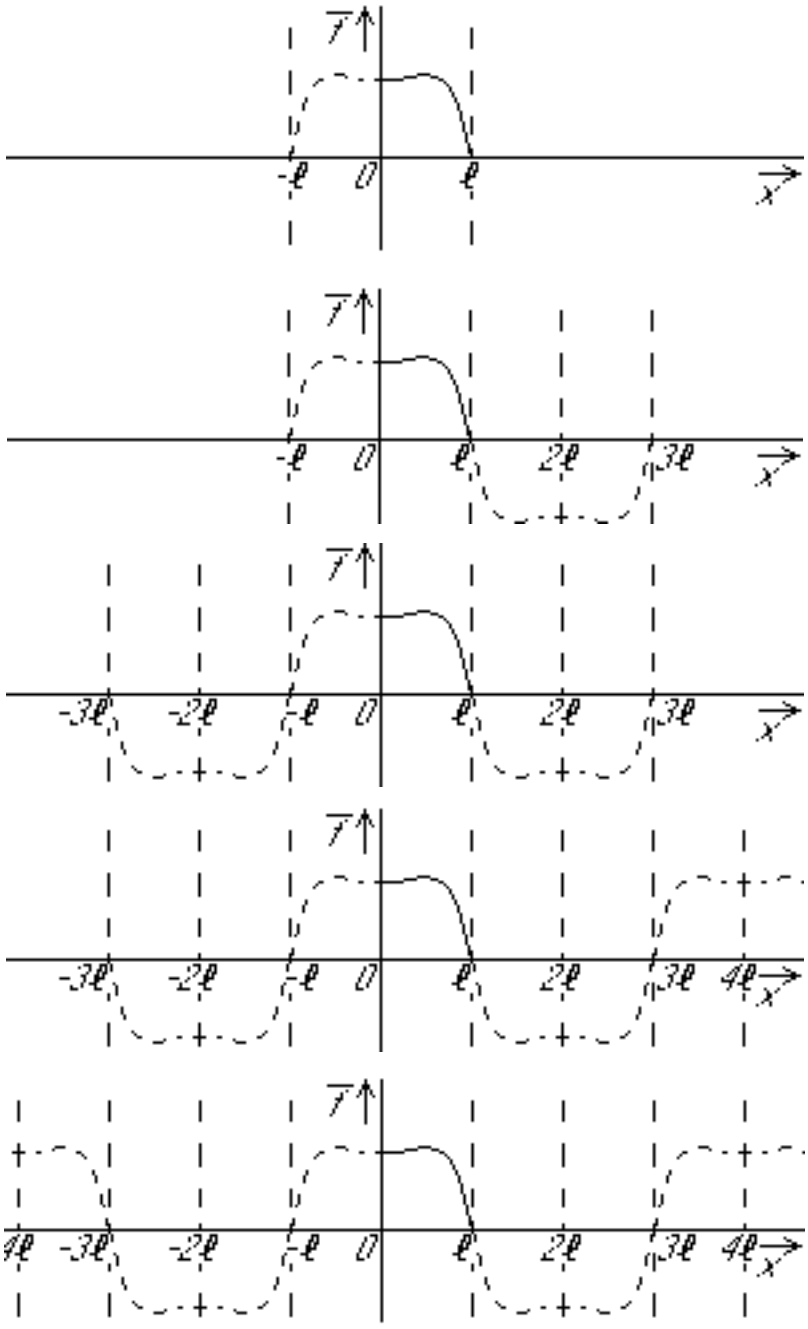
You must select the extension $\bar{f}(x)$ of $f(x)$ to all x so that the correct boundary conditions become automatic.

The way to do it is as follows:

- To make the boundary condition $u_x = 0$ at $x = 0$ automatic, create *symmetry* around $x = 0$. Symmetric functions have zero derivative at the symmetry point.

- To make the boundary condition $u = 0$ at $x = \ell$ automatic, create *anti-symmetry* (odd symmetry) around $x = \ell$. Anti-symmetric functions have zero derivative at the symmetry point.

The process is shown for $f(x)$ below:



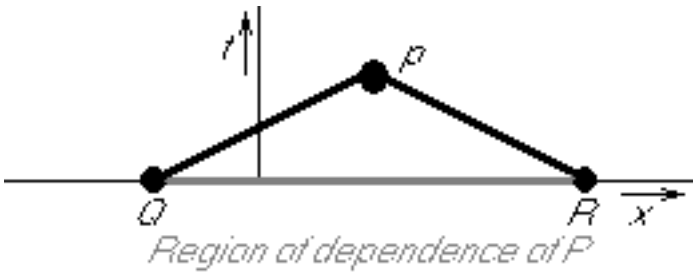
Create the extended function $\bar{g}(x)$ or G the same way.

It is OK if you get kinks or discontinuities in your functions $\bar{f}$ and $\bar{g}$ while creating (anti)symmetry. This happens when f and/or g does not satisfy the given boundary conditions. While then u or u_x may not have a unique value at the initial time, that problem will disappear when the time becomes greater than zero.

21.2.4 The final solution

$$u(x,t) = \frac{\bar{f}(x-at) + \bar{f}(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \bar{g}(\xi)\,\mathrm{d}\xi$$

This is pretty easy to evaluate for simple functions f and g. You will have fun doing it.



In the range $0 \leqslant x \leqslant \ell$, the found solution is exactly the same as for the finite pipe! The solution outside that range can simply be ignored.

Chapter 22

Separation of Variables

Separation of variables is a standard way of solving simple partial differential equations in simple regions. In general, the boundaries will have to be at constant values of the coordinates. In Cartesian coordinates, that works out to rectangles. Using cylindrical or polar coordinates, or similar, allows somewhat more general regions.

The idea is to write the solution as an infinite sum of chosen functions of one coordinate times coefficients that depend on the other coordinate(s). How exactly this works will be demonstrated by example.

22.1 A simple example

The method of separation of variables shall first be demonstrated for a simple example. The method as described here will work as long as the spatial region is finite and has homogeneous boundary conditions.

22.1.1 The physical problem

The problem is to find the unsteady pressure field $u(x,t)$ in a pipe with one end closed and the other open to the atmosphere:

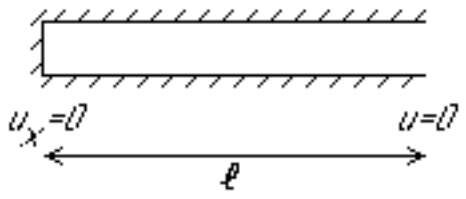


Figure 22.1: Acoustics in a pipe.

22.1.2 The mathematical problem

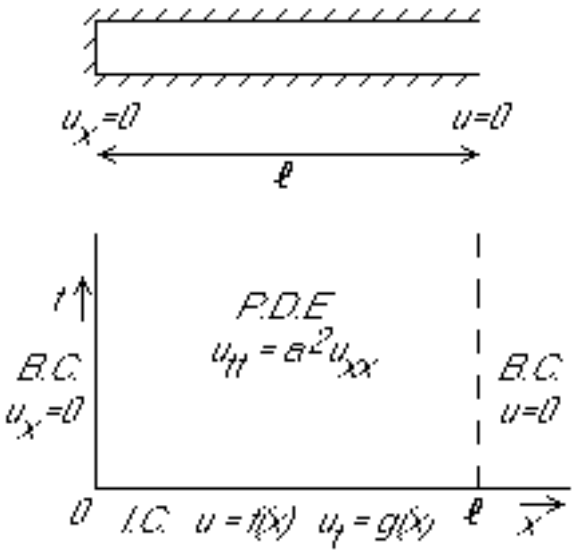


Figure 22.2: Dependent variables.

- There is a finite domain $\bar{\Omega}$ given by $0 \leqslant x \leqslant \ell$
- There is an unknown pressure $u = u(x,t)$ to be found from the wave equation
$$u_{tt} = a^2 u_{xx}$$
- The constant a is the speed of sound a. The equation is only valid for normal acoustics in which the gas velocities are much less than the speed of sound.
- This equation is second order in time so it needs two initial conditions. They are:
$$u(x,0) = f(x) \quad u_t(x,0) = g(x) \qquad \text{for} 0 \leq x \leq \ell$$
- This equation is second order in space so it needs two boundary conditions. Given are one homogeneous Neumann boundary condition at $x = 0$ and one homogeneous Dirichlet condition at $x = \ell$.

22.1.3 Outline of the procedure

We will try to find a solution of this problem in the form

$$u = \sum_n u_n(t) X_n(x)$$

Here the X_n will be cleverly chosen functions called "eigenfunctions." The u_n are coefficients, depending on time, that are found from plugging the expression for u into the partial differential equation and the initial conditions.

There are two big reasons why the X_n must be the eigenfunctions, rather than the u_n:

- You want the independent variable in the eigenfunction to have a finite range. That applies for x, but the time t runs from zero to infinity.
- The time coordinate has initial conditions at $t = 0$. You want the independent variable in the eigenfunctions to have boundary conditions that apply at two different points. In this example, there is one boundary condition at $x = 0$ and the other at $x = \ell$.

(If the spatial range is infinite or semi-infinite, you may be able to use a Fourier transform. Alternatively, you may be able to use a Laplace transform in time.)

22.1.4 Step 1: Find the eigenfunctions

The first step is to find the eigenfunctions X_n.

> *The eigenfunctions are found from requiring that each individual term of the form $u_n(T)X_n(x)$ is capable of satisfying the homogeneous partial differential equation and the homogeneous boundary conditions.*

In this particular example the partial differential equation *is* homogeneous. But even if it is not, i.e. if the partial differential equation was something like

$$u_{tt} = a^2 u_{xx} + q$$

with q a given function of x and t, then still in this step you would use the homogeneous equation

$$u_{tt} = a^2 u_{xx}$$

By convention, $u_n(t)$ is usually written as $T(t)$ and $X_n(x)$ as $X(x)$ in this step. To see when $X(x)T(t)$ satisfies the homogeneous partial differential equation, plug it in:

$$[X(x)T(t)]_{tt} = a^2[X(x)T(t)]_{xx} \qquad \Rightarrow \qquad X(x)T''(t) = a^2 X''(x)T(t)$$

where primes indicate derivatives of the function with respect to its argument.

The trick is now to take the terms containing time to one side of the equation and the terms containing x to the other side.

$$\frac{1}{a^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

This trick is why this solution procedure is called the "method of separation of variables."

While the right hand side, $X''(x)/X(x)$, does not depend on t, you would think that it would depend on the position x; both X and X'' change when x changes. But actually, X''/X does *not* change with x; after all, if we change x, it does nothing to t, so the left hand side does not change. And since the right hand side is the same, it too does not change. So the right hand side does not depend on either x or t; it must be a constant. By convention, we call the constant $-\lambda$:

$$\frac{T''}{a^2T} = \frac{X''}{X} = \text{ constant } = -\lambda$$

If we also require X to satisfy the same homogeneous boundary conditions as u. In this case, that means that at $x = 0$, its x-derivative is zero, and that at $x = \ell$, X itself is zero. So we get the following problem for X:

$$X'' + \lambda X = 0 \qquad X'(0) = 0 \qquad X(\ell) = 0$$

This is a boundary value problem involving an ordinary differential equation. Not a partial differential equation.

Note that the problem is completely homogeneous: $X(x) = 0$ satisfies both the partial differential equation *and* the boundary conditions. This is similar to the eigenvalue problem for vectors $A\vec{v} = \lambda\vec{v}$, which is certainly always true when $\vec{v} = 0$. But for the eigenvalue problem, we are interested in *nonzero* vectors $\vec{v}$ for which $A\vec{v} = \lambda v$. That only occurs for special values λ_1, λ_2, ... of λ.

Similarly, we are interested only in nonzero solutions X(x) of the above ordinary differential equation and boundary conditions. Eigenvalue problems for functions such as the one above are called "Sturm-Liouville problems." The biggest differences from matrix eigenvalue problems are:

- There are infinitely many eigenvalues $\lambda_1, \lambda_2, \ldots$ and corresponding eigenfunctions $X_1(x)$, $X_2(x)$, ... rather than just n eigenvalues and eigenvectors.
- We cannot write a determinant to find the eigenvalues. Instead we must solve the problem using our methods for solving ordinary differential equations.

Fortunately, the above ordinary differential equation is simple: it is a constant coefficient one, so we write its characteristic polynomial:

$$k^2 + \lambda = 0 \qquad \Rightarrow \qquad k = \pm\sqrt{-\lambda} = \pm \mathrm{i}\sqrt{\lambda}$$

We must now find *all* possible eigenvalues λ and all corresponding eigenfunctions that satisfy the required boundary conditions. We must look at all possibilities, one at a time.

1. *Case $\lambda < 0$:*

   Since $k = \pm\sqrt{-\lambda}$

   $$X = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$$

   We try to satisfy the boundary conditions:

   $$X'(0) = 0 = A\sqrt{-\lambda} - B\sqrt{-\lambda} \qquad \Rightarrow \qquad B = A$$

   $$X(\ell) = 0 = A\left(e^{\sqrt{-\lambda}\ell} + e^{-\sqrt{-\lambda}\ell}\right) \qquad \Rightarrow \qquad A = 0$$

   So $A = B = 0$; there are *no* nontrivial solutions for $\lambda < 0$.

2. *Case $\lambda = 0$:*

   Since $k_1 = k_2 = 0$ we have a multiple root of the characteristic equation, and the solution is

   $$X = Ae^{0x} + Bxe^{0x} = A + Bx$$

   We try to satisfy the boundary conditions again:

   $$X'(0) = 0 = B \qquad X(\ell) = 0 = A$$

   So $A = B = 0$; there are again no nontrivial solutions.

3. *Case $\lambda > 0$:*

   Since $k = \pm\sqrt{-\lambda} = \pm\mathrm{i}\sqrt{\lambda}$, the solution of the ordinary differential equation is after cleanup:

   $$X = A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right)$$

   We try to satisfy the first boundary condition:

   $$X'(0) = 0 = A\sqrt{\lambda}$$

   Since we are looking at the case $\lambda > 0$, this can only be true if $A = 0$. So, we need

   $$X = B\cos\left(\sqrt{\lambda}x\right)$$

   We now try to also satisfy the second boundary condition:

   $$X(\ell) = 0 = B\cos\left(\sqrt{\lambda}\ell\right) = 0$$

   For a nonzero solution, B may not be zero, so the cosine must be zero. For positive argument, a cosine is zero at $\frac{1}{2}\pi, \frac{3}{2}\pi, \ldots$, so that our eigenvalues are

   $$\sqrt{\lambda_1} = \frac{\pi}{2\ell}, \sqrt{\lambda_2} = \frac{3\pi}{2\ell}, \sqrt{\lambda_3} = \frac{5\pi}{2\ell}, \ldots$$

The same as for eigenvectors, for our eigenfunctions we *must choose* the one undetermined parameter B. Choosing each $B = 1$, we get the eigenfunctions:

$$X_1 = \cos\left(\frac{\pi x}{2\ell}\right), X_2 = \cos\left(\frac{3\pi x}{2\ell}\right), X_3 = \cos\left(\frac{5\pi x}{2\ell}\right), \ldots$$

The eigenvalues and eigenfunctions have been found. If we want to evaluate them on a computer, we need a general formula for them. You can check that it is:

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4\ell^2} \qquad X_n = \cos\left(\frac{(2n-1)\pi x}{2\ell}\right) \qquad (n = 1, 2, 3, \ldots)$$

Just try a few values for n. We have finished finding the eigenfunctions.

22.1.5 Should we solve the other equation?

If you look back to the beginning of the previous subsection, you may wonder about the function $T(t)$. It satisfied

$$\frac{T''}{a^2 T} = -\lambda$$

Now that we have found the values for λ from the X-problem, we could solve this ordinary differential equation too, and find functions $T_1(t), T_2(t), \ldots$.

However, it is far more straightforward not to do so. Now that the eigenfunctions X_n have been found, the general expression for the solution,

$$u = \sum_n u_n(t) X_n(x)$$

can simply be plugged into the partial differential equation and its initial conditions to find the u_n, completing the solution.

However, most people do solve for the T_n corresponding to each eigenvalue λ_n. If you want to follow the crowd, please keep in mind the following:

1. The values of λ can only be found from the Sturm-Liouville problem for X. The problem for T is *not* a Sturm-Liouville problem and *cannot* produce the correct values for λ.

2. The functions $T(t)$ do *not* satisfy the same initial conditions at time $t = 0$ as u does. That is unlike the X_n which must satisfy the homogeneous boundary conditions.

3. Finding T is useless if the partial differential equation is inhomogeneous; it simply does not work. Unless you add still more artificial tricks to the mix, as the book does.

22.1.6 Step 2: Solve the problem

Now that the eigenfunctions are known, the problem may be solved. To do so, everything needs to be written in terms of the eigenfunctions. And that means everything, including the partial differential equation and the initial conditions.

We first write our solution $u(x,t)$ in terms of the eigenfunctions:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x)$$

The coefficients $u_n(t)$ are called the "Fourier coefficients" of u. The complete sum is called the "Fourier series" for u.

We know our eigenfunctions $X_n(x)$, but not yet our Fourier coefficients $u_n(t)$. In fact, the $u_n(t)$ are what is still missing; if we know the $u_n(t)$, we can find the solution u that we want by doing the sum above. On a computer probably, if we want to get high accuracy. Or just the first few terms by hand, if we accept some numerical error.

Next we write the complete partial differential equation, $u_{tt} = a^2 u_{xx}$, in terms of the eigenfunctions:

$$\sum_{n=1}^{\infty} \ddot{u}_n(t) X_n(x) = a^2 \sum_{n=1}^{\infty} u_n(t) X_n''(x)$$

This equation will *always* simplify; that is how the method of separation of variables works. Look up the differential equation for X_n in the second last subsection; it was

$$X_n''(x) = -\lambda_n X_n(x)$$

Using this expression for X_n'', we can get rid of the x-derivatives in the partial differential equation to get

$$\sum_{n=1}^{\infty} \ddot{u}_n(t) X_n(x) = a^2 \sum_{n=1}^{\infty} \left(-\lambda_n u_n(t)\right) X_n(x)$$

Now if two functions are equal, all their Fourier coefficients must be equal, so we have, for any value of n,

$$\ddot{u}_n(t) = -a^2 \lambda_n u_n(t) \qquad (\text{for } n = 1, 2, 3, \ldots)$$

That no longer contains x at all. The partial differential equation has become a set of ordinary differential equations in t only. And those are much easier to solve than the original partial differential equations. Getting rid of x is really what the method of separation variables does for us.

The above ordinary differential equations can be solved easily. For each value of n it is a constant coefficient equation. So you write the characteristic equation $k^2 = -a^2\lambda_n$. That give $k = \pm ia\sqrt{\lambda}_n$. Then the solution is

$$u_n(t) = C_{1n}e^{ia\sqrt{\lambda}_n t} + C_{2n}e^{-ia\sqrt{\lambda}_n t}$$

or after cleaning up,

$$u_n(t) = D_{1n}\cos\left(a\sqrt{\lambda}_n t\right) + D_{2n}\sin\left(a\sqrt{\lambda}_n t\right)$$

So, we have already found our pressure a bit more precisely:

$$u(x,t) = \sum_{n=1}^{\infty}\left[D_{1n}\cos\left(a\sqrt{\lambda}_n t\right) + D_{2n}\sin\left(a\sqrt{\lambda}_n t\right)\right]X_n(x)$$

but we still need to figure out what the integration constants D_{1n} and D_{2n} are.

To do so, we also write the initial condition $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ in terms of the eigenfunctions:

$$f(x) = \sum_{n=1}^{\infty} f_n X_n(x) \qquad g(x) = \sum_{n=1}^{\infty} g_n X_n(x)$$

Sometimes, when f or g is a simple function, like function 1, students do not write a Fourier series for it. But that does not work.

Using the Fourier series for u, f, and g above, the two initial conditions become

$$\sum_{n=1}^{\infty} D_{1n}X_n(x) = \sum_{n=1}^{\infty} f_n X_n(x)$$

$$\sum_{n=1}^{\infty} a\sqrt{\lambda}_n D_{2n}X_n(x) = \sum_{n=1}^{\infty} g_n X_n(x).$$

The Fourier coefficients must again be equal, so we conclude that the coefficients we are looking for are

$$D_{1n} = f_n \qquad D_{2n} = \frac{g_n}{a\sqrt{\lambda}_n}$$

The Fourier series for u becomes now

$$u(x,t) = \sum_{n=1}^{\infty}\left[f_n\cos\left(a\sqrt{\lambda}_n t\right) + \frac{g_n}{a\sqrt{\lambda}_n}\sin\left(a\sqrt{\lambda}_n t\right)\right]X_n(x)$$

where

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4\ell^2} \qquad X_n = \cos\left(\frac{(2n-1)\pi x}{2\ell}\right)$$

So, if we can find the Fourier coefficients f_n and g_n of functions $f(x)$ and $g(x)$, we are done.

Now $f(x)$ and $g(x)$ are, supposedly, given functions, but how do we find their Fourier coefficients? The answer is the following important formula:

$$\boxed{f_n = \frac{\int_0^l f(x)X_n(x)\mathrm{d}x}{\int_0^l X_n(x)^2\mathrm{d}x}}$$

This is called the "orthogonality relation". Even if f is some simple function like $f = 1$, we still need to do those integrals. Only if $f = 0$ we can immediately say that each Fourier coefficient f_n is zero. The same for g:

$$g_n = \frac{\int_0^l g(x)X_n(x)\mathrm{d}x}{\int_0^l X_n(x)^2\mathrm{d}x}$$

(These formulae work as long as the ordinary differential equation for the X_n is of the form $AX_n'' + BX_n = 0$. What you do for more general differential equations will be covered later.)

We are done! Or at least, we have done as much as we can do until someone tells us the actual functions $f(x)$ and $g(x)$. If they do, we just do the integrals above to find all the f_n and g_n, (maybe analytically or on a computer), and then we can sum the expression for $u(x,t)$ for any x and t that strikes our fancy.

Note that we did not have to do anything with the boundary conditions $u_x(0,t) = 0$ and $u(\ell,t) = 0$. Since every eigenfunction X_n satisfies them, the expression for u above automatically also satisfies these homogeneous boundary conditions.

22.2 Comparison with D'Alembert

The example problem of the previous section was also solved in chapter 21 using D'Alembert. It is interesting to compare the two solutions.

The separation of variables solution took the form:

$$u = \sum_{n=1}^{\infty} \left[f_n \cos\frac{(2n-1)\pi a t}{2\ell} + \frac{2\ell g_n}{(2n-1)\pi a} \sin\frac{(2n-1)\pi a t}{2\ell} \right] \cos\frac{(2n-1)\pi x}{2\ell}$$

Some of its nice features are:

- It shows the natural frequencies (tones) to be $\pi a/2\ell$, $3\pi a/2\ell$,...
- It shows the energy, so the strength, of each harmonic.
- The method is not restricted to the one-dimensional wave equation.

The D'Alembert solution took the form

$$u(x,t) = \frac{\bar{f}(x-at) + \bar{f}(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \bar{g}(\xi)\,\mathrm{d}\xi$$

Some of its nice features are

- The pressure can be evaluated at any point without doing infinite sums.
- It shows how wave fronts propagate.
- It shows regions of influence and dependence.

In short, each method has its advantages and disadvantages.

22.3 Understanding the Procedure

All the different steps in the separation of variable procedure as described may seem totally arbitrary. This section tries to explain why the steps are not arbitrary, but really quite logical. To understand this section does require that you have a good understanding of vectors and linear algebra. Otherwise you may as well skip this.

22.3.1 An ordinary differential equation as a model

Partial differential equations are relatively difficult to understand. Therefore we will instead consider an ordinary differential equation, but for a vector unknown:

$$\vec{u}_{tt} = -A\vec{u}$$

Here $\vec{u}$ is assumed to be a vector of N scalar unknowns u_1, u_2, $\ldots u_N$. Also A is some given constant matrix. Later on, it will be assumed that matrix A is positive definite. That means that A is symmetric and has positive eigenvalues.

The initial conditions are:

$$\vec{u} = \vec{f}, \quad \vec{u}_t = \vec{g} \qquad \text{at} \quad t = 0$$

If you want to solve this problem, the trick is to write $\vec{u}$ in terms of the so-called eigenvectors of matrix A:

$$\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3 + \ldots$$

Here u_1, u_2, $\ldots$ are numerical coefficients that will depend on time. Further e_1, e_2, $\ldots$, are the eigenvectors of matrix A. By definition, these satisfy

$$A\vec{e}_1 = \lambda_1\vec{e}_1 \qquad A\vec{e}_2 = \lambda_2\vec{e}_2 \qquad \ldots$$

where λ_1, λ_2, ... are numbers called the eigenvalues of matrix A. If A is not a defective matrix, a complete set of independent eigenvectors will exist. That then means that the solution $\vec{u}$ of the problem can indeed be written as a combination of the eigenvectors. For simplicity, in this discussion it will be assumed that A is not defective. That is certainly true if A is symmetric; symmetric matrices are never defective.

Now if you substitute the expression for $\vec{u}$ into the ordinary differential equation

$$\vec{u}_{tt} = -A\vec{u}$$

you get

$$\ddot{u}_1\vec{e}_1 + \ddot{u}_2\vec{e}_2 + \ldots = -a^2\lambda_1 u_1\vec{e}_1 - a^2\lambda_2 u_2\vec{e}_2 + \ldots$$

Here the dots in the left hand side indicate time derivatives. Also, in the right hand side, use was made of the fact that $A\vec{e}_i$ is the same as $\lambda_i\vec{e}_i$ for every value of $i = 1, 2, \ldots$.

The above equation can only be true if the coefficients of each individual eigenvector is the same in the left hand side as in the right hand side:

$$\ddot{u}_1 = -\lambda_1 u_1 \qquad \ddot{u}_2 = -\lambda_2 u_2 \qquad \ldots$$

That are ordinary differential equations. You can solve these particular ones relatively easily. The solution is

$$u_1 = C_1\cos(\sqrt{\lambda_1}t) + D_1\sin(\sqrt{\lambda_1}t) \quad u_2 = C_2\cos(\sqrt{\lambda_2}t) + D_2\sin(\sqrt{\lambda_2}t) \quad \ldots$$

Each solution u_n has two integration constants C_n and D_n that still remain unknown. To get them, use the initial conditions

$$\vec{u} = \vec{f}, \quad \vec{u}_t = \vec{g} \qquad \text{at} \quad t = 0$$

where $\vec{f}$ and $\vec{g}$ are given vectors. You need to write these vectors also in terms of the eigenfunctions,

$$\vec{f} = f_1\vec{e}_1 + f_2\vec{e}_2 + \ldots \qquad \vec{g} = g_1\vec{e}_1 + g_2\vec{e}_2 + \ldots$$

Then you can see that what you need is

$$u_1(0) = f_1 \quad \dot{u}_1(0) = g_1 \qquad u_2(0) = f_2 \quad \dot{u}_2(0) = g_2 \qquad \ldots$$

That allows you to figure out the integration constants. So u_1, u_2, ... are now fully determined. And that means that the solution

$$\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3 + \ldots$$

is now fully determined. Just perform the summation at any time you want. So that is it.

The entire process becomes much easier if the matrix A is what is called symmetric. For one, you do not have to worry about the matrix being defective. Symmetric matrices are never defective. Also, you do not have to worry about the eigenvalues possibly being complex numbers. The eigenvalues of a symmetric matrix are always real numbers.

And finally, the eigenvectors of a symmetric matrix can always be chosen to be unit vectors that are mutually orthogonal. In other words, they are like the unit vectors $\hat{\imath}'$, $\hat{\jmath}'$, $\hat{k}'$, ..., of a rotated Cartesian coordinate system.

