SOLUTION MANUAL
Analysis in Mechanical Engineering

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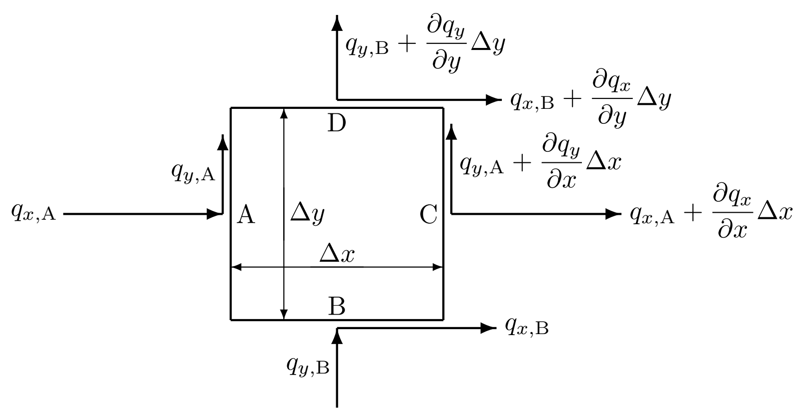
18.2.1 The Laplace equation

18.2.1.1 Solution stanexl-a

Question:

Derive the Laplace equation for steady heat conduction in a two-dimensional plate of constant thickness δ. Do so by considering a little Cartesian rectangle of dimensions $\Delta x \times \Delta y$. A sketch is shown below:



Assume Fourier's law:

$$\vec{q} = (q_x, q_y) \qquad q_x = -k\frac{\partial u}{\partial x} \qquad q_y = -k\frac{\partial u}{\partial y}$$

Here u is the temperature, assumed independent of z. Also, k is the heat conduction coefficient of the material. The vector $\vec{q}$ is the heat flux density. Vector $\vec{q}$ is in the direction of the heat flow. Its magnitude $|\vec{q}|$ equals the heat flowing per unit area normal to the direction of flow.

If you want the heat flow $\dot{Q}$ through an area element $\mathrm{d}S$ that is not normal to the direction of heat flow, the expression is

$$\dot{Q} = \vec{q} \cdot \vec{n}\,\mathrm{d}s$$

Here $\vec{n}$ is the unit vector normal to the surface element $\mathrm{d}S$. Positive $\dot{Q}$ means a heat flow through the surface element in the same direction as $\vec{n}$.

Assume that no heat is added to the little rectangle from external sources.

Answer:

If the temperature distribution is steady, then the *net* heat flowing into the rectangle must be zero. If the net heat flow in was positive, the rectangle would heat up. So it would not be steady. The same way, if the net heat flow in was negative, so net heat coming out, the rectangle would cool down and not be steady.

So what you need to do is find the expression for the *net* heat flowing into the little rectangle.

To get this net heat flow, you need to sum the contributions of all the four segments of the perimeter. It is here accurate enough to assume that the heat flux on each side can be approximated by the heat flux at the center point of the segment, point A, B, C, or D. It is also accurate enough to ignore the variations of the derivatives of the heat fluxes over the rectangle.

Note also that heat flow *along* the boundary does not move heat in or out.

Next plug in Fourier's law as given. Divide by the volume $\Delta x \Delta y \delta$ of the rectangular piece and take the limit that the dimensions go to zero. That then produces the Laplace equation.

You should be able to find arguments like the above in many books on engineering mathematics.

18.2.1.2 Solution stanexl-b

Question:

Derive the Laplace equation for steady heat conduction using vector analysis. Assume Fourier's law as given in the previous question. In vector form

$$\vec{q} = -k\nabla u$$

Assume that no heat is added to the solid from external sources.

Answer:

Consider an arbitrary volume V of the solid. The net heat flow out of the volume V per unit time is given by

$$Q_{\text{net out}} = \int_S \vec{q} \cdot \vec{n} \, \mathrm{d}S$$

where S is the outside surface of the volume and $\vec{n}$ the unit vector normal to the surface element $\mathrm{d}S$.

This heat flux must be zero. If there was net heat flow in or out, the volume would heat up, respectively cool down.

Now use the Gauss-Ostrogradski theorem to convert the surface integral to a volume integral. Then note that if an integrand always integrates to zero, *regardless of which volume is integrated over*, then that integrand must be zero.

Show that this means that

$$\nabla \cdot \vec{q} = \text{div}\vec{q} = 0$$

Plug in Fourier's law, and there you have the Laplace equation, assuming that k is constant. (If it is not, you get an equation given in earlier examples.)

18.2.1.3 Solution stanexl-b1

Question:

Consider the Laplace equation within a unit circle:

$$u_{xx} + u_{yy} = 0 \qquad \text{for} \qquad x^2 + y^2 < 1$$

The boundary condition on the perimeter of the circle is

$$u = (y^2 + 1)x \qquad \text{for} \qquad x^2 + y^2 = 1$$

To find the value of u at the point (0.1,0.2), can I just plug in the coordinates of that point into the boundary condition? If not, what is the correct value of u at the point, and what would I get from the boundary condition?

Also answer the above questions for the following problem:

$$u_{xx} + u_{yy} = 0 \qquad \text{for} \qquad x^2 + y^2 < 1$$

The boundary condition on the perimeter of the circle is

$$u = 2 + 3x + 5y \qquad \text{for} \qquad x^2 + y^2 = 1$$

Find the value of u at the point (0.1,0.2). Fully defend your solution.

Answer:

First verify that the given boundary condition expression, $u = (y^2 + 1)x$ does *not* satisfy the Laplace equation. So this expression is *not* valid for u *inside* the circle.

Of course, the value for u at the given point might still be right by coincidence. To check that, first verify that the correct solution to the problem is

$$u = \tfrac{5}{4}x + \frac{3}{4}xy^2 - \tfrac{1}{4}x^3$$

Do so by plugging it into the partial differential equation and boundary condition. Then compute the value of u at the given point and compare with what you would get from the given boundary condition.

Next verify that in the second case, the function given for u on the boundary does satisfy the Laplace equation. So it must be the correct solution u at all x and y.

So, now just plug in the given coordinates.

18.2.1.4 Solution stanexl-b2

Question:

Suppose you have a Laplace equation problem where the boundary is symmetric around the y-axis, like, say, in the previous two problems. In general, such a symmetric boundary means that if (x, y) is a boundary point, then so is $(-x, y)$. Also assume that u is given as an antisymmetric function of x on this boundary; $u(-x, y) = --u(x, y)$ for any boundary point. Show that in that case, u is antisymmetric function of x everywhere, i.e. $u(-x, y) = --u(x, y)$ everywhere.

Then show that this means that the solution u will be zero on the y axis.