The orthogonality helps greatly when you are trying to write $\vec{f}$ and $\vec{g}$ in terms of the eigenvectors. For example, you need to write $\vec{f}$ in the form

$$\vec{f} = f_1\vec{e}_1 + f_2\vec{e}_2 + \ldots$$

If the eigenvectors $\vec{e}_1$, $\vec{e}_1$, ..., are orthonormal, then f_1, f_2, ... can simply be found using dot products:

$$f_1 = \vec{e}_1 \cdot \vec{f} \qquad f_2 = \vec{e}_2 \cdot \vec{f} \qquad \ldots$$

Usually, however, you do not normalize the eigenvectors to length one. In that case, you can still write

$$\vec{f} = f_1\vec{e}_1 + f_2\vec{e}_2 + \ldots$$

but now you must find the coefficients as

$$f_1 = \frac{\vec{e}_1 \cdot \vec{f}}{\vec{e}_1 \cdot \vec{e}_1} \qquad f_2 = \frac{\vec{e}_2 \cdot \vec{f}}{\vec{e}_2 \cdot \vec{e}_2} \qquad \ldots$$

In short you must divide by the square length of the eigenvector. The values for $g_1, g_2, \ldots$ can be found similarly.

The next subsections will now show how all of the above carries over directly to the method of separation of variables for simple partial differential equations.

22.3.2 Vectors versus functions

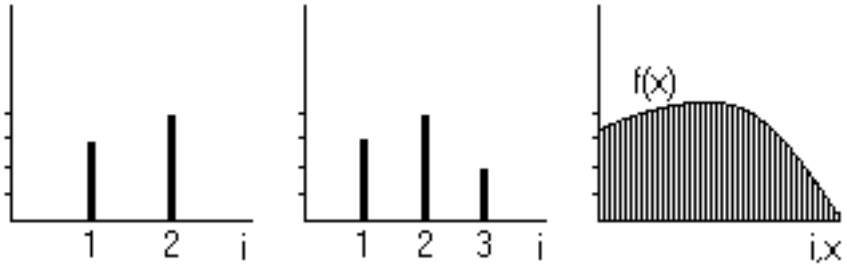
The previous subsection showed how to solve an example ordinary differential for a vector unknown. The procedure had clear similarities to the separation of variables procedure that was used to solve the example partial differential equation in section 22.1.

However, in the ordinary differential equation, the unknown was a vector $\vec{u}$ at any given time t. In the partial differential equation, the unknown $u(x,t)$ was a function of x at any given time. Also, the initial conditions for the ordinary differential equation were given vectors $\vec{f}$ and $\vec{g}$. For the partial differential equation, the initial conditions were given functions $f(x)$ and $g(x)$. The ordinary

differential equation problem had eigenvectors $\vec{e}_1, \vec{e}_2, \ldots$. The partial differential equation problem had eigenfunctions $X_1(x), X_2(x), \ldots$.

The purpose of this subsection is to illustrate that it does not make that much of a difference. The differences between vectors and functions are not really as great as they may seem.

Let's start with a vector in two dimensions, like say the vector $\vec{v} = (3, 4)$. You can represent this vector graphically as a point in a plane, but you can also represent it as the 'spike function', as in the left-hand sketch below:



The first coefficient, v_1, is 3. That corresponds to a spike of height of 3 when the subscript, call it i, is 1. The second coefficient $v_2 = 4$, so there is a spike of height 4 at $i = 2$. Similarly, the three-dimensional vector $\vec{v} = (3, 4, 2)$ can be graphed as the three-spike function in the middle figure. If you keep adding more dimensions, going to the limit of infinite-dimensional space, the spike graph v_i approaches a function f with a continuous coordinate x instead of i.

Phrased differently, you can think of a function $f(x)$ as an infinite column vector of numbers, with the numbers being the successive values of $f(x)$. In this way, vectors become functions. And vector analysis turns into functional analysis.

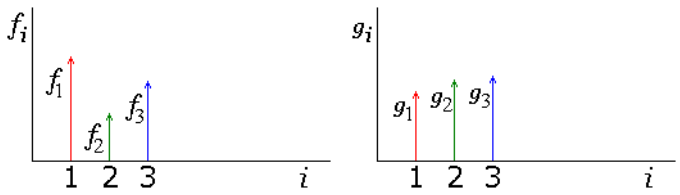
22.3.3 The inner product

You are not going to do much with vectors without the dot product. The dot product makes it possible to find the length of a vector, by multiplying the vector by itself and taking the square root. The dot product is also used to check if two vectors are orthogonal: if their dot product is zero, they are orthogonal. In this subsection, the dot product is generalized to functions.

The usual dot product of two arbitrary vectors $\vec{f}$ and $\vec{g}$ can be found by multiplying components with the same index i together and summing that:

$$\vec{f} \cdot \vec{g} \equiv f_1 g_1 + f_2 g_2 + f_3 g_3$$

The below figure shows multiplied components using equal colors.



The three term sum above can be written more compactly as:

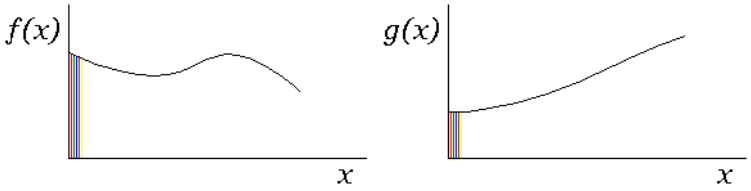
$$\vec{f} \cdot \vec{g} \equiv \sum_{\text{all } i} f_i g_i$$

The Σ is called the "summation symbol."

The dot (or "inner") product of functions is defined in exactly the same way as for vectors, by multiplying values at the same x position together and summing. But since there are infinitely many x-values, the sum becomes an integral:

$$(f, g) = \int_{\text{all } x} f(x) g(x) \, \mathrm{d}x \tag{22.1}$$

It is conventional to put a comma between the functions instead of a dot like for vectors. Also, people like to enclose the functions inside parentheses. But the idea is the same, as illustrated in the figure below:



As an example, the ordinary differential equation model problem involved a given initial condition $\vec{f}$ for $\vec{u}$. To solve the problem, vector $\vec{f}$ had to be written in the form

$$\vec{f} = f_1 \vec{e}_1 + f_2 \vec{e}_2 + \ldots$$

Here the vectors $\vec{e}_1, \vec{e}_2, \ldots$ were the eigenvectors of the matrix A in the problem. The coefficients $f_1, f_2, \ldots$ could be found using dot products:

$$f_1 = \frac{\vec{e}_1 \cdot \vec{f}}{\vec{e}_1 \cdot \vec{e}_1} \qquad f_2 = \frac{\vec{e}_2 \cdot \vec{f}}{\vec{e}_2 \cdot \vec{e}_2} \qquad \ldots$$

This can be done this way as long as the eigenvectors are orthogonal. The dot product between any two different eigenvectors must be zero. The eigenvectors were indeed orthogonal, because it was assumed that the matrix A in the problem was symmetric.

Similarly, the partial differential equation problem of section 22.1 involved a given initial condition $f(x)$ for $u(x,t)$. To solve the problem, this initial condition had to be written in the form:

$$f(x) = f_1 X_1(x) + f_2 X_2(x) + \ldots$$

Here $X_1(x), X_2(x), \ldots$ were the so-called eigenfunctions found in the separation of variables procedure. The coefficients $f_1, f_2, \ldots$ can be found using inner products

$$f_1 = \frac{(X_1, f)}{(X_1, X_1)} \qquad f_2 = \frac{(X_2, f)}{(X_2, X_2)} \qquad \ldots$$

This can be done this way as long as the eigenfunctions are orthogonal. The inner product between any two different eigenfunctions must be zero. The next section explains why that is indeed the case.

22.3.4 Matrices versus operators

This section compares the solution procedure for the ordinary differential equation

$$\vec{u}_{tt} = -A\vec{u} \qquad \text{where } A \text{ is a matrix}$$

to that for the partial differential equation

$$u_{tt} = -Lu \quad \text{where } L = -a^2 \frac{\partial^2}{\partial x^2}$$

You may wonder whether that makes any sense. A matrix is basically a table of numbers. The "linear operator" L is shorthand for "take two derivatives and multiply the resulting function by the constant $-a^2$."

But the difference between matrices and operators is not as great as it seems. One way of defining a matrix A is as a thing that, given a vector $\vec{u}$, can produce a different vector $A\vec{u}$;

$$\vec{u}(t) \quad \xrightarrow{\text{matrix } A} \quad A\vec{u}(t)$$

Similarly you can define an operator L as a thing that, given a function u, produces another function Lu:

$$u(x,t) \quad \xrightarrow{\text{operator } L} \quad Lu(x;t) = -a^2 u_{xx}(x,t)$$

After all, taking derivatives of functions simply produces another function. And multiplying a function by a constant simply produces another function.

Since it was already seen that vectors and functions are closely related, then so are matrices and operators.

Like matrices have eigenvectors, linear operators have eigenfunctions. In particular, section 22.1 found the appropriate eigenfunctions of the operator above to be

$$X_n = \cos\left(\frac{(2n-1)\pi x}{2\ell}\right) \qquad \text{for } n = 1, 2, 3, \ldots$$

(This also depended on the boundary conditions, but that point will be ignored for now.) You can check by differentiation that for these eigenfunctions

$$LX_n = -a^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} X_n = \lambda_n X_n \qquad \text{where } \lambda_n = a^2 \left(\frac{(2n-1)\pi}{2\ell}\right)^2$$

So they are indeed eigenfunctions of operator L.

But, as the previous subsection pointed out, it was also assumed that these eigenfunctions are orthogonal. And that is not automatic. For a matrix the eigenvectors can be taken to be orthogonal if the matrix is symmetric. Similarly, for an operator the eigenfunctions can be taken to be orthogonal if the operator is symmetric.

But how do you check that for an operator? For a matrix, you simply write down the matrix as a table of numbers and check that the rows of the table are the same as the columns. You cannot do that with an operator. But there is another way. A matrix is also symmetric if for any two vectors $\vec{f}$ and $\vec{g}$,

$$\vec{f} \cdot (A\vec{g}) = (A\vec{f}) \cdot \vec{g}$$

In other words, symmetric matrices can be taken to the other side in a dot product. (In terms of linear algebra

$$\vec{f} \cdot (A\vec{g}) = \vec{f}^{\,T} A\vec{g} \qquad (A\vec{f}) \cdot \vec{g} = (A\vec{f})^T \vec{g} = \vec{f}^{\,T} A^T \vec{g}$$

where the superscript T indicates transpose. For the two expression always to be the same requires $A = A^T$.)

Symmetry for operators can be checked similarly by whether they can be taken to the other side in inner products involving any two functions f and g:

$$(f, Lg) = (Lf, g) \qquad \text{iff } L \text{ is symmetric.}$$

To check that for the operator above, write out the first inner product:

$$(f, Lg) = -a^2 \int_0^\ell f(x) g''(x)\, \mathrm{d}x$$

Now use integration by parts twice to get

$$(f, Lg) = -a^2 \int_0^\ell f''(x) g(x)\, \mathrm{d}x = (Lf, g)$$

So operator L is symmetric and therefore it has orthogonal eigenfunctions. (It was assumed in the integrations by parts that the functions f and g satisfy the homogeneous boundary conditions at $x = 0$ and $x = \ell$ given in section 22.1. All functions of interest here must satisfy them.)

22.3.5 Some limitations

Some limitations to the similarity between vectors and functions should be noted.

One difference is that the functions in partial differential equations must normally satisfy boundary conditions. The ones in the example problem were

$$u_x(0,t) = 0 \qquad u(\ell,t) = 0$$

Usually you do not have boundary conditions on vectors. But in principle you could create an analogue to the first boundary condition by demanding that the first component of vector $\vec{u}$ is the same as the second. An analogue to the second boundary condition would be that the very last component of vector $\vec{u}$ would be zero.

As long as matrix A respects these boundary conditions, there is no problem with that. In terms of linear algebra, you would be working in a subspace of the complete vector space; the subspace of vectors that satisfy the boundary conditions.

There is another problem with the analogy between vectors and functions. Consider the initial condition $\vec{f}$ for the solution $\vec{u}$ of the ordinary differential equation. You can give the components of $\vec{f}$ completely arbitrary values and you will still get a solution for $\vec{u}$.

But now consider the initial condition $f(x)$ for the solution $u(x,t)$ of the ordinary differential equation. If you simply give a random value to the function f at every individual value of x, then the function will not be differentiable. The partial differential equation for such a function will then make no sense at all. For functions to be meaningful in the solution of a partial differential equation, they must have enough smoothness that derivatives make some sense.

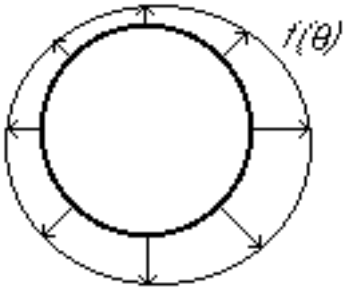
Note that this does not mean that the initial condition cannot have some singularities, like kinks or jumps, say. Normally, you are OK if the initial conditions can be approximated in a meaningful sense by a sum of the eigenfunctions of the problem. Because the functions that can be approximated in this way exclude the extremely singular ones, a partial differential equation will always work in a subspace of all possible functions. A subspace of reasonably smooth functions. Often when you see partial differential equations in literature, they also list the subspace in which it applies. That is beyond the scope of this book.

22.4 Handling Periodic Boundary Conditions

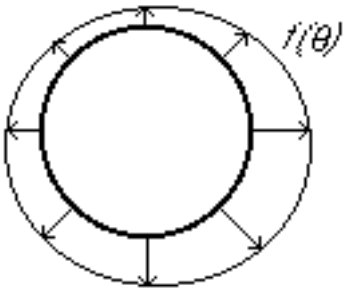
In this section the method of separation of variables will be applied to a problem in polar coordinates. The selected problem turns out to have two eigenfunctions for each eigenvalue other than the lowest.

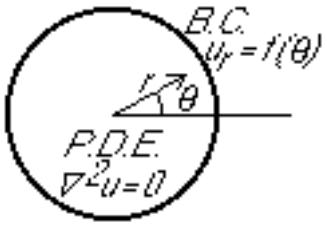
22.4.1 The physical problem

The problem is to find the ideal flow in a unit circle if the normal (radial) velocity on the perimeter is known.



22.4.2 The mathematical problem





- Finite domain $\bar{\Omega}$: $0 \leqslant r \leqslant 1, 0 \leqslant \vartheta < 2\pi$
- Unknown velocity potential $u = u(r, \vartheta)$
- Elliptic equation
$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\vartheta\vartheta} = 0$$
- One Neumann boundary condition at $r = 1$.

22.4.3 Outline of the procedure

We will try to find a solution of this problem in the form

$$u = \sum_n R_n(r)\Theta_n(\theta)$$

Here the Θ_n will be the eigenfunctions.

The reason to take the Θ_n as the eigenfunctions and not the R_n is because separation of variables needs homogeneous boundary conditions. The r direction has an inhomogeneous boundary condition $u_r(1,\theta) = f(\theta)$ at $r = 1$.

22.4.4 Step 1: Find the eigenfunctions

This follows the same procedures as in the first example. We substitute a single term $u = R(r)\Theta(\vartheta)$ into the homogeneous partial differential equation

$$u_{rr} + frac1ru_r + \frac{1}{r^2}u_{\vartheta\vartheta} = 0$$

That gives:

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

which separates into

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \text{ constant } = \lambda$$

Make sure that *all* r terms are at the same side of the equation! Some students leave an r^2 in the θ side.

Now which ordinary differential equation gives us the Sturm-Liouville problem, and thus the eigenvalues? Not the one for $R(r)$; u has an *inhomogeneous* boundary condition on the perimeter $r = 1$. Eigenvalue problems must be homogeneous; they simply don't work if anything is inhomogeneous.

We are in luck with $\Theta(\vartheta)$ however. The unknown $u(r,\vartheta)$ has "periodic" boundary conditions in the ϑ-direction. If ϑ increases by an amount 2π, $u(r,\vartheta)$ returns to exactly the same values as before: it is a "periodic function" of ϑ. Periodic boundary conditions are homogeneous: the zero solution satisfies them. After all, zero remains zero however many times you go around the circle.

The Sturm-Liouville problem for Θ is:

$$-\Theta'' = \lambda\Theta$$

$$\Theta(0) = \Theta(2\pi) \qquad \Theta'(0) = \Theta'(2\pi)$$

Note that for a second order ordinary differential equation, we need two boundary conditions. So we wrote down that both Θ, as well as its derivative are exactly the same at $\vartheta = 0$ and 2π.

Pretend that we do not know the solution of this Sturm-Liouville problem! Write the characteristic equation of the ordinary differential equation:

$$k^2 + \lambda = 0 \qquad \Rightarrow \qquad k = \pm \mathrm{i}\sqrt{\lambda}$$

Lets look at all possibilities:

1. *Case $\lambda = 0$:*

   Since $k_1 = k_2 = 0$:

   $$\Theta = A + B\vartheta$$

   Boundary conditions:

   $$\Theta(0) = \Theta(2\pi) \qquad \Rightarrow \qquad A = A + B2\pi$$

   That can only be true if $B = 0$. Then the second boundary condition is

   $$\Theta'(0) = \Theta'(2\pi) \qquad \Rightarrow \qquad 0 = 0$$

   hence $\Theta = A$. No undetermined constants in eigenfunctions! Simplest is to choose $A = 1$:

   $$\Theta_0(\vartheta) = 1$$

2. *Case $\lambda \neq 0$:*

   We will be lazy and try to do the cases of positive and negative λ at the same time. For positive λ, the cleaned-up solution is

   $$\Theta = A\cos\left(\sqrt{\lambda}\vartheta\right) + B\sin\left(\sqrt{\lambda}\vartheta\right)$$

   This also applies for negative λ, except that the square roots are then imaginary.

   Lets write down the boundary conditions first:

   $$\Theta(0) = \Theta(2\pi) \quad \Rightarrow \quad A = A\cos\left(\sqrt{\lambda}2\pi\right) + B\sin\left(\sqrt{\lambda}2\pi\right)$$

   $$\Theta'(0) = \Theta'(2\pi) \quad \Rightarrow \quad B\sqrt{\lambda} = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}2\pi\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}2\pi\right)$$

   These two equations are a bit less simple than the ones we saw so far. Rather than directly trying to solve them and make mistakes, this time let us write out the augmented matrix of the system of equations for A and B:

   $$\left(\begin{array}{cc|c} 1-\cos\left(\sqrt{\lambda}2\pi\right) & -\sin\left(\sqrt{\lambda}2\pi\right) & 0 \\ \sin\left(\sqrt{\lambda}2\pi\right) & 1-\cos\left(\sqrt{\lambda}2\pi\right) & 0 \end{array}\right)$$

Any nontrivial solution must be nonunique (since zero is also a solution). So the determinant of the matrix must be zero, which is:

$$1 - 2\cos\left(\sqrt{\lambda}2\pi\right) + \cos^2\left(\sqrt{\lambda}2\pi\right) + \sin^2\left(\sqrt{\lambda}2\pi\right) = 0$$

or

$$\cos\left(\sqrt{\lambda}2\pi\right) = 1$$

A cosine is only equal to 1 when its argument is an integer multiple of 2π. Hence the only possible eigenvalues are

$$\sqrt{\lambda_1} = 1 \quad \sqrt{\lambda_2} = 2 \quad \sqrt{\lambda_3} = 3 \quad \ldots$$

If λ is negative, $\cos\left(i\sqrt{-\lambda}2\pi\right) = \cosh\left(\sqrt{-\lambda}2\pi\right)$ which is always greater than one for nonzero λ.

For the found eigenvalues, the system of equations for A and B becomes:

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Hence we can find *neither* A or B; there are *two* undetermined constants in the solution:

$$\Theta_n = A\cos(n\vartheta) + B\sin(n\vartheta)$$

We had this situation before with eigenvector in the case of double eigenvalues, where an eigenvalue gave rise two linearly independent eigenvectors. Basically we have the same situation here: each eigenvalue is double. Similar to the case of eigenvectors of symmetric matrices, here we want two linearly independent, and more specifically, orthogonal eigenfunctions. A suitable pair is

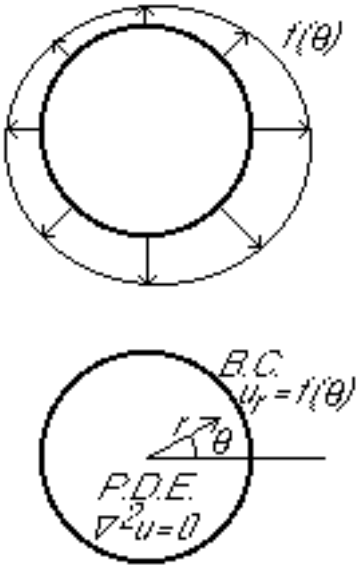
$$\Theta_n^1(\vartheta) = \cos(n\vartheta)$$

$$\Theta_n^2(\vartheta) = \sin(n\vartheta)$$

We can now tabulate the complete set of eigenvalues and eigenfunctions now as:

$$\begin{array}{ccc} \lambda_0 = 0 & \Theta_0 = 1 & \\ \lambda_1 = 1 & \Theta_1^1 = \cos(\vartheta) & \Theta_1^2 = \sin(\vartheta) \\ \lambda_2 = 4 & \Theta_2^1 = \cos(2\vartheta) & \Theta_2^2 = \sin(2\vartheta) \\ \lambda_3 = 9 & \Theta_3^1 = \cos(3\vartheta) & \Theta_3^2 = \sin(3\vartheta) \\ \lambda_4 = 16 & \Theta_4^1 = \cos(4\vartheta) & \Theta_4^2 = \sin(4\vartheta) \\ \vdots & \vdots & \vdots \end{array}$$

22.4.5 Step 2: Solve the problem



We will again expand all variables in the problem in a Fourier series. Let's start with the function $f(\vartheta)$ giving the outflow through the perimeter.

$$f(\vartheta) = f_0 + \sum_{n=1}^{\infty} f_n^1 \cos(n\vartheta) + \sum_{n=1}^{\infty} f_n^2 \sin(n\vartheta)$$

This is the way a Fourier series of a periodic function with period 2π always looks.

Since $f(\vartheta)$ is supposedly known, we should again be able to find its Fourier coefficients using orthogonality. The formulae are as before.

$$f_0 = \frac{\int_{\vartheta=0}^{2\pi} f(\vartheta) 1 \,\mathrm{d}\vartheta}{\int_{\vartheta=0}^{2\pi} 1^2 \,\mathrm{d}\vartheta}$$

(the bottom is of course equal to 2π,)

$$f_n^1 = \frac{\int_{\vartheta=0}^{2\pi} f(\vartheta) \cos(n\vartheta) \,\mathrm{d}\vartheta}{\int_{\vartheta=0}^{2\pi} \cos^2(n\vartheta) \,\mathrm{d}\vartheta} \quad (n = 1, 2, \ldots)$$

$$f_n^2 = \frac{\int_{\vartheta=0}^{2\pi} f(\vartheta) \sin(n\vartheta) \,\mathrm{d}\vartheta}{\int_{\vartheta=0}^{2\pi} \sin^2(n\vartheta) \,\mathrm{d}\vartheta} \quad (n = 1, 2, \ldots)$$

(the bottoms are equal to π.)

Since I hate typing big formulae, allow me to write the Fourier series for $f(\vartheta)$ much more compactly as

$$f(\vartheta) = \sum_{n,i}^{\infty} f_n^i \Theta_n^i(\vartheta)$$

where $\Theta_n^1 = \cos(n\vartheta)$ and $\Theta_n^2 = \sin(n\vartheta)$. Also, all three formulae for the Fourier coefficients can be summarized as

$$f_n^i = \frac{\int_{\vartheta=0}^{2\pi} f(\vartheta)\Theta_n^i(\vartheta)\,\mathrm{d}\vartheta}{\int_{\vartheta=0}^{2\pi} \Theta_n^{i2}(\vartheta)\,\mathrm{d}\vartheta} \quad (n = 0, 1, 2, \ldots; i = 1, 2)$$

For $n = 0$, only the value $i = 1$ is relevant, of course; $\Theta_0^1 = \cos(0\vartheta) = 1 = \Theta_0$. There is no $\Theta_0^2 = \sin(0\vartheta) = 0$.

Next, let's write the unknown $u(r, \vartheta)$ as a compact Fourier series:

$$u(r, \vartheta) = \sum_{n,i} u_n^i(r)\Theta_n^i(\vartheta)$$

We put this into partial differential equation $u_{rr} + u_r/r + u_{\vartheta\vartheta}/r^2 = 0$:

$$\sum_{n,i} u_n^i(r)''\Theta_n^i(\vartheta) + \frac{1}{r}\sum_{n,i} u_n^i(r)'\Theta_n^i(\vartheta) + \frac{1}{r^2}\sum_{n,i} u_n^i(r)\Theta_n^i(\vartheta)'' = 0$$

Using the Sturm-Liouville equation $\Theta_n^i(\vartheta)'' = -\lambda\Theta_n^i(\vartheta)$, where λ was found to be n^2, this simplifies to

$$\sum_{n,i} u_n^i(r)''\Theta_n^i(\vartheta) + \frac{1}{r}\sum_{n,i} u_n^i(r)'\Theta_n^i(\vartheta) - \frac{1}{r^2}\sum_{n,i} n^2 u_n^i(r)\Theta_n^i(\vartheta) = 0$$

We get the following ordinary differential equation for $u_n^i(r)$:

$$u_n^i(r)'' + \frac{1}{r}u_n^i(r)' - \frac{n^2}{r^2}u_n^i(r) = 0$$

or multiplying by r^2:

$$r^2 u_n^i(r)'' + r u_n^i(r)' - n^2 u_n^i(r) = 0$$

This is *not* a constant coefficient equation. Writing down a characteristic equation is no good.

Fortunately, we have seen this one before: it is the Euler equation. You solved that one by changing to the logarithm of the independent variable, in other words, by rewriting the equation in terms of

$$\rho \equiv \ln r$$

instead of r. The r-derivatives can be converted as in:

$$\frac{\mathrm{d}u_n^i}{\mathrm{d}r} = \frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{1}{r}$$

$$\frac{\mathrm{d}^2 u_n^i}{\mathrm{d}r^2} = \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{1}{r}\right] = \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\right]\frac{1}{r} - \frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{1}{r^2}$$
$$= \frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\right]\frac{\mathrm{d}\rho}{\mathrm{d}r}\frac{1}{r} - \frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{1}{r^2} = \frac{\mathrm{d}^2 u_n^i}{\mathrm{d}\rho^2}\frac{1}{r^2} - \frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{1}{r^2}$$

The ordinary differential equation becomes in terms of ρ:

$$\frac{\mathrm{d}^2 u_n^i}{\mathrm{d}\rho^2} - n^2 u_n^i = 0$$

This is now a constant coefficient equation, so we can write the characteristic polynomial, $k^2 - n^2 = 0$, or $k = \pm n$, which has a double root when $n = 0$. So we get for $n = 0$:

$$u_0^1 = A_0^1 + B_0^1 \rho = A_0^1 + B_0^1 \ln r$$

while for $n \neq 0$:

$$u_n^i = A_n^i e^{n\rho} + B_n^i e^{-n\rho} = A_n^i r^n + B_n^i r^{-n}$$

Now both $\ln r$ as well as r^{-n} are infinite when $r = 0$. But that is in the middle of our flow region, and the flow is obviously not infinite there. So from the 'boundary condition' at $r = 0$ that the flow is not singular, we conclude that all the B-coefficients must be zero. Since $r^0 = 1$, all coefficients are of the form $A_n^i r^n$, including the one for $n = 0$.

Hence our solution can be more precisely written

$$u(r, \vartheta) = \sum_{n,i} A_n^i r^n \Theta_n^i(\vartheta)$$

Next we expand the boundary condition $u_r(1, \vartheta) = f(\vartheta)$ at $r = 1$ in a Fourier series:

$$\sum_{n,i} n A_n^i \Theta_n^i(\vartheta) = \sum_{n,i} f_n^i \Theta_n^i(\vartheta)$$

producing

$$n A_n^i = f_n^i$$

For $n = 0$, we see immediately that A_0 can be anything, but we need $f_0 = 0$ for a solution to exist! According to the orthogonality relationship for f_0, this requires:

$$\int_0^{2\pi} f(\vartheta)\,\mathrm{d}\vartheta = 0$$

Are you surprised that the net outflow through the perimeter must be zero for this steady flow?

For nonzero n:

$$A_n^i = \frac{f_n^i}{n}$$

and our solution becomes

$$u = A_0 + \sum_{n,i} f_n^i \frac{r^n}{n} \Theta_n^i(\vartheta)$$

where A_0 can be anything.

22.4.6 Summary of the solution

Let's summarize our results, and write the eigenfunctions out in terms of the individual sines and cosines.

Required for a solution is that:

$$\int_0^{2\pi} f(\vartheta)\,\mathrm{d}\vartheta = 0$$

Then:

$$f_n^1 = \frac{1}{\pi}\int_{\vartheta=0}^{2\pi} f(\vartheta)\cos(n\vartheta)\,\mathrm{d}\vartheta \quad (n = 1, 2, \ldots)$$

$$f_n^2 = \frac{1}{\pi}\int_{\vartheta=0}^{2\pi} f(\vartheta)\sin(n\vartheta)\,\mathrm{d}\vartheta \quad (n = 1, 2, \ldots)$$

$$u = A_0 + \sum_{n=1}^{\infty}\left\{ f_n^1 \frac{r^n}{n}\cos(n\vartheta) + f_n^2 \frac{r^n}{n}\sin(n\vartheta)\right\}$$

where A_0 can be anything.

22.5 Finding the Green's function

The previous section found the solution to the ideal flow in a circle in the form

$$u = A_0 + \sum_{n=1}^{\infty}\left\{ f_n^1 \frac{r^n}{n}\cos(n\vartheta) + f_n^2 \frac{r^n}{n}\sin(n\vartheta)\right\}$$

We can write it directly in terms of the given $f(x)$ if we substitute in the expressions for the Fourier coefficients:

$$u = A_0 + \sum_{n=1}^{\infty}\int_0^{2\pi} f(\phi)\cos(n\phi)\,\mathrm{d}\phi\frac{r^n}{n\pi}\cos(n\vartheta) + \int_0^{2\pi} f(\phi)\sin(n\phi)\,\mathrm{d}\phi\frac{r^n}{n\pi}\sin(n\vartheta)$$

We can clean it up by combining terms and interchanging integration and summation:

$$u = A_0 + \int_0^{2\pi}\sum_{n=1}^{\infty}\left\{\frac{r^n}{n\pi}[\cos(n\phi)\cos(n\vartheta) + \sin(n\phi)\sin(n\vartheta)]\right\} f(\phi)\,\mathrm{d}\phi$$

$$u = A_0 + \int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} \frac{r^n}{n\pi} \cos(n[\vartheta - \phi]) \right\} f(\phi)\, \mathrm{d}\phi$$

This we can clean up even more by giving a name to the function within the curly brackets:

$$u = A_0 + \int_0^{2\pi} G(r, \vartheta - \phi) f(\phi)\, \mathrm{d}\phi$$

Nice, not? We can even simplify G by converting to complex exponentials and differentiating:

$$G(r,\vartheta) = \sum_{n=1}^{\infty} \frac{r^n}{n\pi} \cos(n\vartheta) = \sum_{n=1}^{\infty} \left\{ \frac{r^n}{2n\pi} e^{in\vartheta} + \frac{r^n}{2n\pi} e^{-in\vartheta} \right\}$$

$$2\pi \frac{\partial G}{\partial r} = \sum_{n=1}^{\infty} \left\{ r^{-1} \left(re^{i\vartheta} \right)^n + r^{-1} \left(re^{-i\vartheta} \right)^n \right\} = \frac{e^{i\vartheta}}{1 - re^{i\vartheta}} + \frac{e^{-i\vartheta}}{1 - re^{-i\vartheta}}$$

The last equation applies because the sums are geometric series.

Integrating and cleaning up produces

$$G(r,\vartheta) = -\frac{1}{2\pi} \ln\left(1 - 2r\cos(\vartheta) + r^2\right)$$

So, we finally have the following Poisson-type integral expression giving u directly in terms of the given $f(\vartheta)$, with no sums:

$$u(r,\vartheta) = A_0 - \frac{1}{2\pi} \int_0^{2\pi} \ln\left(1 - 2r\cos(\vartheta - \phi) + r^2\right) f(\phi)\, \mathrm{d}\phi$$

Neat!

22.6 Inhomogeneous boundary conditions

The method of separation of variables needs homogeneous boundary conditions. More precisely, the eigenfunctions *must* have homogeneous boundary conditions. (Even if in a set of functions each function satisfies the given inhomogeneous boundary conditions, a combination of them will in general not do so.)

In the previous example, this problem could be circumvented by choosing θ instead of r as the variable of the eigenfunctions. For the example in this section, however, this does not work.

22.6.1 The physical problem

The problem is to find the unsteady temperature distribution in a bar for any arbitrary position x and time t. The initial temperature distribution at time

zero equals a given function $f(x)$. The heat flux out of the left end equals a given function $g_0(t)$, and the temperature of the right end a given function $g_1(t)$. Heat is added to the bar from an external source at a rate described by a given function $q(x)$.

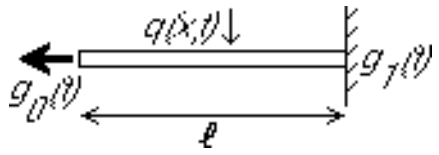


Figure 22.3: Heat conduction in a bar.

22.6.2 The mathematical problem

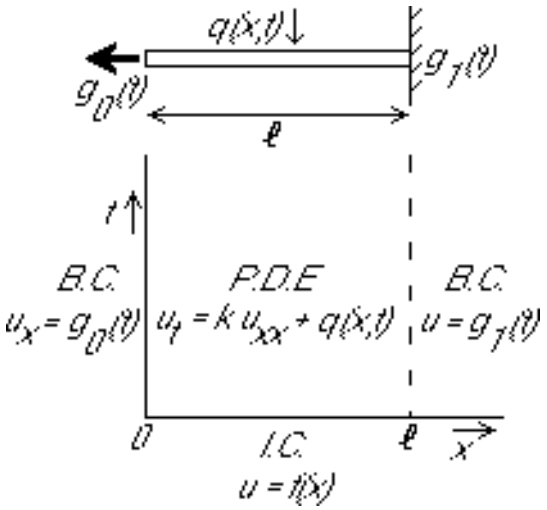


Figure 22.4: Heat conduction in a bar.

- Finite domain $\bar{\Omega}$: $0 \leqslant x \leqslant \ell$
- Unknown temperature $u = u(x, t)$
- Constant κ, so a linear constant coefficient partial differential equation.
- Parabolic
- Inhomogeneous
- One initial condition
- One Neumann boundary condition

- One Dirichlet boundary condition
- All of f, g_0, g_1, and q are given functions.

22.6.3 Outline of the procedure

We would like to use separation of variables to write the solution in a form that looks roughly like:

$$u(x,t) = \sum_n u_n(t) X_n(x)$$

Here the X_n would be the eigenfunctions.

The u_n cannot be eigenfunctions since the time axis is semi-infinite. Also, Sturm-Liouville problems require boundary conditions at both ends, not initial conditions.

Unfortunately, eigenfunctions must have homogeneous boundary conditions. So if u was simply written as a sum of eigenfunctions, it could not satisfy inhomogeneous boundary conditions.

Fortunately, we can apply a trick to get around this problem. The trick is to write u as the sum of a function u_0 that satisfies the inhomogeneous boundary conditions plus a remainder v:

$$u(x,t) = u_0(x,t) + v(x,t)$$

Since u_0 produces the inhomogeneous term in the boundary conditions, the remainder v satisfies homogeneous boundary conditions. Therefore v can be written as

$$v(x,t) = \sum_n v_n(t) X_n(x)$$

using separation of variables. Add u_0 to get u.

22.6.4 Step 0: Fix the boundary conditions

The first thing to do is find a function u_0 that satisfies the same boundary conditions as u. In particular, u_0 must satisfy:

$$u_{0,x}(0,t) = g_0(t) \qquad u_0(\ell,t) = g_1(t)$$

The function u_0 *does not* have to satisfy the either the partial differential equation or the initial condition. That allows you to take something simple for it. The choice is not unique, but you want to select something simple.

A function that is linear in x,

$$u_0(x,t) = A(t) + B(t)x$$

is surely the simplest possible choice. In this example, it works fine too.

Plug this expression for u_0 into the boundary conditions for u,

$$u_x(0,t) = g_0(t) \qquad u(\ell,t) = g_1(t)$$

That produces the requirements

$$B(t) = g_0(t) \qquad A(t) + B(t)\ell = g_1(t)$$

The solution is $B(t) = g_0(t)$ and $A(t) = g_1(t) - B(t)\ell$. So our u_0 is

$$u_0(x,t) = g_1(t) + g_0(t)(x - \ell)$$

Keep track of what we know, and what we do not know. Since we (supposedly) have been given functions $g_0(t)$ and $g_1(t)$, function u_0 is from now on considered a *known* quantity, as given above.

You could use something more complicated than a linear function if you like to make things difficult for yourself. Go ahead and use $A(t)\text{erf}(x) + B(t)J_0(x)$ if you really love to integrate error functions and Bessel functions. It will work. I prefer a linear function myself, though. (For some problems, you may need a quadratic instead of a linear function.)

Under certain conditions, there may be a better choice than a low order polynomial in x. If the problem has steady boundary conditions and a simple steady solution, go ahead and take u_0 to be that steady solution. It will work great. However, in the example here the boundary conditions are not steady; we are assuming that g_0 and g_1 are arbitrary given functions of time.

Next, having found u_0, define a new unknown v as the remainder when u_0 is subtracted from u:

$$v \equiv u - u_0$$

We now solve the problem by finding v. When we have found v, we simply add u_0, already known, back in to get u.

To do so, first, of course, we need the problem for v to solve. We get it from the problem for u by everywhere replacing u by $u_0 + v$. Let's take the picture of the problem for u in front of us and start converting.

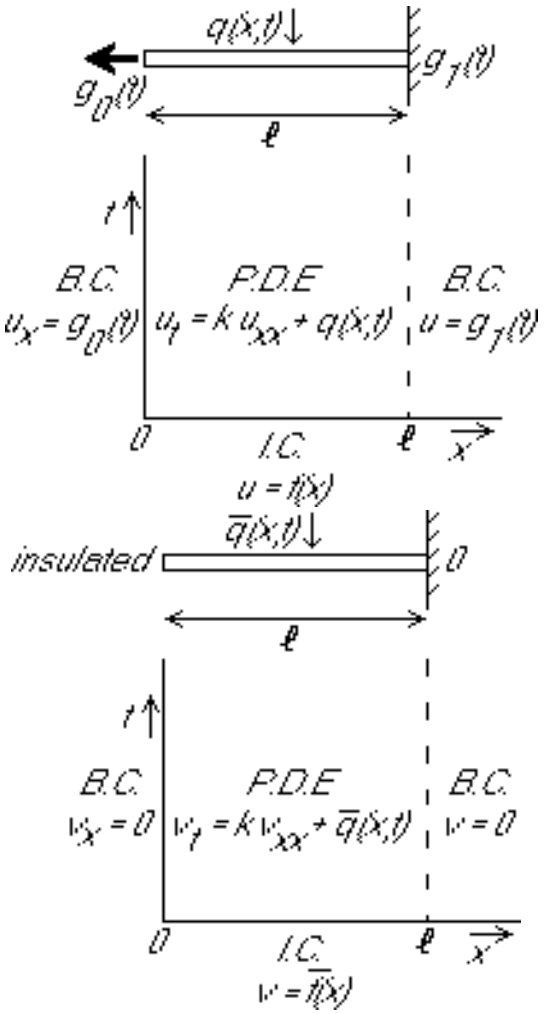


Figure 22.5: Heat conduction in a bar.

First take the boundary conditions at $x = 0$ and $x = \ell$:

$$u_x(0,t) = g_0(t) \qquad u(\ell,t) = g_1(t)$$

Replacing u by $u_0 + v$:

$$u_{0x}(0,t) + v_x(0,t) = g_0(t) \qquad u_0(\ell,t) + v(\ell,t) = g_1(t)$$

But since by construction $u_{0x}(0,t) = g_0$ and $u_0(\ell,t) = g_1$,

$$v_x(0,t) = 0 \qquad v(\ell,t) = 0$$

Note the big thing: while the boundary conditions for v are similar to those for u, they are *homogeneous*. We will get a Sturm-Liouville problem in the x-direction for v where we did not for u. That is what u_0 does for us.

We continue finding the rest of the problem for v. We replace u by $u_0 + v$ into the partial differential equation $u_t = \kappa u_{xx} + q$,

$$u_{0,t} + v_t = \kappa(u_{0,xx} + v_{xx}) + q$$

and take all u_0 terms to the right hand side:

$$v_t = \kappa v_{xx} + \bar{q}$$

where $\bar{q} = \kappa u_{0,xx} + q - u_{0,t}$, or, written out

$$\boxed{\bar{q}(x,t) = q(x,t) - g_1'(t) - g_0'(t)(x-\ell)}$$

Hence $\bar{q}$ is now a *known* function, just like q.

The final part of the problem for u that we have not converted yet is the initial condition. We replace u by $u_0 + v$ in $u(x,0) = f(x)$,

$$u_0(x,0) + v(x,0) = f(x)$$

and take u_0 to the other side:

$$v(x,0) = \bar{f}(x)$$

where $\bar{f}(x)$ is $f(x) - u_0(x,0)$, or written out:

$$\boxed{\bar{f}(x) = f(x) - g_1(0) - g_0(0)(x-\ell)}$$

Again, $\bar{f}$ is now a known function.

The problem for v is now the same as the one for u, except that the boundary conditions are homogeneous and functions f and q have changed into known functions $\bar{f}$ and $\bar{q}$.

Using separation of variables, we can find the solution for v in the form:

$$v(x,t) = \sum_n v_n(t) X_n(x).$$

We already know how to do that! (Don't worry, we will go over the steps anyway.) Having found v, we will simply add u_0 to find the asked temperature u.

22.6.5 Step 1: Find the eigenfunctions

To find the eigenfunctions X_n, substitute a trial solution $v = T(t)X(x)$ into the *homogeneous part* of the partial differential equation, $v_t = \kappa v_{xx} + \bar{q}$. Remember:

ignore the inhomogeneous part $\bar{q}$ when finding the eigenfunctions. Putting $v = T(t)X(x)$ into $v_t = \kappa v_{xx}$ produces:

$$T'X = \kappa TX''$$

Separate variables:

$$\frac{T'}{\kappa T} = \frac{X''}{X} = \text{ constant } = -\lambda$$

As always, λ cannot depend on x since the left hand side does not. Also, λ cannot depend on t since the middle does not. So λ must be a constant.

We then get the following Sturm-Liouville problem for any eigenfunctions $X(x)$:

$$-X'' = \lambda X \qquad X'(0) = 0 \qquad X(\ell) = 0$$

The last two equations are the boundary conditions on v which we made homogeneous.

This is the exact same eigenvalue problem that we had in an earlier example, so I can just take the solution from there. The eigenfunctions are:

$$\boxed{\lambda_n = \frac{(2n-1)^2\pi^2}{4\ell^2} \qquad X_n = \cos\left(\frac{(2n-1)\pi x}{2\ell}\right) \qquad (n = 1, 2, 3, \ldots)}$$

22.6.6 Step 2: Solve the problem

We expand in the problem for v in a Fourier series:

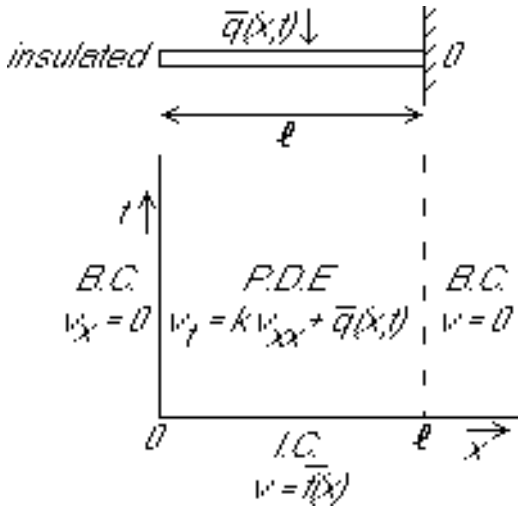


Figure 22.6: Heat conduction in a bar.

In particular,

$$v = \sum_{n=1}^{\infty} v_n(t) X_n(x) \quad \bar{f} = \sum_{n=1}^{\infty} \bar{f}_n X_n(x) \quad \bar{q} = \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

Since $\bar{q}(x)$ and $\bar{f}(x)$ are known functions, we can find their Fourier coefficients from orthogonality:

$$\bar{f}_n = \frac{\int_0^\ell \bar{f}(x) X_n(x)\,\mathrm{d}x}{\int_0^\ell X_n^2(x)\,\mathrm{d}x}$$

$$\bar{q}_n(t) = \frac{\int_0^\ell \bar{q}(x,t) X_n(x)\,\mathrm{d}x}{\int_0^\ell X_n^2(x)\,\mathrm{d}x}$$

or with the eigenfunctions written out

$$\boxed{\bar{f}_n = \frac{\int_0^\ell \bar{f}(x) \cos((2n-1)\pi x/2\ell)\,\mathrm{d}x}{\int_0^\ell \cos^2((2n-1)\pi x/2\ell)\,\mathrm{d}x}}$$

$$\boxed{\bar{q}_n(t) = \frac{\int_0^\ell \bar{q}(x,t) \cos((2n-1)\pi x/2\ell)\,\mathrm{d}x}{\int_0^\ell \cos^2((2n-1)\pi x/2\ell)\,\mathrm{d}x}}$$

The integrals in the bottom equal $\frac{1}{2}\ell$.

So the Fourier coefficients $\bar{f}_n$ are now known constants, and the $\bar{q}_n(t)$ are now known functions of t. Though in actual application, numerical integration may be needed to find them. During finals, I usually make the functions f, g_0 and g_1 simple enough that you can do the integrals analytically.

Now write the partial differential equation $v_t = \kappa v_{xx} + \bar{q}$ using the Fourier series:

$$\sum_{n=1}^{\infty} \dot{v}_n(t) X_n(x) = \kappa \sum_{n=1}^{\infty} v_n(t) X_n''(x) + \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

Looking in the previous section, the Sturm-Liouville equation was $-X'' = \lambda X$, so the partial differential equation simplifies to:

$$\sum_{n=1}^{\infty} \dot{v}_n(t) X_n(x) = -\kappa \sum_{n=1}^{\infty} \lambda_n v_n(t) X_n(x) + \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

It will always simplify or you made a mistake.

For the sums to be equal for any x, the coefficients of every individual eigenfunction must balance. So we get

$$\dot{v}_n(t) + \kappa \lambda_n v_n(t) = q_n(t)$$

We have obtained an ordinary differential equation for each v_n. It is again constant coefficient, but inhomogeneous.

Solve the homogeneous equation first. The characteristic polynomial is

$$k + \kappa\lambda_n = 0$$

so the homogeneous solution is

$$v_{nh} = A_n e^{-\kappa\lambda_n t}$$

For the inhomogeneous equation, undetermined constants is not a possibility since we do not know the actual form of the functions q. So we use variation of parameter:

$$v_n = A_n(t)e^{-\kappa\lambda_n t}$$

Plugging into the ordinary differential equation produces

$$A_n' e^{-\kappa\lambda_n t} + 0 = q_n(t) \quad \Longrightarrow \quad A_n' = q_n(t)e^{\kappa\lambda_n t}$$

We integrate this equation to find A_n. I could write the solution using an indefinite integral:

$$A_n(t) = \int q_n(t)e^{\kappa\lambda_n t}\,\mathrm{d}t$$

But that has the problem that the integration constant is not explicitly shown. That makes it impossible to apply the initial condition. It is better to write the anti-derivative using an integral with limits plus an explicit integration constant as:

$$A_n(t) = \int_{\tau=0}^{t} q_n(\tau)e^{\kappa\lambda_n \tau}\,\mathrm{d}\tau + A_{n0}$$

You can check using the Leibniz rule for differentiation of integrals (or really, just the fundamental theorem of calculus,) that the derivative is exactly what it should be. (Also, the lower limit does not really have to be zero; you could start the integration from 1, if it would be simpler. The important thing is that the upper limit is the independent variable t.)

Putting the found solution for $A_n(t)$ into

$$v_n = A_n(t)e^{-\kappa\lambda_n t}$$

we get, cleaned up:

$$v_n(t) = \int_{\tau=0}^{t} q_n(\tau)e^{-\kappa\lambda_n(t-\tau)}\,\mathrm{d}\tau + A_{n0}e^{-\kappa\lambda_n t}$$

We still need to find the integration constant A_{n0}. To do so, write the initial condition $v(x,0) = \bar{f}(x)$ using Fourier series:

$$\sum_{n=0}^{\infty} v_n(0)X_n(x) = \sum_{n=0}^{\infty} \bar{f}_n X_n(x)$$

This gives us initial conditions for the v_n:

$$v_n(0) = \bar{f}_n = A_{n0}$$

the latter from above, and hence

$$v_n(t) = \int_{\tau=0}^{t} q_n(\tau) e^{-\kappa\lambda_n(t-\tau)}\,\mathrm{d}\tau + \bar{f}_n e^{-\kappa\lambda_n t}$$

or writing out the eigenvalue:

$$\boxed{v_n(t) = \int_{\tau=0}^{t} q_n(\tau) e^{-\kappa(2n-1)^2\pi^2(t-\tau)/4\ell^2}\,\mathrm{d}\tau + \bar{f}_n e^{-\kappa(2n-1)^2\pi^2 t/4\ell^2}}$$

We have v_n in terms of known quantities, so we are done.

22.6.7 Summary of the solution

Collecting all the boxed formulae together, the solution is found by first computing the coefficients $\bar{f}_n$ from:

$$\bar{f}_n = \frac{2}{\ell}\int_0^\ell \bar{f}(x)\cos((2n-1)\pi x/2\ell)\,\mathrm{d}x \qquad (n = 1, 2, 3, \ldots)$$

where

$$\bar{f}(x) = f(x) - g_1(0) - g_0(0)(x-\ell)$$

Also compute the functions $\bar{q}_n(t)$ from:

$$\bar{q}_n(t) = \frac{2}{\ell}\int_0^\ell \bar{q}(x,t)\cos((2n-1)\pi x/2\ell)\,\mathrm{d}x \qquad (n = 1, 2, 3, \ldots)$$

where

$$\bar{q}(x,t) = q(x,t) - g_1'(t) - g_0'(t)(x-\ell)$$

Then the temperature is:

$$u(x,t) = g_1(t) + g_0(t)(x-\ell) + \sum_{n=1}^{\infty}\left[\int_{\tau=0}^{t} q_n(\tau) e^{-\kappa(2n-1)^2\pi^2(t-\tau)/4\ell^2}\,\mathrm{d}\tau + \bar{f}_n e^{-\kappa(2n-1)^2\pi^2 t/4\ell^2}\right]\cos\frac{(2n-1)\pi x}{2\ell}$$

22.7 Finding the Green's functions

We can, if we want, write the solution for v in other ways that may be more efficient numerically. The solution was, rewritten more concisely in terms of the eigenvalues and eigenfunctions:

$$v(x,t) = \sum_n \left[\int_{\tau=0}^{t} \bar{q}_n(\tau) e^{-\kappa\lambda_n(t-\tau)} \,\mathrm{d}\tau + \bar{f}_n e^{-\kappa\lambda_n t} \right] X_n(x).$$

The first part is due to the inhomogeneous term $\bar{q}$ in the partial differential equation, the second due to the initial condition $v(x,0) = \bar{f}(x)$

Look at the second term first, let's call it v_f,

$$v_f \equiv \sum_n \bar{f}_n e^{-\kappa\lambda_n t} X_n(x).$$

We can substitute in the orthogonality relationship for $\bar{f}(x)$:

$$v_f = \sum_n \frac{\int_0^\ell \bar{f}(\xi) X_n(\xi)\,\mathrm{d}\xi}{\int_0^\ell X_n^2(\zeta)\,\mathrm{d}\zeta} \, e^{-\kappa\lambda_n t} X_n(x)$$

and change the order of the terms to get:

$$v_f = \int_0^\ell \left[\sum_n \frac{X_n(\xi) X_n(x)}{\int_0^\ell X_n^2(\zeta)\,\mathrm{d}\zeta} \, e^{-\kappa\lambda_n t} \right] \bar{f}(\xi) d\xi$$

We define a shorthand symbol for the term within the square brackets:

$$G(x,t,\xi) \equiv \sum_n \frac{X_n(\xi) X_n(x)}{\int_0^\ell X_n^2(\zeta)\,\mathrm{d}\zeta} \, e^{-\kappa\lambda_n t}$$

Since this does not depend on what function $\bar{f}(x)$ is, we can evaluate G once and for all. For any $\bar{f}(x)$, the corresponding temperature is then simply found by integration:

$$v_f(x,t) = \int_0^\ell G(x,t,\xi)\bar{f}(\xi) d\xi$$

Function $G(x,t,\xi)$ by itself is the temperature $v(x,t)$ if $\bar{f}$ is a single spike of heat initially located at $x = \xi$. Mathematically, G is the solution for v if $\bar{f}(x)$ is the "delta function" $\delta(x-\xi)$.

Now look at the first term in v, due to $\bar{q}$, let's call it v_q:

$$v_q \equiv \sum_n \int_{\tau=0}^{t} \bar{q}_n(\tau) e^{-\kappa\lambda_n(t-\tau)} \,\mathrm{d}\tau X_n(x)$$

We plug in the orthogonality expression for $\bar{q}_n(\tau)$:

$$v_q = \sum_{n=0}^{\infty} \int_{\tau=0}^{t} \frac{\int_0^\ell \bar{q}(\xi,\tau) X_n(\xi)\,\mathrm{d}\xi}{\int_0^\ell X_n^2(\zeta)\,\mathrm{d}\zeta}\, e^{-\kappa\lambda_n(t-\tau)}\,\mathrm{d}\tau X_n(x)$$

and rewrite

$$v_q = \int_{\tau=0}^{t} \int_0^\ell \left[\sum_n \frac{X_n(\xi)X_n(x)}{\int_0^\ell X_n^2(\zeta)\,\mathrm{d}\zeta}\, e^{-\kappa\lambda_n(t-\tau)} \right] \bar{q}(\xi,\tau)\,\mathrm{d}\xi\mathrm{d}\tau$$

We see that

$$v_q(x,t) = \int_{\tau=0}^{t} \int_0^\ell G(x,t-\tau,\xi)\bar{q}(\xi,\tau)\,\mathrm{d}\xi\mathrm{d}\tau$$

where the function G is exactly the same as it was before. However, $G(x,t-\tau,\xi)$ describes the temperature due to a spike of heat added to the bar at a time τ and position ξ; it is called the Green's function.