Also explain why the above would no longer be true if you had a first order x derivative in the PDE, like for example $u_{xx} + u_{yy} + u_x = 0$.

Answer:

To show that u is zero on the y-axis if it is antisymmetric in x is easy. Just apply $u(-x, y) = u(x, y)$ at $x = 0$ to get $u(0, y) = -u(0, y)$. Then note that something can only be equal to its negative if it is zero.

However, it is suprisingly messy to show that u is indeed antisymmetric everywhere if it is on the boundary. To do it, define a couple of new variables:

$$\bar{x} = -x \qquad \bar{u}(\bar{x}, y) = -u(x, y)$$

Here $\bar{x}$ is the x-coordinate flipped over around the y-axis, and $\bar{u}$ is u with its sign flipped over.

Graphically, this may be pictured as follows:

In terms of this picture, $\bar{u}$ at a point P in the $\bar{x}, y$-plane is defined as $-u$ at the point P' in the x, y-plane.

Show that $\bar{u}(\bar{x}, y)$ satisfies the Laplace equation just like $u(x, y)$. To do so, use the chain rule in

$$\bar{u}(\bar{x}, y) = -u(x, y)$$

where x is a function of $\bar{x}$ given by $x = x(\bar{x}) = -\bar{x}$.)

Show also that $\bar{u}(\bar{x}, y)$ satisfies the exact same boundary condition as $u(x, y)$, (in terms of $\bar{x}$ of course.)

Then, since Dirichlet boundary value problems for the Laplace equation have unique solutions, $\bar{u}(\bar{x}, y)$ must be the exact same function as $u(x, y)$. In terms of the picture above, $\bar{u}$ at the point P is the same as u at the point P". Since from the original definition it is also equal to $-u$ at P', it follows that u at P' is $-u$ at P". So u is antisymmetric.

To see that the above no longer holds when there is a first order x derivative in the equation, just try to repeat the above analysis and then observe where it goes wrong.

18.2.1.5 Solution stanexl-b3

Question:

Consider the Laplace equation within a unit circle, but now in polar coordinates:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad \text{for} \qquad r < 1$$

The boundary condition on the perimeter of the circle is

$$u(1, \theta) = f(\theta)$$

where f is a given function.

The solution is the Poisson integral formula

$$u(r,\theta) = \frac{1-r^2}{2\pi} \oint \frac{f(\bar\theta)\,\mathrm{d}\bar\theta}{1-2r\cos(\bar\theta-\theta)+r^2}$$

Now suppose that function $f(\theta)$ is increased slightly, by an amount δf, and *only* in a very small interval $\theta_1 < \theta < \theta_2$.

Does the solution u change everywhere in the circle, or only in the immediate vicinity of the interval on the boundary at which f was changed. What is the sign of the change in u if δf is positive?

Answer:

To simplify this, assume that the new solution is $u_2 = u_1 + \delta u$ where u_1 is the old solution. So δu is the change in the solution.

You now have for the original solution

$$u(r,\theta) = \frac{1-r^2}{2\pi} \oint \frac{f(\bar\theta)\,\mathrm{d}\bar\theta}{1-2r\cos(\bar\theta-\theta)+r^2}$$

and for the changed solution

$$u(r,\theta)+\delta u(r,\theta) = \frac{1-r^2}{2\pi} \oint \frac{[f(\bar\theta)+\delta f(\bar\theta)]\,\mathrm{d}\bar\theta}{1-2r\cos(\bar\theta-\theta)+r^2}$$

Subtract the two to get

$$\delta u(r,\theta) = \frac{1-r^2}{2\pi} \oint \frac{\delta f(\bar\theta)\,\mathrm{d}\bar\theta}{1-2r\cos(\bar\theta-\theta)+r^2}$$

Now $\delta f(\bar\theta)$ is only nonzero in the interval from θ_1 to θ_2, so

$$\delta u(r,\theta) = \frac{1-r^2}{2\pi} \int_{\theta_1}^{\theta_2} \frac{\delta f(\bar\theta)\,\mathrm{d}\bar\theta}{1-2r\cos(\bar\theta-\theta)+r^2}$$

Now use a little graph to show that for any point not extremely close to the segment from θ_1 to θ_2 on the boundary, the denominator of the integrand is about constant, so

$$\delta u(r,\theta) = \frac{1-r^2}{2\pi} \frac{\int_{\theta_1}^{\theta_2} \delta f\,\mathrm{d}\bar\theta}{1-2r\cos(\theta_{12}-\theta)+r^2} \qquad \theta_{12} = \frac{\theta_1+\theta_2}{2}$$

Now show that

$$1-2r\cos(\theta_{12}-\theta)+r^2 \geq (1-r)^2$$

which is positive in the interior of the unit circle.

Then argue that this means that δu is positive everwhere inside the circle. So the region of influence of the little segment is the interior of the circle, and the temperature increases everywhere.

18.2.1.6 Solution stanexl-b5

Question:

- Show that if u is a harmonic function in a finite domain, and positive on the boundary, then it is positive everywhere in the domain.
- Show by example that this does not need to be true for an infinite domain.
- Let u, v, and w be harmonic functions. Show that if $u \leqslant v \leqslant w$ on the boundary of a finite domain, then $u \leqslant v \leqslant w$ everywhere inside the domain.

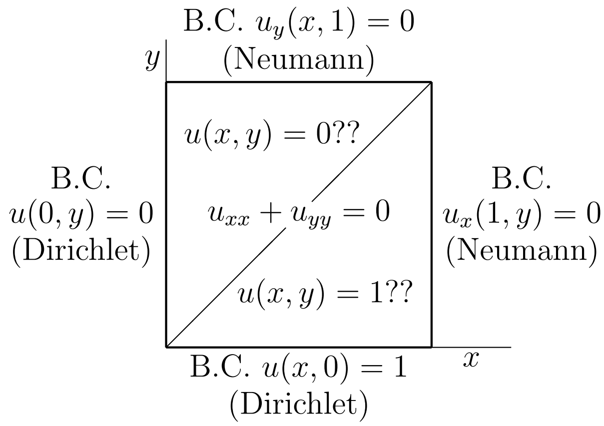
Answer:

- Use the minimum principle. The minimum of u must be on the boundary. So can u be negative or zero inside the region?
- Take the domain, for example, to be $y > 0$, the boundary to be the x axis, and the boundary condition on the x-axis to be $u = 1$. Now consider the solution $u = 1 - y$. Verify that it satisfies the Laplace equation, and that it is positive on the boundary. Verify that it is negative within the region.
- Look at the difference $v - u$. Explain that if u and v are harmonic functions, then so is their difference. Now apply the earlier result about harmonic functions that are posirtive on the boundary.

18.2.1.7 Solution stanexl-c

Question:

Consider the following Laplace equation problem in a unit square:



The problem as shown has a unique solution. It is relevant to a case of heat conduction in a square plate, with u the temperature. Someone proposed that the solution should be simple: in the upper triangle the solution $u(x,t)$ is 0, and in the lower triangle, it is 1.

Thoroughly discuss this proposed solution. Determine whether the boundary conditions and initial conditions are satisfied. Is the partial differential equation satisfied in both triangles?

Explain why all isotherms except 0 and 1 coincide with the 45° line. And why the zero and 1 isotherms are indeterminate.

Finally discuss whether the solution is right.

Answer:

Plug the two given expressions for u into the Laplace equation and boundary conditions. Show that they are satisfied.

An entire triangle has temperature 0, and the inside of a triangle cannot be shown as a single line. The same for temperature 1.

Finally, show that the solution has a jump singularity *inside* the plate. That is inconsistent with the requirement that solutions of the Laplace equation are smooth inside the considered region. That means that the solution is no good.

18.2.1.8 Solution stanexl-d

Question:

If for the problem of the previous question, the proposed solution is wrong, then so are the described isotherms.

To get a clue about the correct solution and isotherms, consider the following simpler problem. In this problem the top and right boundaries have been distorted into a quarter circle:

$$\text{BC:} \quad u(x,0) = 1 \qquad u(0,y) = 0 \qquad \frac{\partial u}{\partial n} = 0 \text{ on } x^2 + y^2 = 1$$

Solve this problem. Then neatly draw the u = 0, 0.25, 0.5, 0.75, and 1 isotherms for this problem.

Also neatly draw u versus the polar angle θ at $r = 0.5$. In a separate graph, draw the solution proposed in the previous section, $u = 1$ for $y < x$ and $u = 0$ for $y > x$, again against θ at r = 0.5.

Now go back to the problem of the previous question and very neatly sketch the correct u = 0, 0.25, 0.5, 0.75, and 1 isotherms for that problem. Pay particular attention to where the 0.25, 0.5, and 0.75 isotherms meet the boundaries and under what angle.

Answer:

You want to convert the simplified problem into polar coordinates r and θ. Show that the Neumann boundary condition on the quarter circle simplifies to $u_r = 0$.

Guess that the solution might be a function of θ only, $u = f(\theta)$. Show that this satisfies the Neumann boundary condition. Show that it satisfies the two Dirichlet boundary conditions if $f(0)$ and $f(\frac{1}{2}\pi)$ have suitable values.

Look up the Laplacian in polar coordinates in a table book. Plug in $u = f(\theta)$. That produces an ordinary differential equation. Solve this equation and plug in the boundary conditions on f. That gives the exact solution $f(\theta)$ to the given problem. Draw the lines with the given values of u in the x, y-plane.

Note that very close to the origin, there is presumably not much difference between the current simplified problem and the original problem of the previous question. In that vicinity the boundary conditions are the same. So draw the isotherms in the original problem the same way near the origin.

The 0.5 isotherm can be drawn all the way based on antisymmetry of $u - \frac{1}{2}$ around the 45° line. Compare with the relevant earlier homework problem. Show that the 0.25 and 0.75 lines cannot hit the left and bottom boundaries. Based on the boundary conditions on the other two boundaries, show that they must hit the boundary normally wherever they leave the square. Then draw them neatly.

18.2.1.9 Solution stanexl-e

Question:

Return once again to the problem of the second-last question.

The correct solution to this problem, that you would find using the so-called method of separation of variables, is:

$$u(x,y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n \cosh(\frac{1}{2}n\pi)} \sin(\tfrac{1}{2}n\pi x) \cosh(\tfrac{1}{2}n\pi(1-y))$$

Verify that this solutions satisfies both the partial differential equation and all boundary conditions.

Now shed some light on the question why this solution is smooth for any arbitrary $y > 0$. To do so, first explain why any sum of sines of the form

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\tfrac{1}{2}n\pi x\right)$$

is smooth as long as the sum is finite. A finite sum means that the coefficients c_n are zero beyond some maximum value of n.

Next, you are allowed to make use of the fact that the function is still smooth if the coefficients c_n go to zero quickly enough. In particular, *if* you can show that

$$\lim_{n\to\infty} n^k c_n = 0$$

for every k, however large, then the function $f(x)$ is infinitely smooth.

Use this to show that u above is indeed infinitely smooth for any $y > 0$. And show that it is not true for $y = 0$, where the solution jumps at the origin.

Answer:

Search through a table book, in the Fourier series section, for the following result:

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(\tfrac{1}{2}n\pi x) = \begin{cases} \text{-1 if } -2 \le x \le 0 \\ 1 \text{ if } 0 \le x \le 2 \end{cases}$$

The book might list this series in a slightly different form; in that case, just rescale the x and f values. If you can only find a saw-tooth, try differentiating it.

Using this result, you can show that the boundary condition at $y = 0$ is satisfied. The other boundary conditions and the partial differential equation can be verified directly from the form of the solution.

A single sine is an infinitely smooth function. You can differentiate it as many times as you want without getting singularities. Argue that according to calculus, the derivatives of a sum of two functions are just the sum of the derivatives of each separate function. So a sum of two sines is an infinitely smooth functions. And then so is the sum of a sum of two sines and another sine. And then so is the sum of the sum of three sines and another sine.

To show that the coefficients c_n go to zero sufficiently quickly, you can use l'Ho[s]pital. You first need to compare u with the generic $f(x)$ to see what the coefficients c_n are that you want to be vanishingly small for large n.

18.2.2 The heat equation

18.2.2.1 Solution stanexh-a

Question:

This is a continuation of a corresponding question in the subsection on the Laplace equation. See there for a definition of terms.

Derive the heat equation for unsteady heat conduction in a two-dimensional plate of thickness δ, Do so by considering a little Cartesian rectangle of dimensions $\Delta x \times \Delta y$.

In particular, derive the heat conduction coefficient κ in terms of the material heat coefficient k, the plate thickness t, and the specific heat of the solid C_p.

Answer:

The amount of thermal energy residing in the little rectangle will equal its volume times its density times its specific heat times its temperature (plus a constant that is not important):

$$E = \rho C_p u \Delta x \Delta y t$$

The time derivative of this energy is of course the net heat energy flowing in per unit time. Which is minus the net heat energy flowing out. That heat flow was derived in the corresponding question for the Laplace equation.

Put the two together and divide by $\Delta x \Delta y \delta$ to get the heat equation.

You should be able to find arguments like the above in many books on engineering mathematics.

18.2.2.2 Solution stanexh-b

Question:

This is a continuation of a corresponding question in the subsection on the Laplace equation. See there for a definition of terms.

Derive the heat equation for unsteady heat conduction using vector analysis.