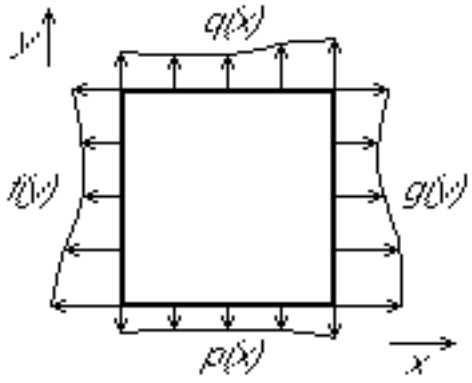
The fact that solving the initial value problem $(\bar{f})$, also solves the inhomogeneous partial differential equation problem $(\bar{q})$ is known as the Duhamel principle. The idea behind this principle is that function $\bar{q}(x,t)$ can be "sliced up" as a cake. The contribution of each slice $\tau \leqslant t \leqslant \tau + \mathrm{d}\tau$ of the cake to the solution v can be found as an initial value problem with $\bar{q}(x,\tau)\,\mathrm{d}\tau$ as the initial condition at time τ.

22.8 An alternate procedure

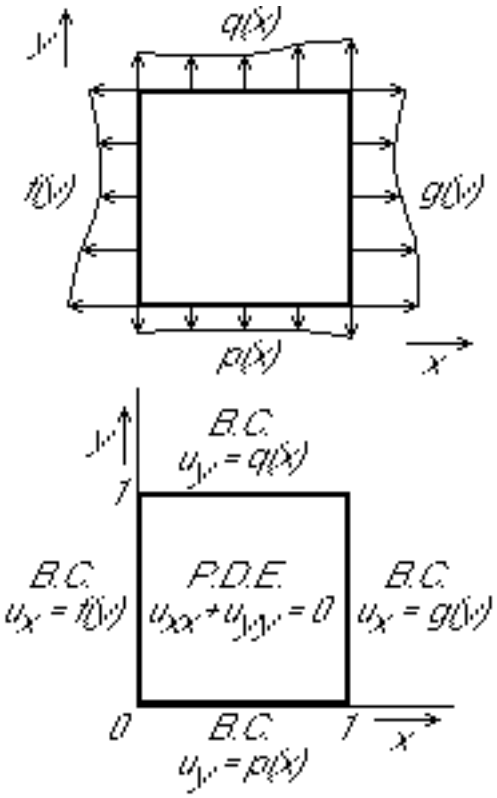
This example tries to be clever about handling inhomogeneous boundary equations for the Laplace equations. It does run into a few problems. But the students will of course explain and fix up the problem.

22.8.1 The physical problem

Find the steady temperature distribution in the square plate/cross section below if the heat fluxes out of the sides are known.



22.8.2 The mathematical problem



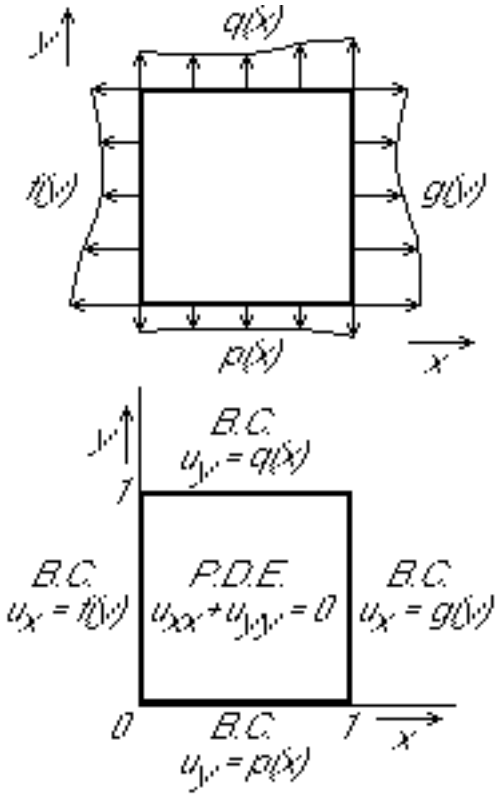
- Finite domain $\bar{\Omega}$: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.
- Unknown temperature $u = u(x, y)$
- Elliptic
- Four Neumann boundary conditions
- Integral constraint due to all Neumann boundary conditions:

$$\int_0^1 p(x)\,\mathrm{d}x - \int_0^1 g(y)\,\mathrm{d}y - \int_0^1 q(x)\,\mathrm{d}x + \int_0^1 f(y)\,\mathrm{d}y = 0$$

Try separation of variables:

$$\sum_n u_n(y)X_n(x) \text{ or } \sum_n u_n(x)Y_n(y)$$

22.8.3 Step 0: Fix the boundary conditions



Standard approach:

All boundary conditions are inhomogeneous. Our standard approach would be to set $u = u_0 + v$ where

$$u_{0x}(0, y) = f(y) \qquad u_{0x}(1, y) = g(y)$$

and then set

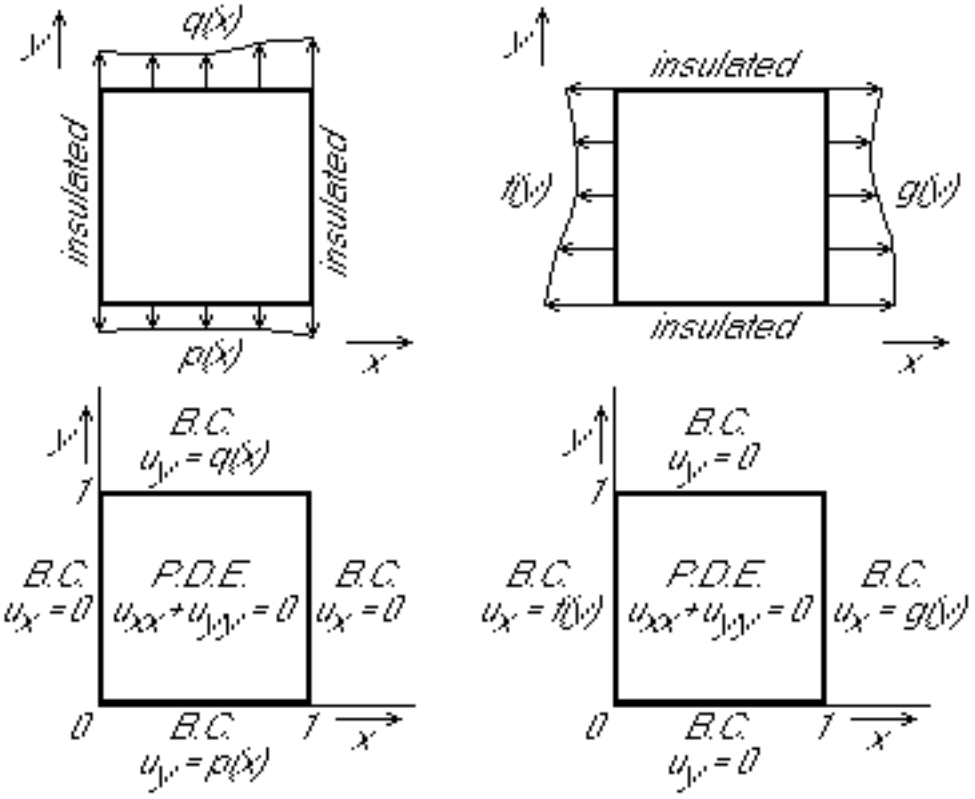
$$v = \sum_n v_n(y) X_n(x)$$

This would work without any problems. A u_0 quadratic in x would be fine. Of course, this choice for u_0 is quite arbitrary.

Alternative approach:

Instead, *we* will follow a more elegant procedure that does not require us to arbitrarily choose a u_0. Unfortunately, this alternative procedure will get us into some trouble.

The idea is that the given problem can be seen as the sum of two problems, each with homogeneous boundary conditions in one direction.



If we add the solutions u to the two problems together, we should get the solution to the original problem.

The *instructor* will solve the left hand problem. The *students* will solve the right hand problem, identify the difficulty, and fix it.

Some people split up the problem into 4, one for each side. That makes the difficulty even worse.

22.8.4 Step 1: Find the eigenfunctions

Substitute $u = T(y)X(x)$ into the homogeneous partial differential equation $u_{xx} + u_{yy} = 0$:

$$TX'' + T''X = 0$$

$$\frac{T''}{T} = -\frac{X''}{X} = \text{ constant } = \lambda$$

Since the instructor's x-boundary conditions are homogeneous, he has a Sturm-Liouville problem for X:

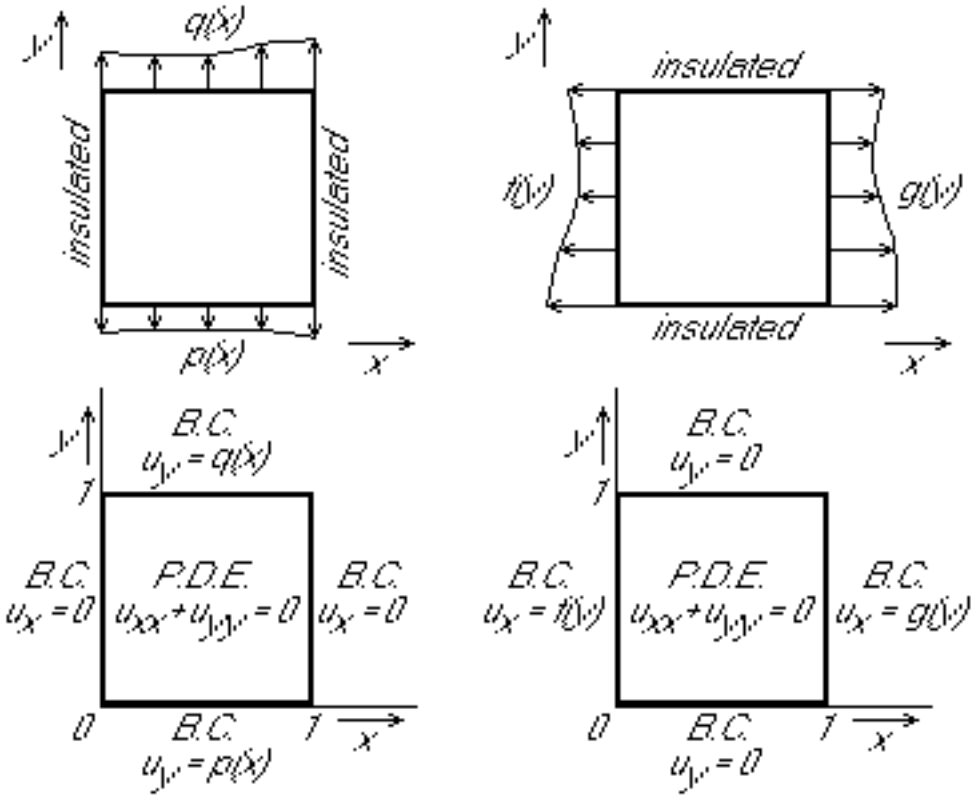
$$-X'' = \lambda X \qquad X'(0) = 0 \qquad X'(1) = 0$$

This was already solved in problem 7.19. Looking back there, substituting $\ell = 1$,

$$\lambda_n = n^2\pi^2 \qquad X_n = \cos(n\pi x) \qquad (n = 0, 1, 2, 3, \ldots)$$

22.8.5 Step 2: Solve the problem

Expand all variables in the problem for u in a Fourier series:



$$u = \sum_{n=0}^{\infty} u_n(y)X_n(x) \quad p(x) = \sum_{n=0}^{\infty} p_n X_n(x) \quad q(x) = \sum_{n=0}^{\infty} q_n X_n(x)$$

$$p_n = \frac{\int_0^1 p(x)X_n(x)\,\mathrm{d}x}{\int_0^1 X_n^2(x)\,\mathrm{d}x}$$

$$q_n = \frac{\int_0^1 q(x)X_n(x)\,\mathrm{d}x}{\int_0^1 X_n^2(x)\,\mathrm{d}x}$$

Remember that the expression you find for the integrals in the bottom, $\frac{1}{2}$, does not work for $n = 0$, in which case it turns out to be 1.

Fourier-expand the partial differential equation $u_{xx} + u_{yy} = 0$:

$$\sum_{n=0}^{\infty} u_n(y) X_n(x)'' + \sum_{n=0}^{\infty} u_n(y)'' X_n(x) = 0$$

Because of the Sturm-Liouville equation in the previous section

$$-\sum_{n=0}^{\infty} \lambda_n u_n(y) X_n(x) + \sum_{n=0}^{\infty} u_n(y)'' X_n(x) = 0$$

giving the ordinary differential equation

$$u_n(y)'' - \lambda_n u_n(y) = 0$$

or substituting in the eigenvalue

$$u_n(y)'' - n^2\pi^2 u_n(y) = 0$$

Fourier-expand the boundary condition $u_y(x, 0) = p(x)$:

$$\sum_{n=0}^{\infty} u_n(0)' X_n(x) = \sum_{n=0}^{\infty} p_n X_n(x) \qquad \Rightarrow \qquad u_n'(0) = p_n$$

Fourier-expand the boundary condition $u_y(x, 1) = q(x)$:

$$\sum_{n=0}^{\infty} u_n(1)' X_n(x) = \sum_{n=0}^{\infty} q_n X_n(x) \qquad \Rightarrow \qquad u_n'(1) = q_n$$

Solve the above ordinary differential equation and boundary conditions for u_n. It is a constant coefficient one, with a characteristic equation

$$k^2 - n^2\pi^2 = 0$$

Caution! Note that both roots are the same when $n = 0$. So we need to do the $n = 0$ case separately.

For $n \neq 0$ the solution is

$$u_n = A_n e^{n\pi y} + B_n e^{-n\pi y}$$

The boundary conditions above give two linear equations for A_n and B_n:

$$\left(\begin{array}{cc|c} n\pi & -n\pi & p_n \\ n\pi e^{n\pi} & -n\pi e^{-n\pi} & q_n \end{array} \right)$$

that are best solved using Gaussian elimination. Rewriting the various exponentials in terms of sinh and cosh, the solution for the Fourier coefficients of u except $n = 0$ is:

$$u_n = -\frac{\cosh(n\pi[y-1])}{n\pi \sinh(n\pi)} p_n + \frac{\cosh(n\pi y)}{n\pi \sinh(n\pi)} q_n \quad (n = 1, 2, 3, \ldots)$$

For $n = 0$ the solution of the ordinary differential equation is

$$u_0 = A_0 + B_0 y$$

Put in the boundary conditions to get equations for the integration constants A_0 and B_0:

$$u_0'(0) = B_0 = p_0 \qquad u_0'(1) = B_0 = q_0$$

Oops! We can only solve this if

$$p_0 = q_0$$

Looking above for the definition of those Fourier coefficients, we see we only have a solution if

$$\int_0^1 p(x)\,\mathrm{d}x = \int_0^1 q(x)\,\mathrm{d}x$$

Unfortunately, these two integrals will normally *not* be equal! Also, A_0 remains unknown.

22.8.6 Summary of the solution

First compute the Fourier coefficients of the given boundary conditions:

$$p_0 = \int_0^1 p(x)\,\mathrm{d}x \qquad p_n = 2\int_0^1 p(x)\cos(n\pi x)\,\mathrm{d}x \quad (n = 1, 2, \ldots)$$

$$q_0 = \int_0^1 q(x)\,\mathrm{d}x \qquad q_n = 2\int_0^1 q(x)\cos(n\pi x)\,\mathrm{d}x \quad (n = 1, 2, \ldots)$$

Then the solution is equal to:

$$u = A_0 + p_0 x + \sum_{n=1}^{\infty}\left[-\frac{\cosh(n\pi[y-1])}{n\pi\sinh(n\pi)}p_n + \frac{\cosh(n\pi y)}{n\pi\sinh(n\pi)}q_n\right]\cos(n\pi x)$$

But this only satisfies the boundary condition on the top of the plate if

$$\int_0^1 q(x)\,\mathrm{d}x = \int_0^1 p(x)\,\mathrm{d}x$$

No problem! Students will explain and fix the problem.

22.9 A Summary of Separation of Variables

After the previous three examples, it is time to give a more general description of the method of separation of variables.

22.9.1 The form of the solution

Before starting the process, you should have some idea of the form of the solution you are looking for. Some experience helps here.

For example, for unsteady heat conduction in a bar of length ℓ, with homogeneous end conditions, the temperature u would be written

$$u(x,t) = \sum_n u_n(t) X_n(x)$$

where the X_n are chosen eigenfunctions and the u_n are computed Fourier coefficients of u. The separation of variables procedure allows you to choose the eigenfunctions cleverly.

For a uniform bar, you will find sines and/or cosines for the functions X_n. In that case the above expansion for u is called a Fourier series. In general it is called a generalized Fourier series.

After the functions X_n have been found, the Fourier coefficients u_n can simply be found from substituting the expression above for u in the given partial differential equation and initial conditions. (The boundary conditions are satisfied when you choose the eigenfunctions X_n.) If there are other functions in the partial differential equation or initial conditions, they too need to be expanded in a Fourier series.

If the problem was axially symmetric heat conduction through the wall of a pipe, the temperature would still be written

$$u(r,t) = \sum_n u_n(t) R_n(r)$$

but the expansion functions R_n would now be found to be Bessel functions, not sines or cosines.

For heat conduction through a pipe wall without axial symmetry, still with homogeneous boundary conditions, the temperature would be written

$$u(r,\theta,t) = \sum_{n,i} u_n^i(r,t)\Theta_n^i(\theta) = \sum_{n,i}\sum_m u_{nm}^i(t) R_{nm}(r)\Theta_n^i(\theta)$$

where the eigenfunctions Θ_n^i turn out to be sines and cosines and the eigenfunctions R_{nm} Bessel functions. Note that in the first sum, the temperature is written as a simple Fourier series in θ, with coefficients u_n that of course depend on r and t. Then in the second sum, these coefficients themselves are written as a (generalized) Fourier series in r with coefficients u_{nm} that depend on t.

(For steady heat conduction, the coordinate "t" might actually be a second spatial coordinate. For convenience, we will refer to conditions at given values of t as "initial conditions", even though they might physically really be boundary conditions.)

22.9.2 Limitations of the method

The problems that can be solved with separation of variables are relatively limited.

First of all, the equation must be linear. After all, the solution is found as an sum of simple solutions.

The partial differential equation does not necessarily have to be a constant coefficient equation, but the coefficients cannot be too complicated. You should be able to separate variables. A coefficient like $\sin(xt)$ in the equation is not separable.

Further, the boundaries must be at constant values of the coordinates. For example, for the heat conduction in a bar, the ends of the bar must be at fixed locations $x = 0$ and $x = \ell$. The bar cannot expand, since then the end points would depend on time.

You may be able to find fixes for problems such as the ones above, of course. For example, the nonlinear Burger's equation can be converted into the linear heat equation. The above observations apply to straightforward application of the method.

22.9.3 The procedure

The general lines of the procedure are to choose the eigenfunctions and then to find the (generalized) Fourier coefficients of the desired solution u. In more detail, the steps are:

1. **Make the boundary conditions for the eigenfunctions X_n homogeneous**

   For heat conduction in a bar, this means that if nonzero end temperatures or heat fluxes through the ends are given, you will need to eliminate these.

   Typically, you eliminate nonzero boundary conditions for the eigenfunctions by subtracting a function u_0 from u that satisfies these boundary conditions. Since u_0 only needs to satisfy the boundary conditions, not the partial differential equation or the initial conditions, such a function is easy to find.

   If the boundary conditions are steady, you can try subtracting the steady solution, if it exists. More generally, a low degree polynomial can be tried, say $u_0 = A + Bx + Cx^2$, where the coefficients are chosen to satisfy the boundary conditions.

   Afterwards, carefully identify the partial differential equation and initial

conditions satisfied by the new unknown $v = u - u_0$. (They are typically different from the ones for u.)

2. **Identify the eigenfunctions X_n**

   To do this substitute a single term TX into the *homogeneous* partial differential equation. Then take all terms involving X and the corresponding independent variable to one side of the equation, and T and the other independent variables to the other side. (If that turns out to be impossible, the partial differential equation cannot be solved using separation of variables.)

   Now, since the two sides of the equation depends on different coordinates, they must both be equal to some constant. The constant is called the eigenvalue.

   Setting the X-side equal to the eigenvalue gives an ordinary differential equation. Solve it to get the eigenfunctions X_n. In particular, you get the complete set of eigenfunctions X_n by finding *all* possible solutions to this ordinary differential equation. (If the ordinary differential equation problem for the X_n turns out to be a regular Sturm-Liouville problem of the type described in the next section, the method is guaranteed to work.)

   The equation for T is usually safest ignored. The book tells you to also solve for the T_n, to get the Fourier coefficients v_n, but if you have an inhomogeneous partial differential equation, you have to mess around to get it right. Also, it is confusing, since the eigenfunctions X_n *do not* have undetermined constants, but the coefficients v_n *do*. It are the undetermined constants in v_n that allow you to satisfy the initial conditions. They probably did not make this fundamental difference between the functions X_n and the coefficients u_n clear in your undergraduate classes.

   There is one case in which you do need to use the equation for the T_n: in problems with more than two independent variables, where you want to expand the T_n themselves in a generalized Fourier series. That would be the case for the pipe wall without axial symmetry. Simply repeat the above separation of variables process for the partial differential equation satisfied by the T_n.

3. **Find the coefficients**

   Now find the Fourier coefficients v_n (or v_{nm} for three independent variables) by putting the Fourier series expansion into the partial differential equation and initial conditions.

While doing this, you will also need to expand the inhomogeneous terms in the partial differential equation and initial conditions into a Fourier series of the same form. You can find the coefficients of these Fourier series using the orthogonality property described in the next section.

You will find that the partial differential equation produces ordinary differential equations for the individual coefficients. And the integration constants in solving those equations follow from the initial conditions.

Afterwards you can play around with the solution to get other equivalent forms. For example, you can interchange the order of summation and integration (which results from the orthogonality property) to put the result in a Green's function form, etcetera.

22.9.4 More general eigenvalue problems

So far, the eigenvalue problems in the examples were of the form $X'' = -\lambda X$. But you might get a different problem in other examples. Usually that produces a different orthogonality expression.

You can figure out what is the correct expression by writing your ordinary differential equation in the standard form of a Sturm-Liouville problem:

$$\boxed{-pX'' - p'X' + qX = \lambda \bar{\mathrm{r}} X,}$$

where $X(x)$ is the eigenfunction to be found and $p(x) > 0$, $q(x)$, and $\bar{\mathrm{r}}(x) > 0$ are given functions. The distinguishing feature is that the coefficient of the second, X', term is the derivative of the coefficient of the first, X'' term.

Starting with an arbitrary second order linear ordinary differential equation, you can achieve such a form by multiplying the entire ordinary differential equation with a suitable factor.

The boundary conditions may either be periodic ones,

$$X(b) = X(a) \qquad X'(b) = X'(a),$$

or they can be homogeneous of the form

$$AX(a) + BX'(a) = 0 \qquad CX(b) + DX'(b) = 0,$$

where A, B, C, and D are given constants. Note the important fact that a Sturm-Liouville problem must be completely homogeneous: $X = 0$ must be a solution.

If you have a Sturm-Liouville problem, simply (well, simply ...) solve it. The solutions only exists for certain values of λ. Make sure you find *all* solutions, or

you are in trouble. They will form an infinite sequence of 'eigenfunctions', say $X_1(x)$, $X_2(x)$, $X_3(x)$, ... with corresponding 'eigenvalues' λ_1, λ_2, λ_3, ... that go off to positive infinity.

You can represent *arbitrary* functions, say $f(x)$, on the interval $[a, b]$ as a generalized Fourier series:

$$f(x) = \sum_n f_n X_n(x).$$

If you know $f(x)$, the orthogonality relation that gives the generalized Fourier coefficients f_n is

$$f_n = \frac{\int_a^b f(x) X_n(x) \bar{\mathrm{r}}(x) dx}{\int_a^b X_n^2(x) \bar{\mathrm{r}}(x) dx}$$

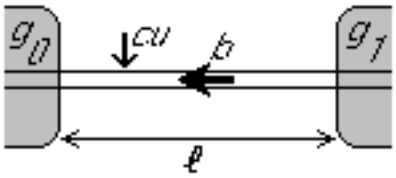
Now you know why you need to write your Sturm-Liouville problem in standard form: it allows you to pick out the weight factor $\bar{\mathrm{r}}$ that you need to put in the orthogonality relation!

22.10 More general eigenfunctions

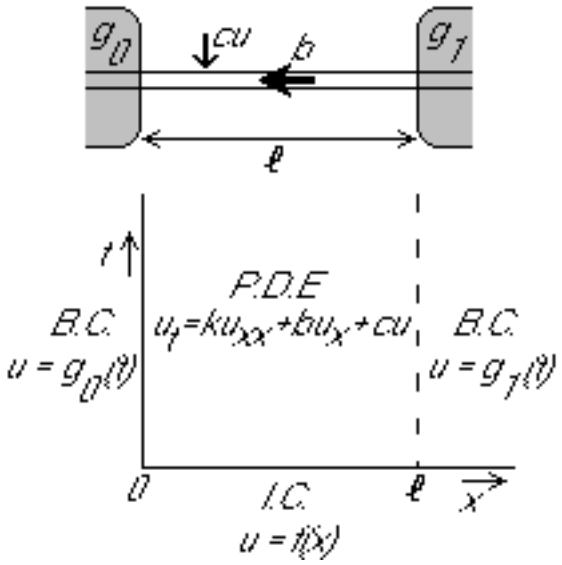
In the simplest problems, the eigenfunctions are sines and cosines. That includes the examples so far. But it is quite easy to get different eigenfunctions. In this example, they will turn out to be products of sines and exponentials.

22.10.1 The physical problem

Find the unsteady temperature distribution in the moving bar below for arbitrary position and time if the initial distribution at time zero and the temperatures of the ends are known.



22.10.2 The mathematical problem



- Finite domain $\bar{\Omega}$: $0 \leqslant x \leqslant \ell$
- Unknown temperature $u = u(x, t)$
- Parabolic
- One initial condition
- Two Dirichlet boundary conditions
- Constant κ

Try separation of variables:

$$\sum_n C_n(t) X_n(x)$$

22.10.3 Step 0: Fix the boundary conditions

The x-boundary conditions are inhomogeneous:

$$u(0, t) = g_0(t) \qquad u(\ell, t) = g_1(t)$$

So we try finding a u_0 satisfying these boundary conditions:

$$u_0(0, t) = g_0(t) \qquad u_0(\ell, t) = g_1(t)$$

A linear expression works:

$$u_0 = A(t) + B(t)x$$

$$A(t) = g_0(t) \qquad A(t) + B(t)\ell = g_1(t)$$

This can be solved to find

$$\boxed{u_0(x,t) = g_0(t) + \frac{g_1(t) - g_0(t)}{\ell} x}$$

To get rid of the inhomogeneous boundary conditions, we subtract u_0 from u. That will produce homogeneous boundary conditions for the remainder $v = u - u_0$. Indeed, if you plug $u = u_0 + v$ into the boundary conditions, you get

$$u_0(0,t) + v(0,t) = g_0(t) \qquad u_0(\ell,t) + v(\ell,t) = g_1(t)$$

And since u_0 satisfies the inhomogeneous boundary conditions, that becomes

$$v(0,t) = 0 \qquad v(\ell,t) = 0$$

Substitute $u = u_0 + v$ into the partial differential equation $u_t = \kappa u_{xx} + b u_x + c u$ to get

$$v_t = \kappa v_{xx} + b v_x + c v + q$$

where

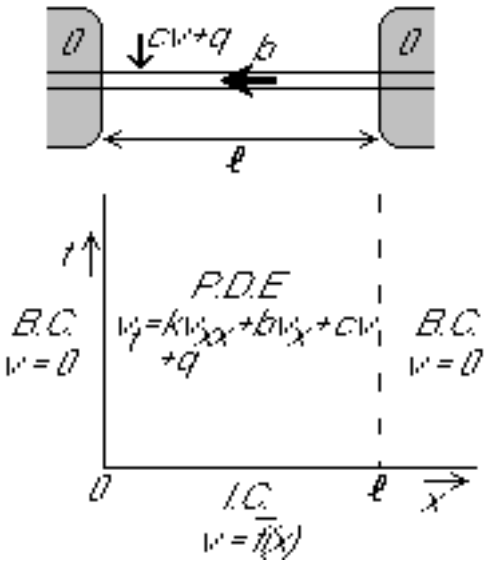
$$\boxed{q(x,t) = -g_0'(t) - \frac{g_1'(t) - g_0'(t)}{\ell} x + b\frac{g_1(t) - g_0(t)}{\ell} + c g_0(t) + c\frac{g_1(t) - g_0(t)}{\ell} x}$$

Substitute $u = u_0 + v$ into the initial condition $u(x,0) = f(x)$:

$$v(x,0) = \bar{f}(x)$$

$$\boxed{\bar{f}(x) = f(x) - g_0(0) - \frac{g_1(0) - g_0(0)}{\ell} x}$$

The problem for v is therefor:



22.10.4 Step 1: Find the eigenfunctions

Substitute $v = T(t)X(x)$ into the *homogeneous* partial differential equation $v_t = \kappa v_{xx} + bv_x + cv$:

$$T'X = \kappa TX'' + bTX' + cTX$$

Separate:

$$\frac{T'}{T} = \kappa\frac{X''}{X} + b\frac{X'}{X} + c = \text{ constant } = -\lambda$$

The Sturm-Liouville problem for X is now:

$$-\kappa X'' - bX' - cX = \lambda X \qquad X(0) = 0 \qquad X(\ell) = 0$$

This is a constant coefficient ordinary differential equation, with a characteristic polynomial:

$$\kappa k^2 + bk + (c + \lambda) = 0$$

The fundamentally different cases are now two real roots (discriminant positive), a double root (discriminant zero), and two complex conjugate roots (discriminant negative.) We do each in turn.