Answer:

The amount of thermal energy residing in a given volume is equal to

$$\int \rho C_p u \, \mathrm{d}V$$

plus a constant that is not important.

The time derivative of this energy is of course the net heat energy flowing in per unit time. Which is minus the heat energy flowing out. That heat flow was derived in the earlier section. Use the divergence theorem to convert it into a volume integral.

Put the two together to get the heat equation. Note that if a volume integral is zero regardless of what you take the volume to be, the integrand must be zero.

18.2.3 The wave equation

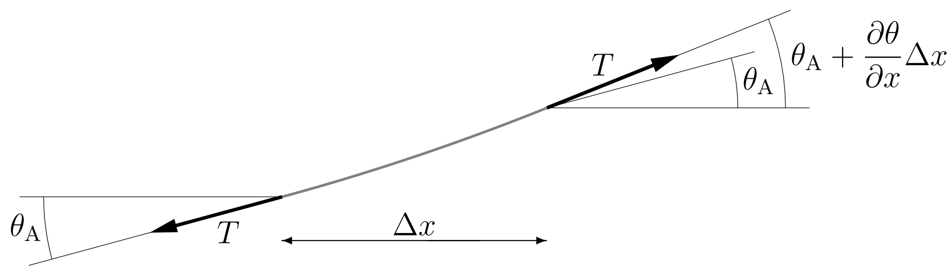
18.2.3.1 Solution stanexw-a

Question:

Derive the wave equation for small transverse vibrations of a string by considering a little string segment of length Δx.

Answer:

A sketch of the string segment is shown below.



The vertical position of the string segment is u and the horizontal coordinate is x. Call the mass of the string per unit length ρ. Call the tension force in the string T. The magnitude of this tension force can be assumed to be constant because there is no significant longitudinal motion. However, the direction of the force varies and that cannot be ignored. The varying direction produces net vertical forces. While small, they are big enough to produce the small vertical vibrations of the string.

Now find the *net* vertical force on the segment by taking vertical components of the tension forces on the two end points of the segment. Then apply Newton's second law. Note that the tension force must be in the direction of the string at each point. Otherwise you get into trouble with the momentum equation for infinitesimal segments; an infinitely thin string has no bending stiffness nor a moment of inertia proportional to the length of string.

Note also that for small angles θ,

$$\sin(\theta) \approx \theta \qquad \tan(\theta) \approx \theta$$

Also remember from calculus that

$$\frac{\partial u}{\partial x} = \tan(\theta)$$

where θ is the angle between the tangent of the curve u versus x and the x-axis for a given time.

You should be able to find arguments like the above in many books on engineering mathematics.

18.2.3.2 Solution stanexw-b

Question:

Maxwell's equations for the electromagnetic field in vacuum are

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad (1) \qquad\qquad \nabla \cdot \vec{B} = 0 \qquad (2)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (3) \qquad\qquad c^2 \nabla \times \vec{B} = \frac{\vec{\jmath}}{\epsilon_0} + \frac{\partial \vec{E}}{\partial t} \qquad (4)$$

Here $\vec{E}$ is the electric field, $\vec{B}$ the magnetic field, ρ the charge density, $\vec{\jmath}$ the current density, c the constant speed of light, and ϵ_0 is a constant called the permittivity of space. The charge and current densities are related by the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{\jmath} = 0 \qquad (5)$$

Show that if you know how to solve the standard wave equation, you know how to solve Maxwell's equations. At least, if the charge and current densities are known.

Identify the wave speed.

Answer:

Use the formulae of vector analysis, as found in, for example, [2].

First show from the Maxwell's equations that the divergence of $\vec{B}$ is zero. Then vector calculus says that it can be written as the curl of some vector. Call that vector $\vec{A}_0$.

$$\vec{B} = \nabla \times \vec{A}_0 \qquad (6a)$$

Next define

$$\vec{E}_\varphi \equiv \vec{E} + \frac{\partial \vec{A}_0}{\partial t}$$

Plug it into the appropriate Maxwell's equation to show that the curl of $\vec{E}_\varphi$ is zero. Then vector calculus says that it can be written as minus the gradient of a scalar. Call this scalar φ_0. Plug that into the expression above to get

$$\vec{E} = -\nabla \varphi_0 - \frac{\partial \vec{A}_0}{\partial t} \qquad (7a)$$

Next verify the following: if you define modified versions $\vec{A}$ and φ of $\vec{A}_0$ and φ_0 by setting

$$\varphi = \varphi_0 - \frac{\partial \Omega}{\partial t} \qquad \vec{A} = \vec{A}_0 + \nabla \Omega$$

where Ω is *any* arbitrary function of x, y, z, and t, then still

$$\vec{B} = \nabla \times \vec{A} \qquad (6)$$

$$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t} \qquad (7)$$

This is the famous "gauge property" of the electromagnetic field. Verify it by plugging in the given definitions of $\vec{A}$ and φ and using (6a) and (7a).

Now argue that you can select Ω so that

$$\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0 \qquad (8)$$

To do so, plug in the definitions of $\vec{A}$ and φ to get

$$\frac{1}{c^2}\frac{\partial^2 \Omega}{\partial t^2} - \nabla^2\Omega = \nabla \cdot \vec{A}_0 + \frac{1}{c^2}\frac{\partial \varphi_0}{\partial t} = 0$$

Argue that this is an inhomogeneous wave equation for Ω. Argue that even if $\vec{A}_0$ and φ_0 do not satisfy (8), after you solve the wave equation for Ω, $\vec{A}$ and φ will.

Now plug the expressions (6) and (7) for $\vec{E}$ and $\vec{B}$ in terms of $\vec{A}$ and φ into the Maxwell's equations. Clean up the expressions you get using (8). That gives uncoupled equations for $\vec{A}$ and φ. Show that they are wave equations. Show that the wave speed is the speed of light.

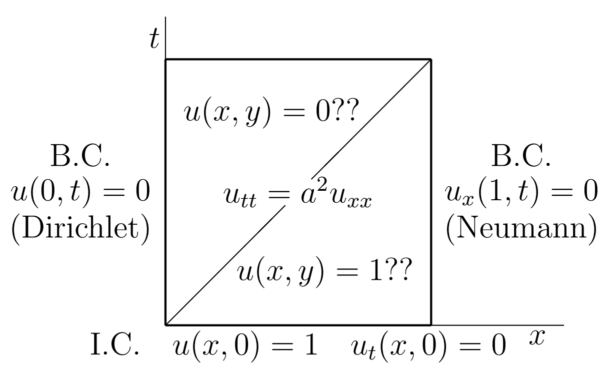
So you have shown that for any solution $\vec{E}$ and $\vec{B}$ of Maxwell's equations, there are potentials $\vec{A}$ and φ that satisfy wave equations.