1.

2. *Case $b^2 - 4\kappa(c + \lambda) > 0$:*

   Roots k_1 and k_2 real and distinct:

   $$X = Ae^{k_1 x} + Be^{k_2 x}$$

   Boundary conditions:

   $$X(0) = 0 = A + B \qquad \Rightarrow \qquad B = -A$$

   $$X(\ell) = 0 = A\left(e^{k_1 \ell} - e^{k_2 \ell}\right) = 0$$

   No nontrivial solutions since the roots are different.

3. *Case $b^2 - 4\kappa(c + \lambda) = 0$:*

   Since $k_1 = k_2 = k$:

   $$X = Ae^{kx} + Bxe^{kx}$$

   Boundary conditions:

   $$X(0) = 0 = A \qquad X(\ell) = 0 = B\ell e^{k\ell}$$

   No nontrivial solutions.

4. *Case $b^2 - 4\kappa(c+\lambda) < 0$:*

   For convenience, we will write the roots of the characteristic polynomial more concisely as:

$$k_1 = -\mu + \mathrm{i}\omega \quad k_2 = -\mu - \mathrm{i}\omega$$

   where according to the solution of the quadratic

$$\mu = \frac{b}{2\kappa} \qquad \omega = \frac{\sqrt{4\kappa(c+\lambda) - b^2}}{2\kappa}$$

   Since it can be confusing to have too many variables representing the same thing, let's agree that μ is our "representative" for b, and ω our "representative" for λ. In terms of these representatives, the solution is, after clean-up,

$$X = e^{-\mu x}\left(A\cos(\omega x) + B\sin(\omega x)\right)$$

   Boundary conditions:

$$X(0) = 0 = A \qquad X(\ell) = 0 = e^{-\mu\ell} B\sin(\omega\ell)$$

   Nontrivial solutions $B \neq 0$ can only occur if

$$\sin(\omega\ell) = 0 \qquad \Rightarrow \qquad \omega_n = n\pi/\ell \quad (n = 1, 2, \ldots)$$

   which gives us our eigenvalues, by substituting in for ω:

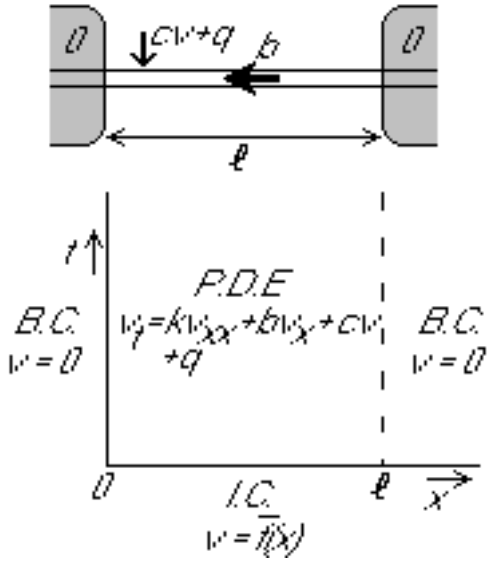
$$\lambda_n = \frac{\kappa n^2\pi^2}{\ell^2} + \frac{b^2}{4\kappa} - c \quad (n = 1, 2, 3, ...)$$

   Also, choosing each $B = 1$:

$$X_n = e^{-\mu x}\sin\left(n\pi x/\ell\right) \quad (n = 1, 2, 3, ...)$$

22.10.5 Step 2: Solve the problem

Expand all variables in the problem for v in a Fourier series:



$$v = \sum_{n=1}^{\infty} v_n(t) X_n(x) \quad \bar{f} = \sum_{n=1}^{\infty} \bar{f}_n X_n(x) \quad q = \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

We want to first find the Fourier coefficients of the known functions $\bar{f}$ and q. Unfortunately, the ordinary differential equation found in the previous section,

$$-\kappa X'' - bX' - cX = \lambda X$$

is *not* in standard Sturm-Liouville form: the derivative of the first, X'', coefficient, $-\kappa$, is zero, not $-b$. Let's try to make it OK by multiplying the entire equation by a factor, which will then be our $\bar{r}$.

$$-\bar{\mathrm{r}}\kappa X'' - \bar{\mathrm{r}} b X' - \bar{\mathrm{r}} c X = \lambda \bar{\mathrm{r}} X$$

We want that the second coefficient is the derivative of the first:

$$\bar{\mathrm{r}} b = \frac{\mathrm{d}}{\mathrm{d}x}(\bar{\mathrm{r}}\kappa)$$

This is a simple ordinary differential equation for the $\bar{\mathrm{r}}$ we are trying to find, and a valid solution is:

$$\bar{\mathrm{r}} = e^{bx/\kappa} = e^{2\mu x}$$

Having found $\bar{\mathrm{r}}$, we can write the orthogonality relationships for the generalized Fourier coefficients of $\bar{f}$ and q (remember that $X_n = e^{-\mu x}\sin(n\pi x/\ell)$):

$$\bar{f}_n = \frac{\int_{x=0}^{\ell} e^{\mu x}\bar{f}(x)\sin(n\pi x/\ell)\,\mathrm{d}x}{\int_{x=0}^{\ell}\sin^2(n\pi x/\ell)\,\mathrm{d}x}$$

$$q_n(t) = \frac{\int_{x=0}^{\ell} e^{\mu x} q(x,t) \sin(n\pi x/\ell)\,\mathrm{d}x}{\int_{x=0}^{\ell} \sin^2(n\pi x/\ell)\,\mathrm{d}x}$$

The integrals in the bottoms equal $\ell/2$.

Expand the partial differential equation $v_t = \kappa v_{xx} + b v_x + cv + q$ in a generalized Fourier series:

$$\sum_{n=1}^{\infty} \dot{v}_n(t) X_n(x) = \kappa \sum_{n=1}^{\infty} v_n(t) X_n''(x) + b \sum_{n=1}^{\infty} v_n(t) X_n'(x) + c \sum_{n=1}^{\infty} v_n(t) X_n(x) + \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

Because of the choice of the X_n, $\kappa X'' + bX' + cX = -\lambda X$:

$$\sum_{n=1}^{\infty} \dot{v}_n(t) X_n(x) = -\sum_{n=1}^{\infty} \lambda_n v_n(t) X_n(x) + \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

So, the ordinary differential equation for the generalized Fourier coefficients of v becomes:

$$\dot{v}_n(t) + \lambda_n v_n(t) = q_n(t)$$

Expand the initial condition $v(x,0) = \bar{f}(x)$ in a generalized Fourier series:

$$\sum_{n=1}^{\infty} v_n(0) X_n(x) = \sum_{n=1}^{\infty} \bar{f}_n X_n(x)$$

so

$$v_n(0) = \bar{f}_n$$

Solve this ordinary differential equation and initial condition for v_n:

Homogeneous equation:

$$v_{nh} = A_n e^{-\lambda_n t}$$

Inhomogeneous equation:

$$A_n' e^{-\lambda_n t} + 0 = q_n(t)$$

$$A_n = \int_{\tau=0}^{t} q_n(\tau) e^{\lambda_n \tau}\,\mathrm{d}\tau + A_{n0}$$

$$v_n = A_n e^{-\lambda_n t}$$

$$v_n = \int_{\tau=0}^{t} q_n(\tau) e^{-\lambda_n (t-\tau)}\,\mathrm{d}\tau + A_{n0} e^{-\lambda_n t}$$

Initial condition: $A_{n0} = \bar{f}_n$.

$$v_n = \int_{\tau=0}^{t} q_n(\tau) e^{-\lambda_n (t-\tau)}\,\mathrm{d}\tau + \bar{f}_n e^{-\lambda_n t}$$

22.10.6 Summary of the solution

Total solution:

$$\mu = \frac{b}{2\kappa} \qquad \lambda_n = \frac{\kappa n^2 \pi^2}{\ell^2} + \lambda_0 \qquad \lambda_0 = \frac{b^2}{4\kappa} - c$$

$$\bar{f}(x) = f(x) - g_0(0) - \frac{g_1(0) - g_0(0)}{\ell} x$$

$$\bar{f}_n = \frac{2}{\ell} \int_{x=0}^{\ell} \bar{f}(x) e^{\mu x} \sin(n\pi x/\ell) \, \mathrm{d}x$$

$$q(x,t) = -g_0'(t) - \frac{g_1'(t) - g_0'(t)}{\ell} x + b \frac{g_1(t) - g_0(t)}{\ell} + c \left(g_0(t) + \frac{g_1(t) - g_0(t)}{\ell} x \right)$$

$$q_n(t) = \frac{2}{\ell} \int_{x=0}^{\ell} q(x,t) e^{\mu x} \sin(n\pi x/\ell) \, \mathrm{d}x$$

$$\begin{aligned} u = {} & g_0(t) + \frac{g_1(t) - g_0(t)}{\ell} x \\ & + \sum_{n=1}^{\infty} \left[\int_{\tau=0}^{t} q_n(\tau) e^{-\lambda_n (t-\tau)} \, \mathrm{d}\tau + \bar{f}_n e^{-\lambda_n t} \right] e^{-\mu x} \sin(n\pi x/\ell) \end{aligned}$$

22.10.7 An alternative procedure

Define a new unknown w by $u = we^{-\alpha x - \beta t}$. Put this in the partial differential equation for u and choose α and β so that the w_x and w terms drop out. This requires:

$$u = we^{-\mu x - \lambda_0 t}$$

Then:

$$w_t = \kappa w_{xx} \quad w(x,0) = e^{\mu x} f(x) \quad w(0,t) = e^{\lambda_0 t} g_0(t) \quad w(\ell,t) = e^{\mu \ell + \lambda_0 t} g_1(t)$$

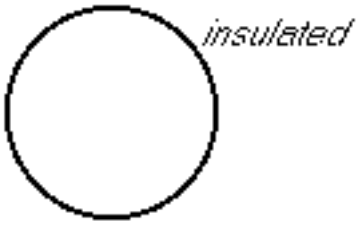
No fun! Note that the generalized Fourier series coefficients for u become normal Fourier coefficients for w.

22.11 A Problem in Three Independent Variables

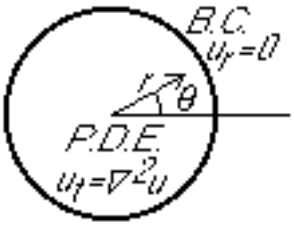
This example addresses a much more complex case. It involves three independent variables and eigenfunctions that turn out to be Bessel functions.

22.11.1 The physical problem

Find the unsteady heat conduction in a disk if the perimeter is insulated. The initial temperature is given.



22.11.2 The mathematical problem



- Finite domain $\bar{\Omega}$: $0 \leqslant r \leqslant a$, $0 \leqslant \vartheta < 2\pi$
- Unknown temperature $u = u(r, \vartheta, t)$
- Parabolic partial differential equation:

$$u_t = \kappa \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\vartheta\vartheta} \right)$$

- One homogeneous Neumann boundary condition at $r = a$:

$$u_r(a, \vartheta, t) = 0$$

- One initial condition at $t = 0$:

$$u(r, \vartheta, 0) = f(r, \vartheta)$$

We will solve using separation of variables in the form

$$u(r, \vartheta, t) = \sum_n \left(\sum_m u_{nm}(t) R_{nm}(r) \right) \Theta_n(\vartheta)$$

The eigenfunctions Θ_n will get rid of the ϑ variable in the partial differential equation, and the eigenfunctions R_{nm} will get rid of the r variable, leaving ordinary differential equations for the Fourier coefficients $u_{nm}(t)$.

22.11.3 Step 1: Find the eigenfunctions

Let's start trying to get rid of one variable first. We might try a solution of the form

$$u(r,\vartheta,t) = \sum_n u_n(\vartheta,t) R_n(r)$$

where the R_n would be the eigenfunctions and the $u_n(\vartheta,t)$ the corresponding Fourier coefficients. Unfortunately, if we try to substitute a single term of the form $C(\vartheta,t)R_n(r)$ into the homogeneous partial differential equation, we are not able to take all r terms to the same side of the equation and θ and t terms to the other side. So we do not get a Sturm-Liouville problem for R_n.

Try again, this time

$$u(r,\vartheta,t) = \sum_n u_n(r,t)\Theta_n(\vartheta)$$

If we substitute $C(r,t)\Theta(\vartheta)$ into the homogeneous partial differential equation $u_t/\kappa = u_{rr} + u_r/r + u_{\vartheta\vartheta}/r^2$ we get:

$$\frac{1}{\kappa}\dot{T}\Theta = C''\Theta + \frac{1}{r}C'\Theta + \frac{1}{r^2}C\Theta''$$

This, fortunately, *can* be separated:

$$r^2\frac{C''}{C} + r\frac{C'}{C} - r^2\frac{\dot{C}}{\kappa C} = -\frac{\Theta''}{\Theta} = \text{ constant } = \lambda$$

So we have a Sturm-Liouville problem for Θ:

$$-\Theta'' = \lambda\Theta$$

with boundary conditions that are periodic of period 2π. This problem was already fully solved in 7.38. It was the standard Fourier series for a function of period 2π. In particular, the eigenfunctions were $\cos(n\vartheta)$, $n = 0, 1, 2, \ldots$, and $\sin(n\vartheta)$, $n = 1, 2, \ldots$.

Like we did in 7.38, in order to cut down on writing, we will indicate those eigenfunctions compactly as Θ_n^i, where $\Theta_n^1 \equiv \cos(n\vartheta)$ and $\Theta_n^2 \equiv \sin(n\vartheta)$.

So we can concisely write

$$u = \sum_{n,i} u_n^i(r,t)\Theta_n^i(\vartheta)$$

Now, if you put this into the partial differential equation, you will see that you get rid of the ϑ coordinate as usual, but that still leaves you with r and t. So instead of ordinary differential equations in t, you get partial differential equations involving both r and t derivatives. That is not good enough.

We must go one step further: in addition we need to expand each Fourier coefficient $u_n^i(r,t)$ in a generalized Fourier series in r:

$$u(r,\vartheta,t) = \sum_{n,i} \left(\sum_m u_{nm}^i(t) R_{nm}^i(r) \right) \Theta_n^i(\vartheta)$$

Now, if you put a single term of the form $T_n(t)R_n(r)\Theta_n(\vartheta)$ into the homogeneous partial differential equation, you get

$$\frac{1}{\kappa}\dot{T}_n^i R_n^i \Theta_n^i = T_n^i R_n^{i\,\prime\prime} \Theta_n^i + \frac{1}{r} T_n^i R_n^{i\,\prime} \Theta_n^i + \frac{1}{r^2} T_n^i R_n^i \Theta_n^{i\,\prime\prime}$$

Since $\Theta_n^{i\,\prime\prime} = -\lambda\Theta_n^i = -n^2\Theta_n^i$, this is separable:

$$\frac{\dot{T}_n^i}{\kappa T_n^i} = \frac{R_n^{i\,\prime\prime}}{R_n^i} + \frac{R_n^{i\,\prime}}{r R_n^i} - n^2 \frac{1}{r^2} = \text{ constant } = -\mu_n$$

So we get a Sturm-Liouville problem for R_n^i with eigenvalue μ_n

$$r^2 R_n^{i\,\prime\prime} + r R_n^{i\,\prime} + (\mu_n r^2 - n^2) R_n^i = 0$$

with again the same homogeneous boundary conditions as u:

$$R_n^i \text{ regular at } r = 0 \qquad R_n^{i\,\prime}(a) = 0$$

We need to find all solutions to this problem.

Unfortunately, the ordinary differential equation above is not a constant coefficient one, so we cannot write a characteristic equation. However, we have seen the special case that $\mu_n = 0$ before, 7.38. It was a Euler equation. We found in 7.38 that the only solutions that are regular at $r = 0$ were found to be $A_n r^n$. But over here, the only one of that form that also satisfies the boundary condition $R_n^{i\,\prime} = 0$ at $r = a$ is the case $n = 0$. So, for $\mu = 0$, we only get a single eigenfunction

$$R_{00} = 1$$

For the case $\mu_n \neq 0$, the trick is to define a stretched r coordinate ρ as

$$\rho = \sqrt{\mu_n} r \qquad \Rightarrow \qquad \rho^2 \frac{\mathrm{d}^2 R_n^i}{\mathrm{d}\rho^2} + \rho \frac{\mathrm{d} R_n^i}{\mathrm{d}\rho} + (\rho^2 - n^2) R_n^i = 0$$

This equation can be found in any mathematical handbook in the section on Bessel functions. It says there that solutions are the Bessel functions of the first kind J_n and of the second kind Y_n:

$$R_n^i = A_n J_n(\sqrt{\mu_n} r) + B_n Y_n(\sqrt{\mu_n} r)$$

Now we need to apply the boundary conditions. Now if you look up the graphs for the functions Y_n, or their power series around the origin, you will see that they are all singular at $r = 0$. So, regularity at $r = 0$ requires $B_n = 0$.

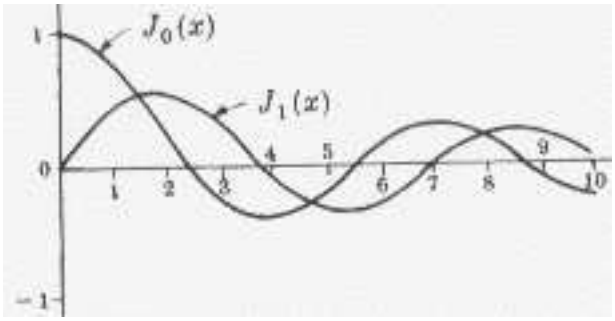
The boundary condition at the perimeter is

$$R_n^{i\,\prime}(a) = 0 = A_n\sqrt{\mu_n}J_n'(\sqrt{\mu_n}a)$$

Since μ_n is nonzero, nontrivial solutions only occur if

$$J_n'(\sqrt{\mu_n}a) = 0$$

Now if you look up the graphs of the various functions J_0, J_1, ..., you will see that they are all oscillatory functions, like decaying sines, and have an infinity of maxima and minima where the derivative is zero.



Each of the extremal points gives you a value of μ_n, so you will get an infinite of values μ_{n1}, μ_{n2}, μ_{n3}, ..., μ_{nm}, ... There is no simple formula for these values, but you can read them off from the graph. Better still, you can find them in tables for low values of n and m. (Schaum's gives a table containing both the zeros of the Bessel functions and the zeros of their derivatives.)

So the r-eigenvalues and eigenfunctions are:

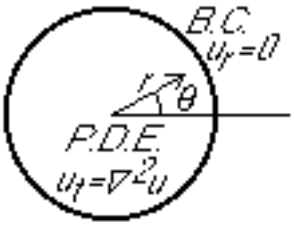
$$\begin{array}{cccccc} \mu_{n1} & & \mu_{n2} & \cdots & \mu_{nm} & \cdots \\ R^i_{n1} = J_n\left(\sqrt{\mu_{n1}}r\right) & R^i_{n2} = J_n\left(\sqrt{\mu_{n2}}r\right) & \cdots & R^i_{nm} = J_n\left(\sqrt{\mu_{n3}}r\right) & \cdots \end{array}$$

where m is the counter over the nonzero stationary points of J_n. To include the special case $\mu_n = 0$, we can simply add $\mu_{00} = 0$, $R^i_{00} = J_0(0) = 1$ to the list above.

In case of negative μ_n, the Bessel function J_n of imaginary argument becomes a modified Bessel function I_n of real argument, and looking at the graph of those, you see that there are no solutions.

22.11.4 Step 2: Solve the problem

We again expand all variables in the problem in generalized Fourier series:



Let's start with the initial condition:

$$f(r,\vartheta) = \sum_{n,i} \sum_{m} f^i_{nm} \Theta^i_n(\vartheta) J_n(\sqrt{\mu_{nm}} r)$$

To find the Fourier coefficients f^i_{nm}, we need orthogonality for both the r and ϑ eigenfunctions. Now the ordinary differential equation for the Θ eigenfunctions was in standard form,

$$-\Theta'' = \lambda\Theta$$

but the one for R_n was not:

$$r^2 R_n^{i\,\prime\prime} + r R_n^{i\,\prime} - n^2 R^i_n = -\mu_n r^2 R^i_n$$

The derivative of the first coefficient is $2r$, not r. To fix it up, we must divide the equation by r. And that makes the weight factor $\bar{r}$ that we need to put in the orthogonality relationship equal to r.

As a result, our orthogonality relation for the Fourier coefficients of initial condition $f(r,\vartheta)$ becomes

$$f^i_{nm} = \frac{\int_0^a J_n(\sqrt{\mu_{nm}} r) \left[\int_0^{2\pi} \Theta^i_n(\vartheta) f(r,\vartheta)\, \mathrm{d}\vartheta \right] r\, \mathrm{d}r}{\int_0^a J_n^2(\sqrt{\mu_{nm}} r)\, r\, \mathrm{d}r \int_0^{2\pi} \Theta^{i2}_n(\vartheta)\, \mathrm{d}\vartheta}$$

The integral within the square brackets turns $f(r,\vartheta)$ into its θ-Fourier coefficient $f^i_n(r)$ and the outer integral turns that coefficient in its generalized r-Fourier coefficient f^i_{nm}. Note that the total numerator is an integral of f over the area of the disk against a mode shape $J_n(\sqrt{\mu_{nm}} r)\Theta^i_n(\vartheta)$.

The r-integral in the denominator can be worked out using Schaum's Mathematical Handbook 24.88/27.88:

$$\int_0^a J_n^2(\sqrt{\mu_{nm}} r)\, r\, \mathrm{d}r = \left(\frac{a^2}{2} - \frac{n^2}{2\mu_{nm}} \right) J_n^2 \left(\sqrt{\mu_{nm}} a \right)$$

(setting the second term to zero for μ_{00}.)

Hence, while akward, there is no fundamental problem in evaluating as many f^i_{nm} as you want numerically. We will therefor consider them now "known".

Next we expand the desired temperature in a generalized Fourier series:

$$u(r,\vartheta,t) = \sum_{n,i}\sum_{m} u^i_{nm}(t)\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)$$

Put into partial differential equation $u_t/\kappa = u_{rr} + u_r/r + u_{\vartheta\vartheta}/r^2$:

$$\begin{aligned}
&\frac{1}{\kappa}\sum_{n,i}\sum_{m} \dot{u}^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)\\
&= \sum_{n,i}\sum_{m} u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)''\\
&+ \frac{1}{r}\sum_{n,i}\sum_{m} u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)'\\
&+ \frac{1}{r^2}\sum_{n,i}\sum_{m} u^i_{nm}\Theta^i_n(\vartheta)''J_n(\sqrt{\mu_{nm}}r)
\end{aligned}$$

Because of the SL equation satisfied by the Θ^i_n:

$$\begin{aligned}
&\frac{1}{\kappa}\sum_{n,i}\sum_{m} \dot{u}^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)\\
&= \sum_{n,i}\sum_{m} u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)''\\
&+ \frac{1}{r}\sum_{n,i}\sum_{m} u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)'\\
&- \frac{1}{r^2}\sum_{n,i}\sum_{m} n^2 u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)
\end{aligned}$$

Because of the SL equation satisfied by the J_n:

$$\begin{aligned}
&\frac{1}{\kappa}\sum_{n,i}\sum_{m} \dot{u}^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)\\
&= -\sum_{n,i}\sum_{m} \mu_{nm} u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)
\end{aligned}$$

Hence the ordinary differential equation for the Fourier coefficients is:

$$\dot{u}^i_{nm} + \kappa\mu_{nm}u^i_{nm} = 0$$

with solution:

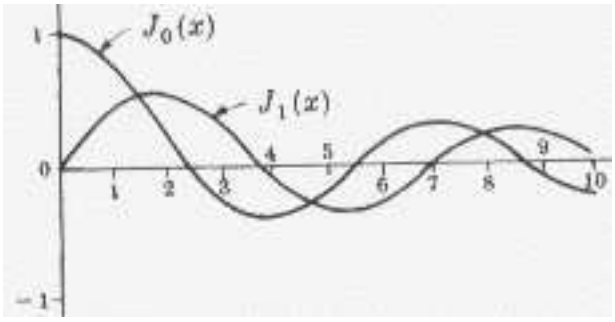
$$u^i_{nm}(t) = u^i_{nm}(0)e^{-\kappa\mu_{nm}t}$$

At time zero, the series expansion for u must be the same as the one for the given initial condition f:

$$u^i_{nm}(0) = f^i_{nm}$$

Hence we have found the Fourier coefficients of u and solved the problem.

22.11.5 Summary of the solution



Find the set $\sqrt{\mu_{nm}}a$ of positive stationary points of the Bessel functions J_n, $n = 0, 1, 2, ...$ and add $\mu_{00} = 0$.

Find the generalized Fourier coefficients of the initial condition:

$$f^1_{0m} = \frac{\int_0^{2\pi}\int_0^a f(r,\vartheta)J_0(\sqrt{\mu_{0m}}r)\,r\,\mathrm{d}\vartheta\mathrm{d}r}{\pi a^2 J_0^2\left(\sqrt{\mu_{0m}}a\right)}$$

$$f^1_{nm} = \frac{2\mu_{nm}\int_0^{2\pi}\int_0^a f(r,\vartheta)\cos(n\vartheta)J_n(\sqrt{\mu_{nm}}r)\,r\,\mathrm{d}\vartheta\mathrm{d}r}{\pi\left(\mu_{nm}a^2 - n^2\right)J_n^2\left(\sqrt{\mu_{nm}}a\right)}$$

$$f^2_{nm} = \frac{2\mu_{nm}\int_0^{2\pi}\int_0^a f(r,\vartheta)\sin(n\vartheta)J_n(\sqrt{\mu_{nm}}r)\,r\,\mathrm{d}\vartheta\mathrm{d}r}{\pi\left(\mu_{nm}a^2 - n^2\right)J_n^2\left(\sqrt{\mu_{nm}}a\right)}$$

Then:

$$\begin{aligned}u(r,\vartheta,t) = &\sum_{m=0}^{\infty} f_{0m}e^{-\kappa\mu_{0m}t}J_0\left(\sqrt{\mu_{0m}}r\right)\\ &+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f^1_{nm}e^{-\kappa\mu_{nm}t}\cos(n\vartheta)J_n\left(\sqrt{\mu_{nm}}r\right)\\ &+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f^2_{nm}e^{-\kappa\mu_{nm}t}\sin(n\vartheta)J_n\left(\sqrt{\mu_{nm}}r\right)\end{aligned}$$

That was not too bad!