You will want to invert that argument. Suppose that you have solutions $\vec{A}$ and φ of the wave equation. Suppose they satisfy (8). Show then that the $\vec{E}$ and $\vec{B}$ as defined by (6) and (7) satisfy Maxwell's equations.

18.2.3.3 Solution stanexw-c

Question:

Consider the following wave equation problem in a unit square:



This is basically identical to a Laplace equation problem in the first subsection. Like that problem, the above wave equation problem has a unique solution. It is relevant to a case of acoustics in a tube, with u the pressure. Someone proposed that the solution should be simple: in the upper triangle the solution $u(x,t)$ is 0, and in the lower triangle, it is 1.

Thoroughly discuss this proposed solution. Determine whether the boundary conditions and initial conditions are satisfied. Is the partial differential equation satisfied in both triangles? Finally discuss whether the solution is right. Consider the value of the wave speed a in your answer.

Sketch the isobars of the correct solution. In particular, sketch the $u = 0$ 0.25, 0.5, 0.75, and 1 isobars, if possible. Sketch both the case that $a = 1$ and that $a = \sqrt{2}$.

Answer:

Plug the given two-region solution into the partial differential equation, boundary, and initial conditions to verify that they are satisfied.

However, singularities should propagate with the wave speed. Show that the speed of propagation $\mathrm{d}x/\mathrm{d}t$ of the singularity is one. So argue that the solution can only be right for $a = 1$.

Draw the 0.25, 0.5, and 0.75 at the locations where the solution changes from 0 to 1, apparently passing through 0.25, 0.5, and 0.75 while doing so.

Argue that an entire region has $u = 0$ and that a region cannot be drawn as a single line. Similarly for $u = 1$.

Obviously, for $a = \sqrt{2}$ the singularity must propagate with that speed. Use that to produce your correct isobars.

18.2.3.4 Solution stanexw-e

Question:

Return again to the problem of the last question. Assume $a = 1$.

The correct solution to this problem, that you would find using the so-called method of separation of variables, is:

$$u = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(\tfrac{1}{2}n\pi x)\cos(\tfrac{1}{2}n\pi t)$$

Verify that this solutions satisfies both the partial differential equation and all boundary and initial conditions.

Explain that it produces the moving jump in the solution as given in the previous question.

The discontinuous solution given in the previous question is right in this case. It is right because it is the proper limiting case of a smooth solution that everywhere satisfies the partial differential equation. In particular, if you sum the above sum for u up to a very high, but not infinite value of n, you get a smooth solution of the partial differential equation that satisfies all initial and boundary conditions, except that the value of u at $t = 0$ still shows small deviations from $u = 1$. The more terms you sum, the smaller those deviations become. (There will always be some differences right at the singularity, but these will be restricted to a negligibly small vicinity of $x = 0$.)

Answer:

From the form of the solution, you will find that the partial differential equation, boundary conditions and initial condition on u_t are satisfied. To verify the initial condition, found in Fourier series table books:

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(\tfrac{1}{2} n\pi x) = \begin{cases} \text{-1 if } -2 \leq x \leq 0 \\ \text{1 if } 0 \leq x \leq 2 \end{cases}$$

The table book might have this result in a slightly different form, but you can rescale it. You might want to give the function above a name, like $h(x)$.

Now simplify the solution by using the fact that

$$\sin\alpha\cos\beta = \tfrac{1}{2}\sin(\alpha+\beta) + \tfrac{1}{2}\sin(\alpha-\beta)$$

This allows you to write the solution as the sum of two simpler ones. Each of these two terms can be written in terms of the function h but with arguments $x - t$ and $x + t$ insteda of x.

Plot these two solutions in the same graph of u versus x for an arbitrary value of t less than one and greater than zero. Add the two curves graphically together. Show in that way that $u = 0$ for $x < t$ and $u = 1$ for $x > t$. That is the solution as given in the previous question.

18.2.3.5 Solution stanexw-f

Question:

Find the possible plane wave solutions for the two-dimensional wave equation

$$u_{tt} = a^2 u_{xx} + a^2 u_{yy}$$

What is the wave speed?

Also find the possible standing wave solutions. Assume homogeneous Dirichlet or Neumann boundary conditions on some rectangle $0 < x < \ell$, $0 < y < h$. What is the frequency?

Repeat for the generalized equation

$$u_{tt} = a_1^2 u_{xx} + a_2^2 u_{yy} + b^2 u$$

where a_1, a_2, and b are positive constants.

Answer:

Plane wave solutions are solutions of the form

$$u = f(\vec{n} \cdot \vec{x} - ct)$$

where c is a constant, giving the speed of the wave, and $\vec{n}$ a constant unit vector, giving the direction of propagation.

Plugging it into the wave equation produces $c^2 = a^2$ so the propagation speed is $\pm a$. The "wave shape," the function f, is completely arbitrary.

These are propagating two-dimensional waves. To make this point, you can show a picture of the wave at both time zero and at a time $t > 0$, assuming an arbitrary wave shape f

There is an alternate solution in which the wave speed is completely arbitrary but the wave shape is not. Since this solution is not bounded, you can argue that it is not what people usually understand to be a wave. You might compare its wave shape with waves on the beach, for example.

If you plug the general expression for a plane wave into the generalized wave equation, things change. You find an ordinary differential equation for the wave shape. You find that that equation has only finite solutions if

$$|c| < \sqrt{a_1^2 n_x^2 + a_2^2 n_y^2}$$

In that case, you find that the waves are sinusoidal:

$$u = A \sin(k[n_x x + n_y y] + \phi)$$

where the amplitude A and the phase angle ϕ are arbitrary constants. The wave speed c is related to the wave number k as

$$c = \pm\sqrt{a_1^2 n_x^2 + a_2^2 n_y^2 - \frac{b^2}{k^2}}$$

Therefore, when b is not zero, the sinusoidal waves in a given direction do not all go at the same speed as they do for the basic wave equation. In particular, waves of smaller wave number, or longer wave length, go slower. And at a lowest value of the wave number, the waves come to a halt. Below that wave number, there are no propagating sinusoidal waves.

The standing wave solutions of interest are solutions of the form

$$u = \sin(k_x x + \phi_1)\sin(k_y y + \phi_2)\sin(\omega t + \phi_3)$$

where the wave numbers k_1 and k_2, and the frequency ω are positive constants. These can satisfy homogeneous Dirichlet or Neumann boundary conditions on some rectangle $0 < x < \ell$, $0 < y < h$. The phase angles ϕ are not important here so you may assume them to be zero.

Use a sketch at different times to show that the shape of the wave does not change. Only the amplitude changes. The wave does not move; for example, draw attention to the locations where $u = 0$.