Chapter 23

Fourier Transforms [None]

Chapter 24

Laplace Transforms

This chapter shows how many simple linear partial differential equation problems can be solved using Laplace transforms. The procedure is illustrated using a few examples.

24.1 Overview of the Procedure

The Laplace transform pairs a function of a real coordinate, call it t, with $0 < t < \infty$, with a different function of a complex coordinate s:

$$u(t,\cdot) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftharpoons}} \hat{u}(s,\cdot)$$

The pairing is designed to get rid of derivatives with respect to t in equations for the function u. This works as long as the coefficients do not depend on t (or at the very most are low degree powers of t.) The transformation is convenient since pairings can be looked up in tables.

24.1.1 Typical procedure

Use tables to find the equations satisfied by $\hat{u}$ from these satisfied by u. Solve for $\hat{u}$ and look up the corresponding u in the tables.

Table 24.1 lists important properties of the Laplace transform and table 24.2 gives example Laplace transform pairs. In the tables, $k > 0$, a, b, and c are constants, normally positive, n is a natural number, and

$$\operatorname{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\xi^2} \, \mathrm{d}\xi$$

Table 24.1 assumes that a and b are positive.

$u(t)$	$\hat{u}(s)$
0. $\dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{-st}\hat{u}(s)\,\mathrm{d}s$	$\displaystyle\int_0^\infty u(t)e^{-st}\,\mathrm{d}t$
1. $C_1u_1(t) + C_2u_2(t)$	$C_1\hat{u}_1(s) + C_2\hat{u}_2(s)$
2. $u(at)$	$a^{-1}\hat{u}(s/a)$
3. $\dfrac{\partial^n u}{\partial t^n}(t)$	$s^n\hat{u}(s) - s^{n-1}u(0) - \ldots - \dfrac{\partial^{n-1}u}{\partial t^{n-1}}(0)$
4. $t^n u(t)$	$(-1)^n\dfrac{\partial^n\hat{u}}{\partial s^n}$
5. $e^{ct}u(t)$	$\hat{u}(s-c)$
6. $\bar{u}(t-b) \equiv H(t-b)u(t-b)$ $= \begin{cases} u(t-b) & (t-b>0) \\ 0 & (t-b<0) \end{cases}$	$e^{-bs}\hat{u}(s)$
7. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau$	$\hat{f}(s)\hat{g}(s)$

Table 24.1: Properties of the Laplace transform.

24.1.2 About the coordinate to be transformed

In many cases, t is physically time, since time is most likely to satisfy the constraints $0 < t < \infty$ and coefficients independent of t. Also, the Laplace transform likes initial conditions at $t = 0$, not boundary conditions at both $t = 0$ and $t = \infty$.

24.2 A parabolic example

This example illustrates Laplace transform solution for a parabolic partial differential equation.

$u(t)$	$\hat{u}(s)$
1. 1	$\frac{1}{s}$
2. t^n	$\frac{n!}{s^{n+1}}$
3. e^{bt}	$\frac{1}{s-b}$
4. $\sin(at)$	$\frac{a}{s^2+a^2}$
5. $\cos(at)$	$\frac{s}{s^2+a^2}$
6. $\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$
7. $\frac{1}{\sqrt{\pi t}} e^{-k^2/(4t)}$	$\frac{1}{\sqrt{s}} e^{-k\sqrt{s}}$
8. $\frac{k}{\sqrt{4\pi t^3}} e^{-k^2/(4t)}$	$e^{-k\sqrt{s}}$
9. $\operatorname{erfc}\left(\frac{k}{2\sqrt{t}}\right)$	$\frac{1}{s} e^{-k\sqrt{s}}$

Table 24.2: Selected Laplace transform pairs.

24.2.1 The physical problem

Find the flow velocity in a viscous fluid being dragged along by an accelerating plate.

Figure 24.1: Viscous flow next to a moving plate

24.2.2 The mathematical problem

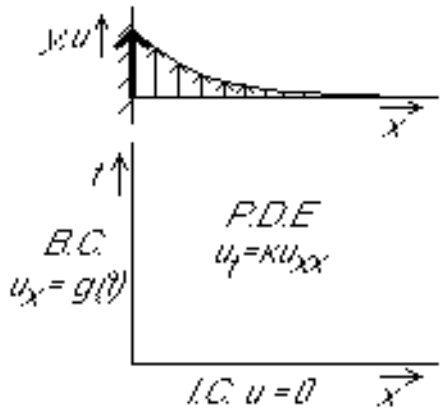


Figure 24.2: Viscous flow next to a moving plate

- Semi-infinite domain $\bar{\Omega}$: $0 \leqslant x < \infty$
- Unknown vertical velocity $u = u(x, t)$
- Parabolic
- One homogeneous initial condition
- One Neumann boundary condition at $x = 0$ and a regularity constraint at $x = \infty$
- Constant kinematic viscosity κ

Try a Laplace transform in t.

24.2.3 Transform the problem

Transform the partial differential equation:

$$u_t = \kappa u_{xx} \quad \xrightarrow{\text{Table 6.3, \# 3}} \quad s\hat{u} - \cancel{u(x,0)} = \kappa \hat{u}_{xx}$$

Transform the boundary condition:

$$u_x = g(t) \quad \Longrightarrow \quad \hat{u}_x = \hat{g}(s)$$

24.2.4 Solve the transformed problem

Solve the partial differential equation:

$$s\hat{u} = \kappa \hat{u}_{xx}$$

This is a constant coefficient ordinary differential equation in x, with s simply a parameter. Solve from the characteristic equation:

$$s = \kappa k^2 \qquad \Rightarrow \qquad k = \pm\sqrt{s/\kappa}$$

$$\hat{u} = Ae^{\sqrt{s/\kappa}\,x} + Be^{-\sqrt{s/\kappa}\,x}$$

Apply the boundary condition at $x = \infty$ that u must be regular there:

$$A = 0$$

Apply the given boundary condition at $x = 0$:

$$\hat{u}_x = \hat{g}(s) \qquad \Rightarrow \qquad -B\sqrt{\frac{s}{\kappa}} = \hat{g}$$

Solving for B and plugging it into the solution of the ordinary differential equation, $\hat{u}$ has been found:

$$\hat{u} = -\sqrt{\frac{\kappa}{s}}e^{-\sqrt{s/\kappa}\,x}\hat{g}$$

24.2.5 Transform back

We need to find the original function u corresponding to the transformed

$$\hat{u} = -\sqrt{\frac{\kappa}{s}}e^{-\sqrt{s/\kappa}\,x}\hat{g}$$

We do not really know what $\hat{g}$ is, just that it transforms back to g. However, we can find the other part of $\hat{u}$ in the tables.

$$-\sqrt{\frac{\kappa}{s}}e^{-\sqrt{s/\kappa}\,x} \quad \xrightarrow{\text{Table 6.4, \# 7}} \quad -\sqrt{\frac{\kappa}{\pi t}}e^{-x^2/4\kappa t}$$

How does $\hat{g}$ times this function transform back? The product of two functions, say $\hat{f}(s)\hat{g}(s)$, does *not* transform back to $f(t)g(t)$. The convolution theorem Table 6.3 # 7 is needed:

$$u(x,t) = -\int_0^t \sqrt{\frac{\kappa}{\pi(t-\tau)}}e^{-x^2/4\kappa(t-\tau)}g(\tau)\,\mathrm{d}\tau$$

24.3 A hyperbolic example

This example illustrates Laplace transform solution for a hyperbolic partial differential equation.

It also illustrates that the transformed coordinate is not always a time.

24.3.1 The physical problem

Find the horizontal perturbation velocity in a supersonic flow above a membrane overlaying a compressible variable medium.

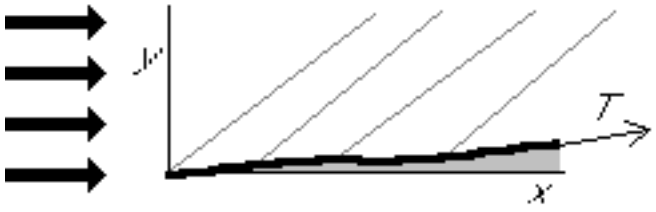


Figure 24.3: Supersonic flow over a membrane.

24.3.2 The mathematical problem

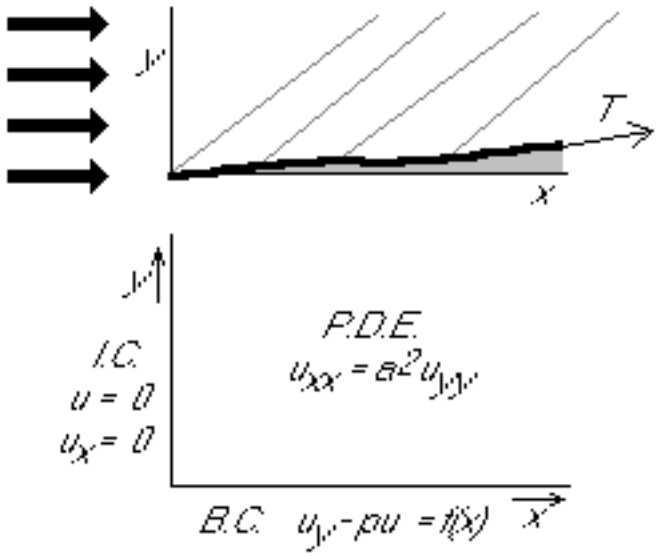


Figure 24.4: Supersonic flow over a membrane.

- Domain $\bar{\Omega}$: $0 \leqslant x < \infty, 0 \leqslant y < \infty$

- Unknown horizontal perturbation velocity $u = u(x, y)$
- Hyperbolic
- Two homogeneous initial conditions
- One mixed boundary condition at $y = 0$ and a regularity constraint at $y = \infty$
- Constant $a = \tan \mu$, where μ is the Mach angle.

Try a Laplace transform. The physics and the fact that Laplace transforms like only initial conditions suggest that x is the one to be transformed. Variable x is our "time-like" coordinate.

24.3.3 Transform the problem

Transform the partial differential equation:

$$u_{xx} = a^2 u_{yy} \quad \xrightarrow{\text{Table 6.3, \# 3}} \quad s^2 \hat{u} - \cancel{su(0, y)} - \cancel{u_x(0, y)} = a^2 \hat{u}_{yy}$$

Transform the boundary condition:

$$u_y - pu = f(x) \quad \Longrightarrow \quad \hat{u}_y - p\hat{u} = \hat{f}(s)$$

24.3.4 Solve the transformed problem

Solve the partial differential equation, again effectively a constant coefficient ordinary differential equation:

$$s^2 \hat{u} = a^2 \hat{u}_{yy}$$

$$s^2 = a^2 k^2 \qquad \Rightarrow \qquad k = \pm s/a$$

$$\hat{u} = Ae^{sy/a} + Be^{-sy/a}$$

Apply the boundary condition at $y = \infty$:

$$A = 0$$

Apply the boundary condition at $y = 0$:

$$\hat{u}_y - p\hat{u} = \hat{f} \qquad \Rightarrow \qquad -\frac{s}{a}B - pB = \hat{f}$$

Solving for B and plugging it into the expression for $\hat{u}$ gives:

$$\hat{u} = -\frac{a\hat{f}}{s + ap} e^{-sy/a}$$

24.3.5 Transform back

We need to find the original to

$$\hat{u} = -\frac{a}{s+ap}\hat{f}e^{-sy/a}$$

Looking in the tables:

$$\frac{1}{s+ap} \quad \xrightarrow{\text{Table 6.4, \# 3}} \quad e^{-apx}$$

The other factor is a shifted function f, restricted to the interval that its argument is positive:

$$e^{-sy/a}\hat{f} \quad \xrightarrow{\text{Table 6.3, \# 6}} \quad \bar{f}\left(x-\frac{y}{a}\right)$$

With the bar, I indicate that I only want the part of the function for which the argument is positive. This could be written instead as

$$f\left(x-\frac{y}{a}\right)H\left(x-\frac{y}{a}\right)$$

where the Heaviside step function $H(x) = 0$ if x is negative and 1 if it is positive.

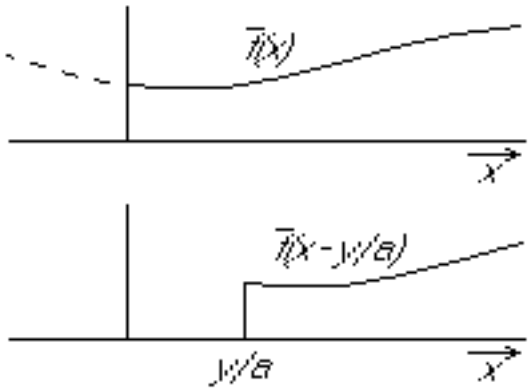


Figure 24.5: Function $\bar{f}$.

Use convolution, Table 6.3, # 7. again to get the product.

$$u(x,y) = -\int_0^x a\bar{f}\left(\xi-\frac{y}{a}\right)e^{-ap(x-\xi)}\,\mathrm{d}\xi$$

This *must* be cleaned up. I do not want bars or step functions in my answer.

I can do that by restricting the range of integration to only those values for which $\bar{f}$ is nonzero. (Or H is nonzero, if you prefer)

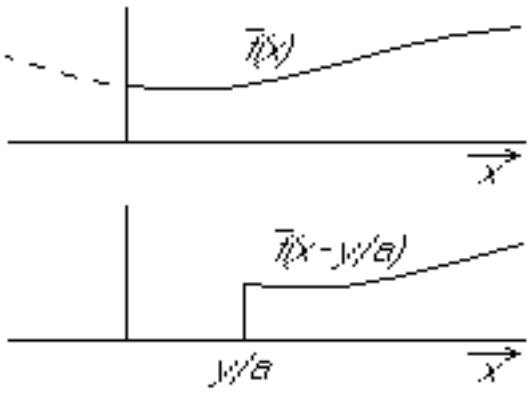


Figure 24.6: Function $\bar{f}$ again.

Two cases now exist:

$$u(x,y) = -\int_{y/a}^{x} af\left(\xi - \frac{y}{a}\right) e^{-ap(x-\xi)}\,\mathrm{d}\xi \qquad (x > \frac{y}{a})$$

$$u(x,y) = 0 \qquad (x < \frac{y}{a})$$

It is neater if the integration variable is the argument of f. So, define $\phi = \xi - y/a$ and convert:

$$u(x,y) = -\int_{0}^{x-y/a} af\left(\phi\right) e^{-apx+py+ap\phi}\,\mathrm{d}\phi \qquad (x > \frac{y}{a})$$

$$u(x,y) = 0 \qquad (x < \frac{y}{a})$$

This allows me to see which physical f values I actually integrate over when finding the flow at an arbitrary point:

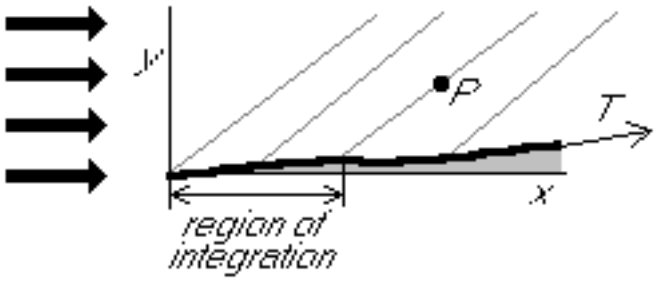


Figure 24.7: Supersonic flow over a membrane.

24.3.6 An alternate procedure

An alternate solution procedure is to define a new unknown:

$$v \equiv u_y - pu$$

You must derive the problem for v:
The boundary condition is simply:

$$v(x,0) = f(x)$$

To get the partial differential equation for v, use

$$\frac{\partial[P.D.E.]}{\partial y} - p[P.D.E.] \qquad \Rightarrow \qquad v_{tt} = a^2 v_{xx}$$

Similarly, for the initial conditions:

$$\frac{\partial[I.C.]}{\partial y} - p[I.C.] \qquad \Rightarrow \qquad v(0,y) = v_x(0,y) = 0$$

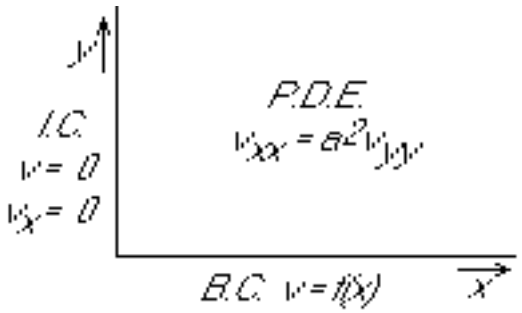


Figure 24.8: Problem for v.

After finding v, I still need to find u from the definition of v:

$$v \equiv u_y - pu$$

Where do you get the integration constant??

Part V

Supplementary Information

Appendix A

Addenda

This appendix describes a number of additional topics. They did not seem important enough to warrant including them in the main text. An addition is always a distraction; at the minimum you have to worry about whether you need to worry about it. However, many of the topics below are covered in well-known other texts. Obviously many other authors disagree about their importance. If they turn out to be right, you can find it here.

A.1 Distributions

A delta function is not a function in the normal sense. Infinity is not a proper number.

However, delta functions have a property that can be used to *define* them. That property is called the "filtering property." If you multiply a delta function by a smooth function $\phi(x)$ and integrate over all x, you get the value of the function at the location of the delta function:

$$\int_{x=-\infty}^{\infty} \phi(x)\delta(x-\xi)\,\mathrm{d}x = \phi(\xi)$$

The reason is that the delta function is everywhere zero except at the single point $x = \xi$. So you can replace $\phi(x)$ by $\phi(\xi)$ without changing anything. And $\phi(\xi)$ is a constant that can be taken out of the integral.

You can reverse that statement and define the delta function as the "distribution" that produces the result above for any smooth function ϕ. (The functions ϕ are normally further constrained by a requirement that they must become zero at their ends.)

In a similar way you can also define the derivative of the delta function, the dipole δ'. It is the distribution for which, for any smooth ϕ,

$$\int_{x=-\infty}^{\infty} \phi(x)\delta'(x-\xi)\,\mathrm{d}x = -\phi'(\xi)$$

To see why you want to define it this way, perform a formal integration by parts.

Appendix D

Derivations

This appendix gives various derivations. Sometimes you need to see the derivation to judge whether a result is applicable in given circumstances. And some people like to see the derivation period.

D.1 Orthogonal coordinate derivatives

Since there is no fundamental difference between the three orthogonal coordinates, it suffices to show that

$$\frac{\partial \hat{\imath}_1}{\partial u_2} = \frac{1}{h_1}\frac{\partial h_2}{\partial u_1}\hat{\imath}_2$$

The other derivatives with $i \neq j$ go the same way, and the derivatives with $i = j$ can then be evaluated by writing $\hat{\imath}_i$ as a cross product of the other two unit vectors.

Now to derive the above result, the only thing we know a priori is that the derivative of a unit vector is normal to the unit vector, so:

$$\frac{\partial \hat{\imath}_1}{\partial u_2} = c_2\hat{\imath}_2 + c_3\hat{\imath}_3$$

But it is not obvious why c_2 would have to be $\partial h_2/h_1\partial u_1$ and c_3 would have to be zero.

Recall however that $\hat{\imath}_1$ was *defined* as

$$\frac{\partial \vec{r}}{\partial u_1} \equiv h_1\hat{\imath}_1$$

and if we differentiate this with respect to u_2, one of the terms will involve the desired derivative of $\hat{\imath}_1$:

$$\frac{\partial^2 \vec{r}}{\partial u_1 \partial u_2} = \frac{\partial h_1}{\partial u_2}\hat{\imath}_1 + h_1\frac{\partial \hat{\imath}_1}{\partial u_2}$$

Since we can change the order of differentiation without changing the derivative, we must have

$$\frac{\partial h_1}{\partial u_2}\hat{\imath}_1 + h_1\frac{\partial \hat{\imath}_1}{\partial u_2} = \frac{\partial h_2}{\partial u_1}\hat{\imath}_2 + h_2\frac{\partial \hat{\imath}_2}{\partial u_1}$$

so, writing out the derivative

$$\frac{\partial h_1}{\partial u_2}\hat{\imath}_1 + h_1(c_2\hat{\imath}_2 + c_3\hat{\imath}_3) = \frac{\partial h_2}{\partial u_1}\hat{\imath}_2 + h_2\frac{\partial \hat{\imath}_2}{\partial u_1}$$

Now compare $\hat{\imath}_2$ components in both sides, noting that the derivative of $\hat{\imath}_2$ is normal to $\hat{\imath}_2$. Then you see that the value of c_2 given above is indeed correct.

But it is not clear from the above why c_3 would have to be zero. To find that out, we dot the derivative $\partial^2\vec{r}/\partial u_1\partial u_2$ with $\partial\vec{r}/\partial u_3$, because the latter derivative is by definition $h_3\hat{\imath}_3$, so the dot product will give

$$\frac{\partial^2\vec{r}}{\partial u_1\partial u_2}\cdot\frac{\partial\vec{r}}{\partial u_3} = h_1c_3h_3$$

So to show that c_3 is zero, we must show that the dot product above is zero. That can be done with a bit of manipulation. The only thing you can do, of course, is shuffle the derivatives around. In particular, if you pull the derivative with respect to u_1 to the front of the entire thing,

$$\frac{\partial^2\vec{r}}{\partial u_1\partial u_2}\cdot\frac{\partial\vec{r}}{\partial u_3} = \frac{\partial}{\partial u_1}\left(\frac{\partial\vec{r}}{\partial u_2}\cdot\frac{\partial\vec{r}}{\partial u_3}\right) - \frac{\partial\vec{r}}{\partial u_2}\cdot\frac{\partial\vec{r}}{\partial u_1\partial u_3}$$

where the final term corrects for the additional term generated by pulling the u_1 derivative out. Note now that the dot product in the parentheses above is zero since the vectors are orthogonal. Only the final term survives.

Next, ask yourself: why pull the u_1 derivative out? Why not u_2? After all, u_1 and u_2 appear completely symmetrically in the expression. That suggests that really, what we should do is pull u_1 out of half the term and u_2 out of the other half. That gives

$$-\tfrac{1}{2}\frac{\partial\vec{r}}{\partial u_2}\cdot\frac{\partial\vec{r}}{\partial u_1\partial u_3} - \tfrac{1}{2}\frac{\partial\vec{r}}{\partial u_1}\cdot\frac{\partial\vec{r}}{\partial u_2\partial u_3}$$

That is seen to be the same as

$$-\tfrac{1}{2}\frac{\partial}{\partial u_3}\left(\frac{\partial\vec{r}}{\partial u_2}\cdot\frac{\partial\vec{r}}{\partial u_1}\right)$$

and the dot product in parenthesis is zero. So indeed c_3 is zero.

D.2 Harmonic functions are analytic

This note shows that harmonic functions have converging Taylor series. The proof uses the Poisson integral formula derived in a later chapter.

Consider a point in the interior of the domain in which a function u is harmonic. Let the largest sphere around the point that stays inside the domain have radius R. It is to be shown that u has a Taylor series around the considered point with a finite radius of convergence.

To do so, scale the coordinates so that the radius of the sphere becomes 1. Move the origin of your coordinate system to the center of the sphere. The Poisson integral formula then says:

$$u = (1 - r^2) \int_S \frac{g}{|\vec{\xi} - \vec{x}|^n} \frac{\mathrm{d}S}{S}$$

where $r = |\vec{x}|$, S is the surface of the sphere, n is the number of dimensions, and g is the value of u on the surface of the sphere. Rotate the coordinate system so that the point at which the solution is to be found is on the x-axis. That gives

$$u = (1 - x^2) \int_S \frac{g}{|1 - 2x\xi + x^2|^{n/2}} \frac{\mathrm{d}S}{S}$$

Take a factor $|1 + x^2|^{n/2}$ out of the denominator of the integrand. For what is left, define a new variable

$$\alpha \equiv \frac{2x}{1 + x^2}$$

(Note that $-1 < x < 1$ corresponds to $-1 < \alpha < 1$.) That gives

$$u = \frac{1 - x^2}{(1 + x^2)^{n/2}} \int_S \frac{g}{|1 - \xi\alpha|^{n/2}} \frac{\mathrm{d}S}{S}$$

Now the first factor is an analytical function of x in the range $-1 < x < 1$ and is of no concern for now. In the integral, do a Taylor series expansion of the denominator. That gives the integral as a power series in α. Note that the convergence of this power series is no worse than that of $(1 - \alpha)^{-n/2}$ since

$$\int_S g\xi^n \frac{\mathrm{d}S}{S} \leq \int_S g \frac{\mathrm{d}S}{S}$$

So the integral is an analytical function of α with a radius of convergence of 1. And α is in turn an analytical function of x. Allow α and x to have complex values. Within a finite distance from $x = 0$, α will be less than one in magnitude. In that range then, the integral will be an analytical function of x. And then so will u, because the factor in front of the integral is analytical too for $|x| < 1$. That means that the Taylor series in terms of x converges within the indicated range.

Presumably, the radius of convergence is 1, like it is in 2 and 3 dimensions. However, the above proof shows only that it is greater than zero. Apparently, you will need to do a separation of variables solution to show the unit radius of convergence.

D.3 Some properties of harmonic functions

The mean value theorem says that if you take a sphere around some point, the average of u on the surface of that sphere is the value of u at the center of the sphere. That is true as long as u satisfies the Laplace equation inside the sphere.

To prove the mean value theorem, take the origin of your coordinate system at the center of the sphere. Then integrate the Laplace equation over the volume of a sphere. Use the divergence theorem to get

$$0 = \int_\Omega \nabla \cdot \nabla u \, \mathrm{d}V = \int_{\delta\Omega} \vec{n} \cdot \nabla u \, \mathrm{d}S = \int_{\delta\Omega} \frac{\partial u}{\partial n} \, \mathrm{d}S$$

But the normal direction is the radial direction, so

$$0 = \int_{\delta\Omega} \frac{\partial u}{\partial r} \, \mathrm{d}S$$

This holds for a spherical surface of any radius r around the origin as long as the Laplace equation is applicable in the sphere. Divide by the total spherical surface:

$$0 = \int_{\delta\Omega} \frac{\partial u}{\partial r} \frac{\mathrm{d}S}{S}$$

The ratio $\mathrm{d}S/S$ does not depend on the radius. So you can take the derivative out of the integral to get

$$0 = \frac{\mathrm{d}}{\mathrm{d}r} \int_{\delta\Omega} u \frac{\mathrm{d}S}{S}$$

The integral is by definition the average of u on the spherical surface. So it does not depend on the radius of the surface. That means it remains the same when you let the radius of the spherica; surface go to zero. But when $r = 0$, you get the average of u at the origin, which is simply u at the origin, since the origin is a single point.

The minimum and maximum properties follow immediately from the mean value theorem. Note that the minimum property implies the maximum property: the maximum of a harmonic function u is the minimum of the harmonic function $-u$.

To show the minimum property, consider an arbitrary point in the interior of the domain. Put a sphere of a sufficiently small radius around the point; the sphere must stay in the domain. Now u is at the considered point equals the average value of u on the spherical surface. And an average is always in between

the minimum and maximum values. So u at the considered point cannot be less than the smallest u value on the spherical surface. So u at the considered point cannot be a *unique* minimum, lower than all other u values.