If you plug it in, for the standard wave equation you find $\omega = \pm ak$ where $k = \sqrt{k_x^2 + k_y^2}$. The ratio ω/k is therefore equal to the wave speed a in magnitude.

For the generalized wave equation, there are only standing wave solutions if the wave number is large enough. In particular you find that if $a_1 = a_2 = a$, then $k > b/a$ is required.

18.3 Properly Posedness

18.3.1 The conditions for properly posedness

18.3.1.1 Solution ppc-a

Question:

Show that the Dirichlet boundary-value problem for the Poisson equation on a finite domain,

$$\nabla^2 u = f \quad \text{on} \quad \Omega \qquad u = g \quad \text{on} \quad \delta\Omega$$

has unique solutions. You cannot have two different solutions u_1 and u_2 to this problem.

Answer:

Suppose that there are two solutions u_1 and u_2. For nonuniqueness, the difference $v = u_2 - u_1$ must be nonzero.

Show that v satisfies the Laplace equation by subtracting the Poisson equations satisfied by u_1 and u_2. Show that v is zero on the boundaries by subtracting the boundary conditions satisfied by u_1 and u_2.

Now use the maximum and minimum properties of the Laplace equation to show that v is zero. That means that $u_2 = u_1 + v$ equals u_1. So the supposed two different solutions are not different.

18.3.1.2 Solution ppc-b

Question:

Assuming that the Dirichlet boundary-value problem for the Laplace equation on a finite domain,

$$\nabla^2 u = 0 \quad \text{on} \quad \Omega \qquad u = f \quad \text{on} \quad \delta\Omega$$

is solvable, show that it depends continuously on the data.

Answer:

Assume that data f_1 produce a solution u_1 and f_2 a solution u_2. Define v as the difference between the two solutions, and g as the difference between the two data.

Show that v satisfies the Laplace equation. Show that on the boundary v equals the change in the data g. Then, using the maximum and minimum properties of the Laplace equation, argue that the (maximum) change in the solution is no larger than the (maximum) change in the data.

18.3.1.3 Solution ppc-c

Question:

Repeat the previous two questions for the Dirichlet initial / boundary value problem for the heat equation,

$$u_t = \kappa \nabla^2 u \quad \text{on} \quad \Omega \qquad u = f \quad \text{on} \quad \delta\Omega \qquad u = g \quad \text{at} \quad t = 0$$

Answer:

Use the maximum and minimum properties of the heat equation.

18.3.2 An improperly posed parabolic problem

18.3.3 An improperly posed elliptic problem

18.3.3.1 Solution ppe-a

Question:

Show that the given solution

$$u(x,y) = \sin(nx)\cosh(ny)$$

for natural n does indeed satisfisfy the Laplace equation

$$u_{yy} + u_{xx} = 0$$

and the boundary conditions

$$u(x,0) = \sin(x) \quad u_y(x,0) = 0 \quad u(0,y) = 0 \quad u(\pi,y) = 0$$

Answer:

Plug it in.

18.3.3.2 Solution ppe-b

Question:

For the Laplace equation

$$u_{yy} + u_{xx} = 0$$

with boundary conditions

$$u(x,0) = f(x) \quad u_y(x,0) = 0 \quad u(0,y) = 0 \quad u(\pi,y) = 0$$

the "separation of variables" solution is

$$u(x,y) = \sum_{n=1}^{\infty} f_n \sin(nx)\cosh(ny)$$

Here the "Fourier coefficients" f_n must chosen so that they satisfy

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(nx)$$

Check this solution.

Can you immediately see that this separation of variables solution is probably no good?

Answer:

Plug the solution in the partial differential equation and all four boundary conditions. You will find that they are all satisfied.

If you are concerned about manipulating infinite sums, then for now simply assume that $f(x)$ is such that f_n is zero above some largest value $n = n_{\max}$. Then the sums are finite and you can manipulate them in the usual ways.

You can see that this separation of variables solution is probably no good, but not from the simple checks above. So no.

18.3.3.3 Solution ppe-c

Question:

For the Laplace equation

$$u_{yy} + u_{xx} = 0$$

with boundary conditions

$$u(x,0) = f(x) \quad u_y(x,0) = 0 \quad u(0,y) = 0 \quad u(\pi,y) = 0$$

assume that $f(x)$ is the triangular profile:

$$f(x) = x \quad \text{if} \quad x \le \tfrac{1}{2}\pi \qquad f(x) = \pi - x \quad \text{if} \quad x \ge \tfrac{1}{2}\pi$$

The "separation of variables" solution for this problem is

$$u(x, y) = \sum_{n=1}^{\infty} f_n \sin(nx) \cosh(ny)$$

where the "Fourier coefficients" f_n must chosen so that they satisfy

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(nx)$$

where $f(x)$ is the triangular profile described above.

Plot this separation of variables solution for $y = 0$ and for a few values greater than zero like $y = 1$, $y = 0.5$, $y = 0.25$. Then comment on whether a solution u exists at $y = 0$ and for $y > 0$.

This example should illustrate that typical improperly posed problems might have solutions if the data are perfectly smooth and their Taylor series have finite radii of convergence. But if there is a singularity, like the kink in the triangular profile, all bets are off.

You might know that if you talk about instability of ordinary differential equations, you wonder about what happens to the solution for infinite time. But in this problem you do not let the "time" coordinate y go to infinity. The problem is not large y, but large "wave number" n. The large wave number problem is really unique to partial differential equations. (If you had a system of infinitely many ordinary differential equations, you might also run into it.)

Include your code, if any.

Answer:

Skim through the Fourier series tables of a table book for an appropriate trangular wave $f(x)$. You will find that the Fourier coefficients f_n are:

$$f_n = \frac{4}{\pi n^2} \sin\left(n\tfrac{1}{2}\pi\right)$$

Given that, you can plot the solution

$$u(x, y) = \sum_{n=1}^{\infty} \frac{4}{\pi n^2} \sin\left(n\tfrac{1}{2}\pi\right) \sin(nx) \cosh(ny)$$

Use a programming language with plotting capability like mathcad, matlab, or octave to plot this solution.

Of course, on a computer you must stop summing at some finite value $n_{\max}$ of n. Try plotting increasing values of $n_{\max}$ to figure out what the infinite sum will look like.

First plot $y = 0$. You should find that the triangular profile is produced just fine by the infinite sum.

Now plot a nonzero value of y, like $y = 0.5$. Try different values $n_{\max} = 3$, then 5, 10, 20, 40, and 100. Keep your eyes on the numbers on the axes. Use the results to argue that the exact solution u at $y = 0.5$ does not exist.

When using matlab or octave, you could use a program like badlap_run.m[1]. You should first look at what is in there. Then you need to save it in your matlab working directory as a file called `badlap_run.m`. You will also need the function that performs the sum, badlap_sum.m[2]. You need to save this in your matlab working directory as a file called `badlap_sum.m`.