You may wonder whether u might be a minimum that is not unique. But for the average of u on the spherical surface to equal the lowest value of u requires that u is everywhere the lowest value on the surface. If u would be above the minimum anywhere, the average would be above the minimum. So any spherical surface around the considered point has the same value of u as the considered point. At least as long as the sphere stays inside the domain. In other word, u is constant within some sphere around the considered point that goes up to the boundary. And for every point inside that sphere there is again a surrounding sphere in which u is constant. You can then readily see from a sketch that this means that u will have to be the minimum everywhere. In other words, u must be a constant for there to be a nonunique minimum in the interior of the region.

D.4 Coordinate transformation derivation

This note derives the coordinate transformation formulae of chapter 18.7.2.

According to the total differential formula from calculus:

$$\frac{\partial u}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial u}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_i}$$

This formula is used to transform the first order derivatives of u.

Differentiating once more, using the product rule of differentiation and again the total differential formulae for the first factor of the product:

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{k=1}^{n} \left[\left(\sum_{l=1}^{n} \frac{\partial^2 u}{\partial \xi_k \partial \xi_l} \frac{\partial \xi_l}{\partial x_j} \right) \frac{\partial \xi_k}{\partial x_i} + \frac{\partial u}{\partial \xi_k} \frac{\partial^2 \xi_k}{\partial x_i \partial x_j} \right]$$

If you plug the formula for the second order derivatives above into the left hand side of the original partial differential equation and rearrange, you get the transformed equation.

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_l}{\partial x_j} \right) \frac{\partial^2 u}{\partial \xi_k \partial \xi_l} = d - \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{\partial^2 \xi_k}{\partial x_i \partial x_j} \right) \frac{\partial u}{\partial \xi_k}$$

The coefficients of the transformed matrix A' and the transformed right hand side can be read off from the above expression.

D.5 2D coordinate transformation derivation

This note gives a more detailed description where the expressions (18.22) for the coefficients a', b', c' and d' comes from.

In terms of the notations for the general case, you have

$$a = a_{11} \qquad b = a_{12} = a_{21} \qquad c = a_{22} \qquad x = x_1 \qquad y = x_2$$

and

$$a' = a'_{11} \qquad b' = a'_{12} = a'_{21} \qquad c' = a'_{22} \qquad \xi = x_1 \qquad \eta = x_2$$

The introduction noted that the new coefficients can be found from

$$a'_{kl} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial \xi_k}{\partial x_i}\frac{\partial \xi_l}{\partial x_j} \qquad d' = d - \sum_{k=1}^{n}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 \xi_k}{\partial x_i \partial x_j}\right)\frac{\partial u}{\partial \xi_k}$$

As an example, let's find the value for a'. In terms of the notations in the general case, a' is the 1,1 element of matrix A': $a' = a'_{11}$. To get $a' = a'_{11}$, put $k = l = 1$ in

$$a'_{kl} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial \xi_k}{\partial x_i}\frac{\partial \xi_l}{\partial x_j}.$$

That turns ξ_k and ξ_l into ξ_1, or ξ for short:

$$a' = a'_{11} = \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}\frac{\partial \xi}{\partial x_i}\frac{\partial \xi}{\partial x_j}$$

If we write out the four terms of the double sum explicitly, that becomes:

$$a' = a_{11}\frac{\partial \xi}{\partial x_1}\frac{\partial \xi}{\partial x_1} + a_{12}\frac{\partial \xi}{\partial x_1}\frac{\partial \xi}{\partial x_2} + a_{21}\frac{\partial \xi}{\partial x_2}\frac{\partial \xi}{\partial x_1} + a_{22}\frac{\partial \xi}{\partial x_2}\frac{\partial \xi}{\partial x_2}$$

Now note that by definition $a_{11} = a$, $a_{12} = a_{21} = b$, $a_{22} = c$, $x_1 = x$, and $x_2 = y$, and you get the expression for a' claimed:

$$a' = a\left(\xi_x\right)^2 + 2b\left(\xi_x\right)\left(\xi_y\right) + c\left(\xi_y\right)^2$$

The expressions for b', c' and d' may be verified similarly.

D.6 2D elliptical transformation

To bring two-dimensional elliptical equations in the two-dimensional canonical form, you need to solve, say, the ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b + i\sqrt{ac - b^2}}{a}$$

Note now that even if you take $\mathrm{d}x$ to be real, $\mathrm{d}y$ will be complex. And that means that you need to know what happens to the coefficients a, b, and c when

you depart the real x, y-plane into the complex domain. That may be fine if you know the coefficients analytically, but otherwise it is a problem.

Assuming that you can solve the system, call the integration constant $\tilde{\xi}$. Assuming that it is a differentiable function of x and y, it will satisfy

$$a(\tilde{\xi}_x)^2 + 2b\tilde{\xi}_x\tilde{\xi}_y + c(\tilde{\xi}_y)^2 = 0$$

Now set

$$\tilde{\xi} = \xi + \mathrm{i}\eta$$

If you plug that in the equation above and multiply out, you get

$$[a(\xi_x)^2 + 2b\xi_x\xi_y + c(\xi_y)^2] - [a(\eta_x)^2 + 2b\eta_x\eta_y + c(\eta_y)^2]$$
$$+2\mathrm{i}[a\xi_x\eta_x + b\xi_x\eta_y + b\xi_y\eta_x + c\xi_y\eta_y] = 0$$

Now within the square brackets above, you find the generic expressions for the coefficients a', c' and b', respectively, of the transformed partial diffential equation. For a complex number to be zero, both its real and its imaginary part must be zero. It follows that $a' = c'$ and that $b' = 0$.

Appendix N

Notes

This appendix collects various notes on the material. This sort of material is often given in footnotes at the bottom of the text. However, such a footnote is distracting. You tend to read them even if they are probably not really that important to you. Also, footnotes have to be concise, or they make a mess of the main text.

N.1 Why this book?

See the preface.

N.2 History and wish list

- Jan. 11, 2011. The first version of this manuscript was posted.
- Jan. 14, 2011. A spell check was done. Various provements in the first chapter were made.
- Jan.-Feb. xx, 2011. Various additions, improvements.
- Feb. 16, 2011. Version 0.7 alpha. Some corrections. Made explanation of classification scheme a separate subsection.
- Feb. 18, 2011. Version 0.9 alpha. Various minor rewrites.
- Feb. 21, 2011. Version 0.10 alpha. Various minor rewrites and corrections.
- Feb. 23, 2011. Version 0.11 alpha. Various minor rewrites and corrections. Added energy method for heat and wave equation.
- Feb. 22, 2011. Version 0.12 alpha. Various minor rewrites and corrections. Added questions for Laplace energy methods.

- Mar. 10, 2011. Version 0.13 alpha. Added new intro for Green's functions. Added questions for Green's functions.
- Mar. 18, 2011. Version 0.14 alpha. Added new intro for Poisson's integral formula. Added questions for Poisson's integral formula.
- Mar. 21, 2011. Version 0.15 alpha. Reorganized notes. Cleaned up first order PDE a bit.

Web Pages

Below is a list of relevant web pages.

1. Wikipedia[1]

   A valuable source source of information on about every loose end, though somewhat uneven. Some great, some confusing, some overly technical.

[1] `http://wikipedia.org`

References

[1] F. Ayres and E. Mendelson. *Calculus*. Schaum's Outline Series. McGraw-Hill, 5th edition, 2009. 4, 7, 9, 18, 19, 22, 23, 25, 27, 29, 33, 38, 41, 46, 50, 52

[2] G. Dahlquist and Å. Björck. *Numerical Methods*. Prentice-Hall, 1974. Translated by Ned Anderson. 63

[3] P. DuChateau and D.W. Zachmann. *Partial Differential Equations*. Schaum's Outline Series. McGraw-Hill, 1986. xvii, 138

[4] M.R. Spiegel and J. Liu. *Mathematical Handbook of Formulas and Tables*. Schaum's Outline Series. McGraw-Hill, second edition, 1999. 295, 301, 311

Notations

The below are the simplest possible descriptions of various symbols, just to help you keep reading if you do not remember/know what they stand for.

Watch it. There are so many ad hoc usages of symbols, some will have been overlooked here. Always use common sense first in guessing what a symbol means in a given context.

· A dot might indicate

- A dot product between vectors, if in between them.
- A time derivative of a quantity, if on top of it.

And also many more prosaic things (punctuation signs, decimal points, ...).

× Multiplication symbol. May indicate:

- An emphatic multiplication.
- Multiplication continued on the next line or from the previous line.
- A vectorial product between vectors. In index notation, the i-th component of $\vec{v} \times \vec{w}$ equals

$$(\vec{v} \times \vec{w})_i = v_{\bar{i}} w_{\bar{\bar{i}}} - v_{\bar{\bar{i}}} w_{\bar{i}}$$

where $\bar{i}$ is the index following i in the sequence 123123..., and $\bar{\bar{i}}$ the one preceding it (or second following). Alternatively, evaluate the determinant

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

! Might be used to indicate a factorial. Example: $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$.

The function that generalizes $n!$ to noninteger values of n is called the gamma function; $n! = \Gamma(n+1)$. The gamma function generalization is

due to, who else, Euler. (However, the fact that $n! = \Gamma(n+1)$ instead of $n!$ $= \Gamma(n)$ is due to the idiocy of Legendre.) In Legendre-resistant notation,

$$n! = \int_0^\infty t^n e^{-t}\,\mathrm{d}t$$

Straightforward integration shows that 0! is 1 as it should, and integration by parts shows that $(n+1)! = (n+1)n!$, which ensures that the integral also produces the correct value of $n!$ for any higher integer value of n than 0. The integral, however, exists for any real value of n above -1, not just integers. The values of the integral are always positive, tending to positive infinity for both $n \downarrow -1$, (because the integral then blows up at small values of t), and for $n \uparrow \infty$, (because the integral then blows up at medium-large values of t). In particular, Stirling's formula says that for large positive n, $n!$ can be approximated as

$$n! \sim \sqrt{2\pi n} n^n e^{-n}\,[1 + \ldots]$$

where the value indicated by the dots becomes negligibly small for large n. The function $n!$ can be extended further to any complex value of n, except the negative integer values of n, where $n!$ is infinite, but is then no longer positive. Euler's integral can be done for $n = -\frac{1}{2}$ by making the change of variables $\sqrt{t} = u$, producing the integral $\int_0^\infty 2e^{-u^2}\,\mathrm{d}u$, or $\int_{-\infty}^\infty e^{-u^2}\,\mathrm{d}u$, which equals $\sqrt{\int_{-\infty}^\infty e^{-x^2}\,\mathrm{d}x \int_{-\infty}^\infty e^{-y^2}\,\mathrm{d}y}$ and the integral under the square root can be done analytically using polar coordinates. The result is that

$$(-\frac{1}{2})! = \int_{-\infty}^\infty e^{-u^2}\,\mathrm{d}u = \sqrt{\pi}$$

To get $\frac{1}{2}!$, multiply by $\frac{1}{2}$, since $n! = n(n-1)!$.

A double exclamation mark may mean every second item is skipped, e.g. $5!! = 1 \times 3 \times 5$. In general, $(2n+1)!! = (2n+1)!/2^n n!$. Of course, 5!! should logically mean (5!)!. Logic would indicate that $5 \times 3 \times 1$ should be indicated by something like 5!'. But what is logic in physics?

| May indicate:

- The magnitude or absolute value of the number or vector, if enclosed between a pair of them.
- The determinant of a matrix, if enclosed between a pair of them.
- The norm of the function, if enclosed between two pairs of them.

$\sum$ Summation symbol. Example: if in three dimensional space a vector $\vec{f}$ has components $f_1 = 2$, $f_2 = 1$, $f_3 = 4$, then $\sum_{\text{all } i} f_i$ stands for $2+1+4 = 7$.

One important thing to remember: the symbol used for the summation index does not make a difference: $\sum_{\text{all } j} f_j$ is exactly the same as $\sum_{\text{all } i} f_i$. So freely rename the index, but always make sure that the new name is not already used for something else in the part that it appears in. If you use the same name for two different things, it becomes a mess.

Related to that, $\sum_{\text{all } i} f_i$ is *not* something that depends on an index i. It is just a combined simple number. Like 7 in the example above. It is commonly said that the summation index "sums away."

$\prod$ Multiplication symbol. Example: if in three dimensional space a vector $\vec{f}$ has components $f_1 = 2$, $f_2 = 1$, $f_3 = 4$, then $\prod_{\text{all } i} f_i$ stands for $2 \times 1 \times 4 = 6$.

One important thing to remember: the symbol used for the multiplications index does not make a difference: $\prod_{\text{all } j} f_j$ is exactly the same as $\prod_{\text{all } i} f_i$. So freely rename the index, but always make sure that the new name is not already used for something else in the part that it appears in. If you use the same name for two different things, it becomes a mess.

Related to that, $\prod_{\text{all } i} f_i$ is *not* something that depends on an index i. It is just a combined simple number. Like 6 in the example above. It is commonly said that the multiplication index "factors away." (By who?)

$\int$ Integration symbol, the continuous version of the summation symbol. For example,

$$\int_{\text{all } x} f(x)\,\mathrm{d}x$$

is the summation of $f(x)\,\mathrm{d}x$ over all infinitesimally small fragments $\mathrm{d}x$ that make up the entire x-range. For example, $\int_{x=0}^{2}(2+x)\,\mathrm{d}x$ equals $3 \times 2 = 6$; the average value of $2+x$ between $x = 0$ and $x = 2$ is 3, and the sum of all the infinitesimally small segments $\mathrm{d}x$ gives the total length 2 of the range in x from 0 to 2.

One important thing to remember: the symbol used for the integration variable does not make a difference: $\int_{\text{all } y} f(y)\,\mathrm{d}y$ is exactly the same as $\int_{\text{all } x} f(x)\,\mathrm{d}x$. So freely rename the integration variable, but always make sure that the new name is not already used for something else in the part it appears in. If you use the same name for two different things, it becomes a mess.

Related to that $\int_{\text{all } x} f(x)\,\mathrm{d}x$ is *not* something that depends on a variable x. It is just a combined number. Like 6 in the example above. It is commonly said that the integration variable "integrates away."

$\rightarrow$ May indicate:

- An approaching process. $\lim_{\varepsilon \to 0}$ indicates for practical purposes the value of the expression following the lim when ε is extremely small. Similarly, $\lim_{r \to \infty}$ indicates the value of the following expression when r is extremely large.
- The fact that the left side leads to, or implies, the right-hand side.

$\vec{}$ Vector symbol. An arrow above a letter indicates it is a vector. A vector is a quantity that requires more than one number to be characterized. Typical vectors in physics include position $\vec{r}$, velocity $\vec{v}$, linear momentum $\vec{p}$, acceleration $\vec{a}$, force $\vec{F}$, angular momentum $\vec{L}$, etcetera.

$'$ May indicate:

- A derivative of a function. Examples: $1' = 0$, $x' = 1$, $\sin'(x) = \cos(x)$, $\cos'(x) = -\sin(x)$, $(e^x)' = e^x$.
- A small or modified quantity.

$\boldsymbol{\nabla}$ The spatial differentiation operator nabla. In Cartesian coordinates:

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \hat{\imath} \frac{\partial}{\partial x} + \hat{\jmath} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

Nabla can be applied to a scalar function f in which case it gives a vector of partial derivatives called the gradient of the function:

$$\operatorname{grad} f = \nabla f = \hat{\imath} \frac{\partial f}{\partial x} + \hat{\jmath} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z}.$$

Nabla can be applied to a vector in a dot product multiplication, in which case it gives a scalar function called the divergence of the vector:

$$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

or in index notation

$$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i}$$

Nabla can also be applied to a vector in a vectorial product multiplication, in which case it gives a vector function called the curl or rot of the vector. In index notation, the i-th component of this vector is

$$(\operatorname{curl} \vec{v})_i = (\operatorname{rot} \vec{v})_i = (\nabla \times \vec{v})_i = \frac{\partial v_{\bar{i}}}{\partial x_{\bar{i}}} - \frac{\partial v_{\bar{i}}}{\partial x_{\bar{i}}}$$

where $\bar{\imath}$ is the index following i in the sequence 123123..., and $\bar{\bar{\imath}}$ the one preceding it (or the second following it).

The operator ∇^2 is called the Laplacian. In Cartesian coordinates:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Sometimes the Laplacian is indicated as Δ.

In non Cartesian coordinates, don't guess; look these operators up in a table book, [4, pp. 124-126]: . For example, in spherical coordinates,

$$\nabla = \hat{\imath}_r \frac{\partial}{\partial r} + \hat{\imath}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\imath}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \tag{N.1}$$

That allows the gradient of a scalar function f, i.e. ∇f, to be found immediately. But if you apply ∇ on a vector, you have to be very careful because you also need to differentiate $\hat{\imath}_r$, $\hat{\imath}_\theta$, and $\hat{\imath}_\phi$. In particular, the correct divergence of a vector $\vec{v}$ is

$$\nabla \cdot \vec{v} = \frac{1}{r^2} \frac{\partial r^2 v_r}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial \sin\theta \, v_\theta}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial v_\phi}{\partial \phi} \tag{N.2}$$

The curl $\nabla \times \vec{v}$ of the vector is

$$\frac{\hat{\imath}_r}{r \sin\theta} \left(\frac{\partial \sin\theta \, v_\phi}{\partial \theta} - \frac{\partial v_\theta}{\partial \phi} \right) + \frac{\hat{\imath}_\theta}{r} \left(\frac{1}{\sin\theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial r \, v_\phi}{\partial r} \right) + \frac{\hat{\imath}_\phi}{r} \left(\frac{\partial r \, v_\theta}{\partial r} - \frac{\partial v_r}{\partial \theta} \right) \tag{N.3}$$

Finally the Laplacian is:

$$\nabla^2 = \frac{1}{r^2} \left\{ \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right\} \tag{N.4}$$

See also "spherical coordinates."

Cylindrical coordinates are usually indicated as r, θ and z. Here z is the Cartesian coordinate, while r is the distance from the z-axis and θ the angle around the z axis. In two dimensions, i.e. without the z terms, they are usually called polar coordinates. In cylindrical coordinates:

$$\nabla = \hat{\imath}_r \frac{\partial}{\partial r} + \hat{\imath}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\imath}_z \frac{\partial}{\partial z} \tag{N.5}$$

$$\nabla \cdot \vec{v} = \frac{1}{r} \frac{\partial r v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \tag{N.6}$$

$$\nabla \times \vec{v} = \hat{\imath}_r \left(\frac{1}{r} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \right) + \hat{\imath}_\theta \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right) + \frac{\hat{\imath}_z}{r} \left(\frac{\partial r v_\theta}{\partial r} - \frac{\partial v_r}{\partial \theta} \right) \tag{N.7}$$

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{N.8}$$

$\Box$ The D'Alembertian is defined as

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

where c is a constant called the wave speed.

* A superscript star normally indicates a complex conjugate. In the complex conjugate of a number, every i is changed into a $-$i.

$<$ Less than.

$\leqslant$ Less than or equal.

$>$ Greater than.

$\geqslant$ Greater than or equal.

$=$ Equals sign. The quantity to the left is the same as the one to the right.

$\equiv$ Emphatic equals sign. Typically means "by definition equal" or "everywhere equal."

$\approx$ Indicates approximately equal. Read it as "is approximately equal to."

$\sim$ Indicates approximately equal. Often used when the approximation applies only when something is small or large. Read it as "is approximately equal to" or as "is asymptotically equal to."

$\propto$ Proportional to. The two sides are equal except for some unknown constant factor.

$\boldsymbol{\Gamma}$ (Gamma) May indicate:

- The Gamma function. Look under "!" for details.

$\boldsymbol{\Delta}$ (capital delta) May indicate:

- An increment in the quantity following it.
- Often used to indicate the Laplacian ∇^2.

$\boldsymbol{\delta}$ (delta) May indicate:

- With two subscripts, the "Kronecker delta", which by definition is equal to one if its two subscripts are equal, and zero in all other cases.

- Without two subscripts, the "Dirac delta function", which is infinite when its argument is zero, and zero if it is not. In addition the infinity is such that the integral of the delta function over its single nonzero point is unity. The delta function is not a normal function, but a distribution. It is best to think of it as the approximate function shown in the right hand side of figure 19.5 for a very, very, small positive value of ε.

  One often important way to create a three-dimensional delta function in spherical coordinates is to take the Laplacian of the function $-1/4\pi r$. In two dimensions, take the Laplacian of $\ln(r)/2\pi$ to get a delta function.
- Often used to indicate a small amount of the following quantity, or of a small change in the following quantity. There are nuanced differences in the usage of δ, ∂ and d that are too much to go in here.
- Often used to indicate a second small quantity in addition to ε.

$\boldsymbol{\partial}$ (partial) Indicates a vanishingly small change or interval of the following variable. For example, $\partial f/\partial x$ is the ratio of a vanishingly small change in function f divided by the vanishingly small change in variable x that causes this change in f. Such ratios define derivatives, in this case the partial derivative of f with respect to x.

$\boldsymbol{\varepsilon}$ (variant of epsilon) May indicate:

- A very small quantity.

$\boldsymbol{\eta}$ (eta) May be used to indicate a y-position.

$\boldsymbol{\Theta}$ (capital theta) Used in this book to indicate some function of θ to be determined.

$\boldsymbol{\theta}$ (theta) May indicate:

- In spherical coordinates, the angle from the chosen z axis, with apex at the origin.
- A z-position.
- A generic angle, like the one between the vectors in a cross or dot product.

$\boldsymbol{\vartheta}$ (variant of theta) An alternate symbol for θ.

$\boldsymbol{\lambda}$ (lambda) May indicate:

- Wave length.

- An eigenvalue.
- Some multiple of something.

$\boldsymbol{\xi}$ (xi) May indicate:

- An x-position.
- An integration variable.

$\boldsymbol{\pi}$ (pi) May indicate:

- A geometrical constant with value 3.141,592,653,589,793,238,462... The area of a circle of radius r is πr^2 and its perimeter is $2\pi r$. The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ and its surface is $4\pi r^2$. A 180° angle expressed in radians is π. Note also that $e^{\pm i\pi} = -1$ and $e^{\pm i2\pi} = 1$.

$\boldsymbol{\rho}$ (rho) May indicate:

- Scaled radial coordinate.
- Radial coordinate.

$\boldsymbol{\tau}$ (tau) May indicate:

- A time or time interval.

$\boldsymbol{\Phi}$ (capital phi) May indicate:

- Some function of ϕ to be determined.

$\boldsymbol{\phi}$ (phi) May indicate:

- In spherical coordinates, the angle around the chosen z axis. Increasing ϕ by 2π encircles the z-axis exactly once.
- A phase angle.
- Something equivalent to an angle.

$\boldsymbol{\varphi}$ (variant of phi) May indicate:

- A change in angle ϕ.
- An alternate symbol for ϕ.

$\boldsymbol{\omega}$ (omega) May indicate:

- Angular frequency.

A May indicate:

- Some generic matrix.
- Some constant.
- Area.

a May indicate:

- Acceleration.
- Start point of an integration interval.
- Some coefficient.
- Some constant.

absolute May indicate:

- The absolute value of a real number a is indicated by $|a|$. It equals a is a is positive or zero and $-a$ if a is negative.
- The absolute value of a complex number a is indicated by $|a|$. It equals the length of the number plotted as a vector in the complex plane. This simplifies to above definition if a is real.

adjoint The adjoint A^H or $A^\dagger$ of a matrix is the complex-conjugate transpose of the matrix.

Alternatively, it is the matrix you get if you take it to the other side of an inner product. (While keeping the value of the inner product the same regardless of whatever two vectors or functions may be involved.)

"Hermitian"matrices are "self-adjoint;"they are equal to their adjoint. "Skew-Hermitian"matrices are the negative of their adjoint.

"Unitary"matrices are the inverse of their adjoint. Unitary matrices generalize rotations and reflections of vectors. Unitary operators preserve inner products.

Fourier transforms are unitary operators on account of the Parseval equality that says that inner products are preserved.

angle Consider two semi-infinite lines extending from a common intersection point. Then the angle between these lines is defined in the following way: draw a unit circle in the plane of the lines and centered at their intersection point. The angle is then the length of the circular arc that is in between the lines. More precisely, this gives the angle in radians, rad.

Sometimes an angle is expressed in degrees, where 2π rad is taken to be $360°$. However, using degrees is usually a very bad idea in science.

In three dimensions, you may be interested in the so-called "solid angle" Ω inside a conical surface. This angle is defined in the following way: draw a sphere of unit radius centered at the apex of the conical surface. Then the solid angle is the area of the spherical surface that is inside the cone. Solid angles are in steradians. The cone does not need to be a circular one, (i.e. have a circular cross section), for this to apply. In fact, the most common case is the solid angle corresponding to an infinitesimal element $\mathrm{d}\theta \times \mathrm{d}\phi$ of spherical coordinate system angles. In that case the surface of the unit sphere inside the conical surface is is approximately rectangular, with sides $\mathrm{d}\theta$ and $\sin(\theta)\mathrm{d}\phi$. That makes the enclosed solid angle equal to $\mathrm{d}\Omega = \sin(\theta)\mathrm{d}\theta\mathrm{d}\phi$.

B May indicate:

- A generic second matrix.
- Some constant.

b May indicate:

- End point of an integration interval.
- Some coefficient.
- Some constant.

basis A basis is a minimal set of vectors or functions that you can write all other vectors or functions in terms of. For example, the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are a basis for normal three-dimensional space. Every three-dimensional vector can be written as a linear combination of the three.

C May indicate:

- A third matrix.
- A constant.

Cauchy-Schwartz inequality The Cauchy-Schwartz inequality describes a limitation on the magnitude of inner products. In particular, it says that for any vectors $\vec{v}$ and $\vec{w}$

$$|\vec{v}^H \vec{w}| \leq |\vec{v}||\vec{w}|$$

For example, if $\vec{v}$ and $\vec{w}$ are real vectors, the inner product is the dot product and we have

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$$

where $|\vec{v}|$ is the length of vector $\vec{v}$ and $|\vec{w}|$ the one of $\vec{w}$, and θ is the angle in between the two vectors. Since a cosine is less than one in magnitude, the Cauchy-Schwartz inequality is therefore true for vectors.

cos The cosine function, a periodic function oscillating between 1 and -1 as shown in [4, pp. 40-]. See also "sin."

curl The curl of a vector $\vec{v}$ is defined as $\operatorname{curl} \vec{v} = \operatorname{rot} \vec{v} = \nabla \times \vec{v}$.

d Indicates a vanishingly small change or interval of the following variable. For example, $\mathrm{d}x$ can be thought of as a small segment of the x-axis.

In three dimensions, $\mathrm{d}^3\vec{r} \equiv \mathrm{d}x\mathrm{d}y\mathrm{d}z$ is an infinitesimal volume element. The symbol $\int$ means that you sum over all such infinitesimal volume elements.

derivative A derivative of a function is the ratio of a vanishingly small change in a function divided by the vanishingly small change in the independent variable that causes the change in the function. The derivative of $f(x)$ with respect to x is written as $\mathrm{d}f/\mathrm{d}x$, or also simply as f'. Note that the derivative of function $f(x)$ is again a function of x: a ratio f' can be found at every point x. The derivative of a function $f(x, y, z)$ with respect to x is written as $\partial f/\partial x$ to indicate that there are other variables, y and z, that do not vary.

determinant The determinant of a square matrix A is a single number indicated by $|A|$. If this number is nonzero, $A\vec{v}$ can be any vector $\vec{w}$ for the right choice of $\vec{v}$. Conversely, if the determinant is zero, $A\vec{v}$ can only produce a very limited set of vectors, though if it can produce a vector w, it can do so for multiple vectors $\vec{v}$.