Then inside matlab or octave issue the command `badlap_run` to plot. Change the values of `n_max` and `y` as needed and run the program again. Use "help print" for information on how to make hardcopies of some of the plots you need to make your case.

18.3.3.4 Solution ppe-d

Question:

Continuing the previous question, show analytically that for the supposed solution

$$u(x, y) = \sum_{n=1}^{\infty} \frac{4}{\pi n^2} \sin\left(n\tfrac{1}{2}\pi\right) \sin(nx) \cosh(ny)$$

the sum does not converge for any x if $y > 0$.

Also show analytically that at the halfway point $x = \frac{1}{2}\pi$, the values that you get while summing increase monotonically to infinity.

Answer:

Note that a requirement for a sum to converge is that the terms in the sum go to zero. Now apply good old l'Ho[s]pital, or equivalent, on $\cosh(ny)/n^2$.

At the halfway point, you can show that the terms being summed are always positive, and that they grow bigger and bigger. So their sum grows bigger and bigger too. Therefore u goes to infinity monotoneously.

[1] `badlap_run.m`
[2] `badlap_sum.m`

18.3.3.5 Solution ppe-e

Question:

Show that the Laplace equation

$$\nabla^2 u = 0 \qquad \text{inside} \qquad \Omega$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial n} = 1 \qquad \text{on} \qquad \delta\Omega$$

has no solution. That makes it an improperly posed problem. To focus your thoughts, you can take an example domain Ω to be the inside of a sphere, and $\delta\Omega$ as its surface.

Explain the lack of solution in physical terms. To do so, consider this a steady heat conduction problem, with u the temperature, and the gradient of u the scaled heat flux.

Generalize the derivation to determine the requirement that

$$\nabla^2 u = f \qquad \text{inside} \qquad \Omega$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial n} = g \qquad \text{on} \qquad \delta\Omega$$

has a solution.

Answer:

Write the integral of the partial differential equation over the domain Ω (like the volume of the sphere). Then use the divergence theorem to convert it to an integral over $\delta\Omega$ (like the surface of the sphere). Identify the integrand in each integral, and hence the integrals themselves.

In the explanation, consider the net heat entering or leaving the domain. Note that the Laplace equation describes steady heat conduction, in which the temperature does not vary with time.

18.3.3.6 Solution ppe-f

Question:

Show that if the Poisson equation

$$\nabla^2 u = f \qquad \text{inside} \qquad \Omega$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial n} = g \qquad \text{on} \qquad \delta\Omega$$

has a solution, it is not unique.

Answer:

Suppose you have a solution u_1. Then $u_2 = u_1 + C$, where C is any arbitrary constant, is also a solution. The constant differentiates away in both the partial differential equation and boundary condition.

18.3.4 Improperly posed hyperbolic problems

18.3.4.1 Solution pph-a

Question:

Show that the given solution

$$u(x, y) = \sin(nx)\sin(nt)$$

with $n = m_2$ does indeed satisfisfy the wave equation

$$u_{tt} = u_{xx}$$

and the boundary conditions

$$u(x, 0) = 0 \quad u(0, t) = 0 \quad u(\pi, t) = 0 \quad u(x, \frac{m_1}{m_2}\pi) = 0$$

How about twice that solution? Ten times? How about if $n = 2m_2$? How about if $n = 10m_2$? So how many solutions are there really to this single problem?

Answer:

Plug it in.

There are infinitely many solutions.

18.3.4.2 Solution pph-b

Question:

For the brave. Show without peeking at the solution that the problem for irrational T is improperly posed by showing that you can make

$$\sin(nT\pi)$$

arbitrarily small by choosing suitable values of n. Then for these values of n, the solution

$$u = \frac{1}{\sin(nT\pi)} \sin(nx)\sin(nt)$$

becomes arbitrarily large in the interior although is is no larger than 1 on the boundary. So the problem for irrational T is improperly posed too, but not because the solution is not unique, but because small data (f, i.e. u on the top boundary) do not produce correspondingly small solutions in the interior.

Answer:

Trying to approximate T by its decimal expansion, as done for $\sqrt{2}$, is not accurate enough.

Instead note that what you need is that nT is arbitrarily close to an integer. That will make the sine arbitrarily small.

To achieve that, build up the desired value of n in stages as a product.

To start, simply take $n = 1$. Note that obviously $nT = T$ will always within a distance of no more that $\frac{1}{2}$ of some integer. Now if nT is also within a distance of no more that $\frac{1}{3}$ of that integer, do not change n, leave it 1. If however the value of nT is more than $\frac{1}{3}$ away from the integer, multiply n by 2, i.e. take the new n equal to 2. That brings the new nT within $\frac{1}{3}$ of a (different) integer.

Next, if the current nT is a distance within $\frac{1}{4}$ of an integer, do nothing. Otherwise multiply the current n by 3. That brings nT within a distance $\frac{1}{4}$ of an integer.

Next, if the current nT is a distance within $\frac{1}{5}$ of an integer, do nothing. Otherwise multiply the current n by 4. That brings nT within a distance $\frac{1}{5}$ of an integer.

Etcetera. In this way, nT can be driven arbitrarily close to an integer. That makes $\sin(nT\pi)$ arbitrarily small.

18.4 Energy methods

18.4.1 The Poisson equation

18.4.1.1 Solution emp-a

Question:

Show that the Poisson equation

$$\nabla^2 u = f$$

with boundary conditions

$$u_y(x,1) = g_1(x) \qquad u_y(x,0) = g_2(x)$$

$$u(0,y) = g_3(y) \qquad u(1,y) + u_x(1,y) = g_4(y)$$

has unique solutions.

Answer:

Follow the lines of the uniqueness proofs above. However, in this case you need to write out all four parts of the boundary integral separately. Then you can follow arguments like the ones in the text to show that the diffence between any two solutions is still a constant, and that that constant is still zero.

18.4.1.2 Solution emp-b

Question:

Using the arguments given in the text, uniqueness can *not* be shown for the Poisson equation

$$\nabla^2 u = f$$

with boundary conditions

$$u_y(x,1) = g_1(x) \qquad u_y(x,0) = g_2(x)$$

$$u(0,y) = g_3(y) \qquad u(1,y) - u_x(1,y) = g_4(y)$$

Of course, just because you cannot *prove* uniqueness does not mean it is not true. But show that this problem never has unique solutions. If it has a solution at all, there are infinitely many different ones.

Answer:

If you can show that the *homogeneous* problem has a nontrivial (nonzero) solution v, you are done. Then if there is any solution u_1 to the original problem, infinitely many more solutions can be obtained by adding arbitrary multiples of v to u_1.