There is a recursive algorithm that allows you to compute determinants from increasingly bigger matrices in terms of determinants of smaller matrices. For a 1×1 matrix consisting of a single number, the determinant is simply that number:

$$|a_{11}| = a_{11}$$

(This determinant should not be confused with the absolute value of the number, which is written the same way. Since you normally do not deal with 1×1 matrices, there is normally no confusion.) For 2×2 matrices, the determinant can be written in terms of 1×1 determinants:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = +a_{11} \begin{vmatrix} & \\ & a_{22} \end{vmatrix} - a_{12} \begin{vmatrix} & \\ a_{21} & \end{vmatrix}$$

so the determinant is $a_{11}a_{22} - a_{12}a_{21}$ in short. For 3×3 matrices, you have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$+a_{11}\begin{vmatrix} & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

and you already know how to work out those 2×2 determinants, so you now know how to do 3×3 determinants. Written out fully:

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

For 4×4 determinants,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} =$$

$$+a_{11}\begin{vmatrix} & a_{22} & a_{23} & a_{24} \\ & a_{32} & a_{33} & a_{34} \\ & a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & & a_{23} & a_{24} \\ a_{31} & & a_{33} & a_{34} \\ a_{41} & & a_{43} & a_{44} \end{vmatrix}$$

$$+a_{13}\begin{vmatrix} a_{21} & a_{22} & & a_{24} \\ a_{31} & a_{32} & & a_{34} \\ a_{41} & a_{42} & & a_{44} \end{vmatrix} - a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} & \\ a_{31} & a_{32} & a_{33} & \\ a_{41} & a_{42} & a_{43} & \end{vmatrix}$$

Etcetera. Note the alternating sign pattern of the terms.

As you might infer from the above, computing a good size determinant takes a large amount of work. Fortunately, it is possible to simplify the matrix to put zeros in suitable locations, and that can cut down the work of finding the determinant greatly. You are allowed to use the following manipulations without seriously affecting the computed determinant:

1. You can "transpose"the matrix, i.e. change its columns into its rows.
2. You can create zeros in a row by subtracting a suitable multiple of another row.
3. You can also swap rows, as long as you remember that each time that you swap two rows, it will flip over the sign of the computed determinant.

4. You can also multiply an entire row by a constant, but that will multiply the computed determinant by the same constant.

Applying these tricks in a systematic way, called "Gaussian elimination" or "reduction to lower triangular form", you can eliminate all matrix coefficients a_{ij} for which j is greater than i, and that makes evaluating the determinant pretty much trivial.

div(ergence) The divergence of a vector $\vec{v}$ is defined as $\operatorname{div} \vec{v} = \nabla \cdot \vec{v}$.

e May indicate:

- The basis for the natural logarithms. Equal to 2.718,281,828,459... This number produces the "exponential function" e^x, or $\exp(x)$, or in words "e to the power x", whose derivative with respect to x is again e^x. If a is a constant, then the derivative of e^{ax} is ae^{ax}. Also, if a is an ordinary real number, then $e^{\mathrm{i}a}$ is a complex number with magnitude 1.

***e*$^{\mathrm{i}ax}$** Assuming that a is an ordinary real number, and x a real variable, $e^{\mathrm{i}ax}$ is a complex function of magnitude one. The derivative of $e^{\mathrm{i}ax}$ with respect to x is $\mathrm{i}ae^{\mathrm{i}ax}$

eigenvector A concept from linear algebra. A vector $\vec{v}$ is an eigenvector of a matrix A if $\vec{v}$ is nonzero and $A\vec{v} = \lambda\vec{v}$ for some number λ called the corresponding eigenvalue.

exponential function A function of the form $e^{\cdots}$, also written as $\exp(\ldots)$. See "function" and "*e*."

F May indicate:

- The anti-derivative of some function f.
- Some function.

f May indicate:

- A generic function.
- A fraction.
- Frequency.

function A mathematical object that associates values with other values. A function $f(x)$ associates every value of x with a value f. For example, the function $f(x) = x^2$ associates $x = 0$ with $f = 0$, $x = \frac{1}{2}$ with $f = \frac{1}{4}$, $x = 1$ with $f = 1$, $x = 2$ with $f = 4$, $x = 3$ with $f = 9$, and more generally, any

arbitrary value of x with the square of that value x^2. Similarly, function $f(x) = x^3$ associates any arbitrary x with its cube x^3, $f(x) = \sin(x)$ associates any arbitrary x with the sine of that value, etcetera.

One way of thinking of a function is as a procedure that allows you, whenever given a number, to compute another number.

functional A functional associates entire functions with single numbers. For example, the expectation energy is mathematically a functional: it associates any arbitrary wave function with a number: the value of the expectation energy if physics is described by that wave function.

g May indicate:

- A second generic function.

Gauss' Theorem This theorem, also called divergence theorem or Gauss-Ostrogradsky theorem, says that for a continuously differentiable vector $\vec{v}$,

$$\int_V \nabla \cdot \vec{v}\, \mathrm{d}V = \int_A \vec{v} \cdot \vec{n}\, \mathrm{d}A$$

where the first integral is over the volume of an arbitrary region and the second integral is over all the surface area of that region; $\vec{n}$ is at each point found as the unit vector that is normal to the surface at that point.

grad(ient) The gradient of a scalar f is defined as $\mathrm{grad}\, f = \nabla f$.

hypersphere A hypersphere is the generalization of the normal three-dimensional sphere to n-dimensional space. A sphere of radius R in three-dimensional space consists of all points satisfying

$$r_1^2 + r_2^2 + r_3^2 \leqslant R^2$$

where r_1, r_2, and r_3 are Cartesian coordinates with origin at the center of the sphere. Similarly a hypersphere in n-dimensional space is *defined* as all points satisfying

$$r_1^2 + r_2^2 + \ldots + r_n^2 \leqslant R^2$$

So a two-dimensional "hypersphere" of radius R is really just a circle of radius R. A one-dimensional "hypersphere" is really just the line segment $-R \leqslant x \leqslant -R$.

The "volume" V_n and surface "area" A_n of an n-dimensional hypersphere is given by

$$V_n = C_n R^n \qquad A_n = n C_n R^{n-1}$$

$$C_n = \begin{cases} (2\pi)^{n/2}/2 \times 4 \times 6 \times \ldots \times n & \text{if } n \text{ is even} \\ (2\pi)^{(n-1)/2} 2/1 \times 3 \times 5 \times \ldots \times n & \text{if } n \text{ is odd} \end{cases}$$

(This is readily derived recursively. For a sphere of unit radius, note that the n-dimensional "volume" is an integration of $n-1$-dimensional volumes with respect to r_1. Then renotate r_1 as $\sin\phi$ and look up the resulting integral in a table book. The formula for the area follows because $V = \int A\mathrm{d}r$ where r is the distance from the origin.) In three dimensions, $C_3 = 4\pi/3$ according to the above formula. That makes the three-dimensional "volume" $4\pi R^3/3$ equal to the actual volume of the sphere, and the three-dimensional "area" $4\pi R^2$ equal to the actual surface area. On the other hand in two dimensions, $C_2 = \pi$. That makes the two-dimensional "volume" πR^2 really the *area* of the circle. Similarly the two-dimensional surface "area" $2\pi R$ is really the perimeter of the circle. In one dimensions $C_1 = 2$ and the "volume" $2R$ is really the length of the interval, and the "area" 2 is really its number of end points.

Often the infinitesimal n-dimensional "volume" element $\mathrm{d}^n\vec{r}$ is needed. This is the infinitesimal integration element for integration over all coordinates. It is:

$$\mathrm{d}^n\vec{r} = \mathrm{d}r_1\mathrm{d}r_2\ldots\mathrm{d}r_n = \mathrm{d}A_n\mathrm{d}r$$

Specifically, in two dimensions:

$$\mathrm{d}^2\vec{r} = \mathrm{d}r_1\mathrm{d}r_2 = \mathrm{d}x\mathrm{d}y = (r\,\mathrm{d}\theta)\mathrm{d}r = \mathrm{d}A_2\mathrm{d}r$$

while in three dimensions:

$$\mathrm{d}^3\vec{r} = \mathrm{d}r_1\mathrm{d}r_2\mathrm{d}r_3 = \mathrm{d}x\mathrm{d}y\mathrm{d}z = (r^2\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi)\mathrm{d}r = \mathrm{d}A_3\mathrm{d}r$$

The expressions in parentheses are $\mathrm{d}A_2$ in polar coordinates, respectively $\mathrm{d}A_3$ in spherical coordinates.

$\Im$ The imaginary part of a complex number. If $c = c_r + \mathrm{i}c_i$ with c_r and c_i real numbers, then $\Im(c) = c_i$. Note that $c - c^* = 2\mathrm{i}\Im(c)$.

$\boldsymbol{i}$ May indicate:

- The number of a particle.
- A summation index.
- A generic index or counter.

Not to be confused with i.

$\hat{\boldsymbol{\imath}}$ The unit vector in the x-direction.

$\mathbf{i}$ The standard square root of minus one: $\mathrm{i} = \sqrt{-1}$, $\mathrm{i}^2 = -1$, $1/\mathrm{i} = -\mathrm{i}$, $\mathrm{i}^* = -\mathrm{i}$.

index notation A more concise and powerful way of writing vector and matrix components by using a numerical index to indicate the components. For Cartesian coordinates, you might number the coordinates x as 1, y as 2, and z as 3. In that case, a sum like $v_x + v_y + v_z$ can be more concisely written as $\sum_i v_i$. And a statement like $v_x \neq 0$, $v_y \neq 0$, $v_z \neq 0$ can be more compactly written as $v_i \neq 0$. To really see how it simplifies the notations, have a look at the matrix entry. (And that one shows only 2 by 2 matrices. Just imagine 100 by 100 matrices.)

iff Emphatic "if." Should be read as "if and only if."

integer Integer numbers are the whole numbers: $\ldots, -2, -1, 0, 1, 2, 3, 4, \ldots$.

inverse (Of matrices.) If a matrix A converts a vector $\vec{v}$ into a vector $\vec{w}$, then the inverse of the matrix, A^{-1}, converts $\vec{w}$ back into $\vec{v}$.

In other words, $A^{-1}A = AA^{-1} = I$ with I the unit, or identity, matrix.

The inverse of a matrix only exists if the matrix is square and has nonzero determinant.

irrotational A vector $\vec{v}$ is irrotational if its curl $\nabla \times \vec{v}$ is zero.

$\boldsymbol{j}$ May indicate:

- A summation index.
- A generic index or counter.

$\hat{\boldsymbol{j}}$ The unit vector in the y-direction.

$\boldsymbol{k}$ May indicate:

- A generic summation index.

$\hat{\boldsymbol{k}}$ The unit vector in the z-direction.

$\boldsymbol{l}$ May indicate:

- A generic summation index.

$\boldsymbol{\ell}$ May indicate:

- A length.

lim Indicates the final result of an approaching process. $\lim_{\varepsilon \to 0}$ indicates for practical purposes the value of the following expression when ε is extremely small.

linear combination A very generic concept indicating sums of objects times coefficients. For example, a position vector $\vec{r}$ is the linear combination $x\hat{\imath}+y\hat{\jmath}+z\hat{k}$ with the objects the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ and the coefficients the position coordinates x, y, and z. A linear combination of a set of functions $f_1(x), f_2(x), f_3(x), \ldots, f_n(x)$ would be the function

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + \ldots c_n f_n(x)$$

where $c_1, c_2, c_3, \ldots, c_n$ are constants, i.e. independent of x.

linear dependence A set of vectors or functions is linearly dependent if at least one of the set can be expressed in terms of the others. Consider the example of a set of functions $f_1(x), f_2(x), \ldots, f_n(x)$. This set is linearly dependent if

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + \ldots c_n f_n(x) = 0$$

where at least one of the constants $c_1, c_2, c_2, \ldots, c_n$ is nonzero. To see why, suppose that say c_2 is nonzero. Then you can divide by c_2 and rearrange to get

$$f_2(x) = -\frac{c_1}{c_2} f_1(x) - \frac{c_3}{c_2} f_3(x) - \ldots - \frac{c_n}{c_2} f_n(x)$$

That expresses $f_2(x)$ in terms of the other functions.

linear independence A set of vectors or functions is linearly independent if none of the set can be expressed in terms of the others. Consider the example of a set of functions $f_1(x), f_2(x), \ldots, f_n(x)$. This set is linearly independent if

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + \ldots c_n f_n(x) = 0$$

only if every one of the constants $c_1, c_2, c_3, \ldots, c_n$ is zero. To see why, assume that say $f_2(x)$ could be expressed in terms of the others,

$$f_2(x) = C_1 f_1(x) + C_3 f_3(x) + \ldots + C_n f_n(x)$$

Then taking $c_2 = 1$, $c_1 = -C_1$, $c_3 = -C_3$, $\ldots c_n = -C_n$, the condition above would be violated. So f_2 cannot be expressed in terms of the others.

m May indicate:

- Number of rows in a matrix.
- A generic summation index or generic integer.

matrix A table of numbers.

As a simple example, a two-dimensional matrix A is a table of four numbers called a_{11}, a_{12}, a_{21}, and a_{22}:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

unlike a two-dimensional (ket) vector $\vec{v}$, which would consist of only two numbers v_1 and v_2 arranged in a column:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

(Such a vector can be seen as a "rectangular matrix" of size 2×1, but let's not get into that.)

In index notation, a matrix A is a set of numbers $\{a_{ij}\}$ indexed by two indices. The first index i is the row number, the second index j is the column number. A matrix turns a vector $\vec{v}$ into another vector $\vec{w}$ according to the recipe

$$w_i = \sum_{\text{all } j} a_{ij} v_j \quad \text{for all } i$$

where v_j stands for "the j-th component of vector $\vec{v}$," and w_i for "the i-th component of vector $\vec{w}$."

As an example, the product of A and $\vec{v}$ above is by definition

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{pmatrix}$$

which is another two-dimensional ket vector.

Note that in matrix multiplications like the example above, in geometric terms you take dot products between the rows of the first factor and the column of the second factor.

To multiply two matrices together, just think of the columns of the second matrix as separate vectors. For example:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

which is another two-dimensional matrix. In index notation, the ij component of the product matrix has value $\sum_k a_{ik}b_{kj}$.

The zero matrix is like the number zero; it does not change a matrix it is added to and turns whatever it is multiplied with into zero. A zero matrix is zero everywhere. In two dimensions:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

A unit matrix is the equivalent of the number one for matrices; it does not change the quantity it is multiplied with. A unit matrix is one on its "main diagonal" and zero elsewhere. The 2 by 2 unit matrix is:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

More generally the coefficients, $\{\delta_{ij}\}$, of a unit matrix are one if $i = j$ and zero otherwise.

The transpose of a matrix A, A^{T}, is what you get if you switch the two indices. Graphically, it turns its rows into its columns and vice versa. The Hermitian "adjoint" A^H is what you get if you switch the two indices and then take the complex conjugate of every element. If you want to take a matrix to the other side of an inner product, you will need to change it to its Hermitian adjoint. "Hermitian matrices" are equal to their Hermitian adjoint, so this does nothing for them.

See also "determinant" and "eigenvector."

n May indicate:

- Number of columns in a matrix.
- A generic summation index or generic integer.
- A natural number.

and maybe some other stuff.

natural Natural numbers are the numbers: $1, 2, 3, 4, \ldots$.

normal A normal operator or matrix is one that has orthonormal eigenfunctions or eigenvectors. Since eigenvectors are not orthonormal in general, a normal operator or matrix is abnormal!

For an operator or matrix A to be "normal," it must commute with its Hermitian adjoint, $[A, A^\dagger] = 0$. Hermitian matrices are normal since they are equal to their Hermitian adjoint. Skew-Hermitian matrices are normal since they are equal to the negative of their Hermitian adjoint. Unitary matrices are normal because they are the inverse of their Hermitian adjoint.

O May indicate the origin of the coordinate system.

opposite The opposite of a number a is $-a$. In other words, it is the additive inverse.

perpendicular bisector For two given points P and Q, the perpendicular bisector consists of all points R that are equally far from P as they are from Q. In two dimensions, the perpendicular bisector is the line that passes through the point exactly half way in between P and Q, and that is orthogonal to the line connecting P and Q. In three dimensions, the perpendicular bisector is the plane that passes through the point exactly half way in between P and Q, and that is orthogonal to the line connecting P and Q. In vector notation, the perpendicular bisector of points P and Q is all points R whose radius vector $\vec{r}$ satisfies the equation:

$$(\vec{r} - \vec{r}_P) \cdot (\vec{r}_Q - \vec{r}_P) = \tfrac{1}{2}(\vec{r}_Q - \vec{r}_P) \cdot (\vec{r}_Q - \vec{r}_P)$$

(Note that the halfway point $\vec{r} - \vec{r}_P = \frac{1}{2}(\vec{r}_Q - \vec{r}_P)$ is included in this formula, as is the half way point plus any vector that is normal to $(\vec{r}_Q - \vec{r}_P)$.)

phase angle Any complex number can be written in "polar form" as $c = |c|e^{\mathrm{i}\alpha}$ where both the magnitude $|c|$ and the phase angle α are real numbers. Note that when the phase angle varies from zero to 2π, the complex number c varies from positive real to positive imaginary to negative real to negative imaginary and back to positive real. When the complex number is plotted in the complex plane, the phase angle is the direction of the number relative to the origin. The phase angle α is often called the argument, but so is about everything else in mathematics, so that is not very helpful.

In complex time-dependent waves of the form $e^{\mathrm{i}(\omega t - \phi)}$, and its real equivalent $\cos(\omega t - \phi)$, the phase angle ϕ gives the angular argument of the wave at time zero.

$\boldsymbol{q}$ May indicate:

- Charge.
- Heat flux density.

$\boldsymbol{R}$ May indicate:

- Some radius.
- Some function of r to be determined.

$\boldsymbol{\Re}$ The real part of a complex number. If $c = c_r + \mathrm{i}c_i$ with c_r and c_i real numbers, then $\Re(c) = c_r$. Note that $c + c^* = 2\Re(c)$.

$\boldsymbol{r}$ May indicate:

- The radial distance from the chosen origin of the coordinate system.

- r_i typically indicates the i-th Cartesian component of the radius vector $\vec{r}$.
- Some ratio.

$\vec{r}$ The position vector. In Cartesian coordinates (x, y, z) or $x\hat{\imath} + y\hat{\jmath} + z\hat{k}$. In spherical coordinates $r\hat{\imath}_r$. Its three Cartesian components may be indicated by r_1, r_2, r_3 or by x, y, z or by x_1, x_2, x_3.

reciprocal The reciprocal of a number a is $1/a$. In other words, it is the multiplicative inverse.

rot The rot of a vector $\vec{v}$ is defined as $\operatorname{curl} \vec{v} \equiv \operatorname{rot} \vec{v} \equiv \nabla \times \vec{v}$.

scalar A quantity characterized by a single number.

sin The sine function, a periodic function oscillating between 1 and -1 as shown in [4, pp. 40-]. Good to remember: $\cos^2 \alpha + \sin^2 \alpha = 1$ and $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ and $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$.

spherical coordinates The spherical coordinates r, θ, and ϕ of an arbitrary point P are defined as

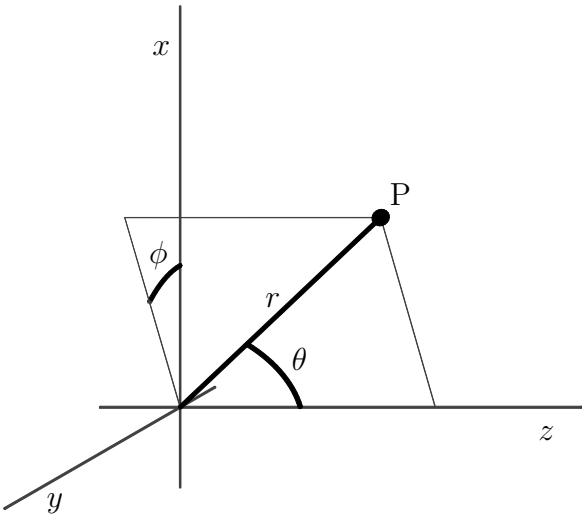


Figure N.1: Spherical coordinates of an arbitrary point P.

In Cartesian coordinates, the unit vectors in the x, y, and z directions are called $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. Similarly, in spherical coordinates, the unit vectors in the r, θ, and ϕ directions are called $\hat{\imath}_r$, $\hat{\imath}_\theta$, and $\hat{\imath}_\phi$. Here, say, the θ direction is defined as the direction of the change in position if you increase θ by an infinitesimally small amount while keeping r and φ the same. Note

therefore in particular that the direction of $\hat{\imath}_r$ is the same as that of $\vec{r}$; radially outward.

An arbitrary vector $\vec{v}$ can be decomposed in components v_r, v_θ, and v_ϕ along these unit vectors. In particular

$$\vec{v} \equiv v_r \hat{\imath}_r + v_\theta \hat{\imath}_\theta + v_\phi \hat{\imath}_\phi$$

Recall from calculus that in spherical coordinates, a volume integral of an arbitrary function f takes the form

$$\int f \, \mathrm{d}^3\vec{r} = \int\int\int f r^2 \sin\theta \, \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$$

In other words, the volume element in spherical coordinates is

$$\mathrm{d}V = \mathrm{d}^3\vec{r} = r^2 \sin\theta \, \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$$

Often it is convenient of think of volume integrations as a two-step process: first perform an integration over the angular coordinates θ and ϕ. Physically, that integrates over spherical surfaces. Then perform an integration over r to integrate all the spherical surfaces together. The combined infinitesimal angular integration element

$$\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta\mathrm{d}\phi$$

is called the infinitesimal "solid angle" $\mathrm{d}\Omega$. In two-dimensional polar coordinates r and θ, the equivalent would be the infinitesimal polar angle $\mathrm{d}\theta$. Recall that $\mathrm{d}\theta$, (in proper radians of course), equals the arclength of an infinitesimal part of the circle of integration divided by the circle radius. Similarly $\mathrm{d}\Omega$ is the surface of an infinitesimal part of the sphere of integration divided by the square sphere radius.

See the "∇" entry for the gradient operator and Laplacian in spherical coordinates.

Stokes' Theorem This theorem, first derived by Kelvin and first published by someone else I cannot recall, says that for any reasonably smoothly varying vector $\vec{v}$,

$$\int_A (\nabla \times \vec{v}) \, \mathrm{d}A = \oint \vec{v} \cdot \mathrm{d}\vec{r}$$

where the first integral is over any smooth surface area A and the second integral is over the edge of that surface. How did Stokes get his name on it? He tortured his students with it, that's how!

One important consequence of the Stokes theorem is for vector fields $\vec{v}$ that are "irrotational," i.e. that have $\nabla \times \vec{v} = 0$. Such fields can be written as

$$\vec{v} = \nabla f \qquad f(\vec{r}) \equiv \int_{\vec{\underline{r}}=\vec{r}_{\rm ref}}^{\vec{r}} \vec{v}(\vec{\underline{r}}) \cdot \mathrm{d}\vec{\underline{r}}$$

Here $\vec{r}_{\rm ref}$ is the position of an arbitrarily chosen reference point, usually the origin. The reason the field $\vec{v}$ can be written this way is the Stokes theorem. Because of the theorem, it does not make a difference along which path from $\vec{r}_{\rm ref}$ to $\vec{r}$ you integrate. (Any two paths give the same answer, as long as $\vec{v}$ is irrotational everywhere in between the paths.) So the definition of f is unambiguous. And you can verify that the partial derivatives of f give the components of $\vec{v}$ by approaching the final position $\vec{r}$ in the integration from the corresponding direction.

symmetry A symmetry is an operation under which an object does not change. For example, a human face is almost, but not completely, mirror symmetric: it looks almost the same in a mirror as when seen directly. The electrical field of a single point charge is spherically symmetric; it looks the same from whatever angle you look at it, just like a sphere does. A simple smooth glass (like a glass of water) is cylindrically symmetric; it looks the same whatever way you rotate it around its vertical axis.

t May indicate:

- Time.

triple product A product of three vectors. There are two different versions:

- The scalar triple product $\vec{a} \cdot (\vec{b} \times \vec{c})$. In index notation,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \sum_i a_i (b_{\bar{\imath}} c_{\bar{\bar{\imath}}} - b_{\bar{\bar{\imath}}} c_{\bar{\imath}})$$

  where $\bar{\imath}$ is the index following i in the sequence 123123..., and $\bar{\bar{\imath}}$ the one preceding it. This triple product equals the determinant $|\vec{a}\vec{b}\vec{c}|$ formed with the three vectors. Geometrically, it is plus or minus the volume of the parallelepiped that has vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ as edges. Either way, as long as the vectors are normal vectors and not operators,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$$

  and you can change the two sides of the dot product without changing the triple product, and/or you can change the sides in the vectorial product with a change of sign.

- The vectorial triple product $\vec{a} \times (\vec{b} \times \vec{c})$. In index notation, component number i of this triple product is

$$a_{\bar{\imath}}(b_i c_{\bar{\imath}} - b_{\bar{\imath}} c_i) - a_{\bar{\bar{\imath}}}(b_{\bar{\bar{\imath}}} c_i - b_i c_{\bar{\bar{\imath}}})$$

  which may be rewritten as

$$a_i b_i c_i + a_{\bar{\imath}} b_i c_{\bar{\imath}} + a_{\bar{\bar{\imath}}} b_i c_{\bar{\bar{\imath}}} - a_i b_i c_i - a_{\bar{\imath}} b_{\bar{\imath}} c_i - a_{\bar{\bar{\imath}}} b_{\bar{\bar{\imath}}} c_i$$

  In particular, as long as the vectors are normal ones,

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

u May indicate:

- The first velocity component in a Cartesian coordinate system.
- An integration variable.

V May indicate:

- Volume.
- "Volume" in n-dimensions (i.e. line segment length in one dimensions, area in two, volume in three, etc.)

v May indicate:

- The second velocity component in a Cartesian coordinate system.
- Magnitude of a velocity (speed).

$\vec{\boldsymbol{v}}$ May indicate:

- Velocity vector.
- Generic vector.

vector A quantity characterized by a list of numbers. A vector $\vec{v}$ in index notation is a set of numbers $\{v_i\}$ indexed by an index i. In normal three-dimensional Cartesian space, i takes the values 1, 2, and 3, making the vector a list of three numbers, v_1, v_2, and v_3. These numbers are called the three components of $\vec{v}$.

vectorial product An vectorial product, or cross product is a product of vectors that produces another vector. If

$$\vec{c} = \vec{a} \times \vec{b},$$

it means in index notation that the i-th component of vector $\vec{c}$ is

$$c_i = a_{\bar{i}} b_{\bar{\bar{i}}} - a_{\bar{\bar{i}}} b_{\bar{i}}$$

where $\bar{i}$ is the index following i in the sequence 123123..., and $\bar{\bar{i}}$ the one preceding it. For example, c_1 will equal $a_2 b_3 - a_3 b_2$.

$\boldsymbol{w}$ May indicate:

- The third velocity component in a Cartesian coordinate system.
- Weight factor.

$\boldsymbol{\vec{w}}$ Generic vector.

$\boldsymbol{X}$ Used in this book to indicate a function of x to be determined.

$\boldsymbol{x}$ May indicate:

- First coordinate in a Cartesian coordinate system.
- A generic argument of a function.
- An unknown value.

$\boldsymbol{Y}$ Used in this book to indicate a function of y to be determined.

$\boldsymbol{y}$ May indicate:

- Second coordinate in a Cartesian coordinate system.
- A second generic argument of a function.
- A second unknown value.

$\boldsymbol{Z}$ Used in this book to indicate a function of z to be determined.

$\boldsymbol{z}$ May indicate:

- Third coordinate in a Cartesian coordinate system.
- A third generic argument of a function.
- A third unknown value.

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