To find a nontrivial solution, guess it. In particular, based on the boundary conditions for v at $y = 0$ and $y = 1$, guess that the nontrivial solution v may be independent of x. If you plug that assumption into the partial differential equation and boundary conditions, you can indeed find a nonzero solution.

If you solve the problem for a general mixed boundary condition, using separation of variables, you find that for many values of the coefficients A and B, but not all, there are nonunique solutions. However, there are none unless A and B have opposite sign.

18.4.2 The heat equation

18.4.3 The wave equation

18.5 Variational methods [None]

18.6 Classification

18.6.1 Introduction

18.6.2 Scalar second order equations

18.6.2.1 Solution clasnd-a

Question:

The equation

$$u_t - \nabla \cdot (p\nabla u) + qu = f$$

is a generic unsteady heat conduction equation, with u the temperature relative to the surroundings. The first term is the rate of temperature change at a point. The second term represents heat accumulation at the point due to conduction of heat. In it, p is the heat conduction coefficient. The third term would in be an approximation to the heat radiated away to the surroundings, either in two-dimensions or for a transparant medium. The right hand side represents heat that is explicitly added from other sources. Classify this equation. Also classify the steady version, i.e. the equation without the u_t term.

Answer:

Note that the unsteady equation is a partial differential equation in four dimensions even though there are no second order derivatives involving time. There is still a first order time derivative.

18.7 Changes of Coordinates

18.7.1 Introduction

18.7.2 The formulae for coordinate transformations

18.7.3 Rotation of coordinates

18.7.3.1 Solution rotcoor-a

Question:

Simplify the partial differential equation

$$10u_{xx} + 6u_{xy} + 2u_{yy} = u_x + x + 1$$

by rotating the coordinate system. Classify the equation. Draw the original and rotated coordinate system and identify the angle of rotation.

Answer:

Identify the matrix A. Find the eigenvalues and orthonormal eigenvectors. You find

$$\lambda_1 = 11 \quad \hat{\imath}' = \begin{pmatrix} 3 \\ 1 \end{pmatrix} / \sqrt{10} \qquad \lambda_2 = 1 \quad \hat{\jmath}' = \begin{pmatrix} -1 \\ 3 \end{pmatrix} / \sqrt{10}$$

The eigenvalues are of the same sign, so it is elliptic.

You then find the relation between the coordinates to be

$$x = \frac{3}{\sqrt{10}}x' - \frac{1}{\sqrt{10}}y' \qquad y = \frac{1}{\sqrt{10}}x' + \frac{3}{\sqrt{10}}y'$$

$$x' = \frac{3}{\sqrt{10}}x + \frac{1}{\sqrt{10}}y' \qquad y' = -\frac{1}{\sqrt{10}}x' + \frac{3}{\sqrt{10}}y'$$

So using the conversion rule for the first derivative u_x, the PDE becomes

$$11u_{x'x'} + u_{y'y'} = \frac{3}{\sqrt{10}}u_{x'} - \frac{1}{\sqrt{10}}u_{y'} + \frac{3}{\sqrt{10}}x' - \frac{1}{\sqrt{10}}y' + 1$$

18.7.4 Explanation of the classification

18.7.4.1 Solution expclass-a

Question:

Convert the equation

$$11u_{x'x'} + u_{y'y'} = \frac{3}{\sqrt{10}}u_{x'} - \frac{1}{\sqrt{10}}u_{y'} + \frac{3}{\sqrt{10}}x' - \frac{1}{\sqrt{10}}y' + 1$$

to be as close as possible to the Laplace equation.

Answer:

Define $\xi = x'/\sqrt{11}$, $\eta = y'$ to give

$$u_{\xi\xi} + u_{\eta\eta} = \frac{3}{\sqrt{110}}u_{\xi} - \frac{1}{\sqrt{10}}u_{\eta} + \frac{3\sqrt{11}}{\sqrt{10}}\xi - \frac{1}{\sqrt{10}}\eta + 1$$

Then define v by the relation $u = v\,e^{a\bar{\xi}+b\bar{\eta}}$. Plug it in and you see that for the values of a and b for which the first order derivatives vanish,

$$v_{\xi\xi} + v_{\eta\eta} = \frac{1}{22}v + \left(\frac{3\sqrt{11}}{\sqrt{10}}\xi - \frac{1}{\sqrt{10}}\eta + 1\right)e^{(\eta\sqrt{11}-3\xi)/(2\sqrt{110})}$$

18.8 Two-Dimensional Coordinate Transforms

18.8.1 Characteristic Coordinates

18.8.2 Parabolic equations in two dimensions

18.8.3 Elliptic equations in two dimensions

18.8.3.1 Solution 2dcanel-a

Question:

Convert the equation

$$10u_{xx} + 6u_{xy} + 2u_{yy} = u_x + x + 1$$

to two-dimensional canonical form.

Using rotation and stretching of the coordinates you would get

$$u_{\xi\xi} + u_{\eta\eta} = \frac{3}{\sqrt{110}} u_\xi - \frac{1}{\sqrt{10}} u_\eta + \frac{3\sqrt{11}}{\sqrt{10}} \xi - \frac{1}{\sqrt{10}} \eta + 1$$

Do you get the same equation? Should you? Comment.

Answer:

You get

$$u_{\xi\xi} + u_{\eta\eta} = \frac{3}{11} u_\xi + \frac{1}{\sqrt{11}} u_\eta + \frac{100}{11\sqrt{11}} \eta + \frac{10}{11}$$

You do not necessarily get the same result. If you rotate the coordinate system, the Laplacian stays the same, but the right hand side changes. The same happens when you scale both coordinates by the same factor.

Chapter 19

Green's Functions

19.1 Introduction

19.1.1 The one-dimensional Poisson equation

19.1.1.1 Solution gf1d-a

Question:

Solve the Poisson equation

$$u_{xx} = -2\frac{\sinh x}{\cosh^3 x}$$

numerically using Green's functions. Experiment with numerical parameters and show convergence.

Include your code.

Answer:

The different right hand side does not change the Green's function of the one-dimensional Poisson equation. However, the exact solution u is of course different; you find it is $\tanh x$.

To numerically find that solution, you might consider adapting the program used to produce the example in the text. That program consisted of the matlab or octave files gf1d_run.m[1], gf1d_f.m[2], gf1d_sumgf.m[3], and gf1d_u_exa.m[4].

You may want to experiment with the size of the interval outside of which you can ignore the right hand side in your Poisson equation. However, the right hand side does fortunately become negligible quickly.

Another thing you could try is evaluate $\int_{\text{spike}} f(\xi')\,\mathrm{d}\xi'$ exactly, rather than approximate it as $f(\xi)\Delta\xi$. I suspect that this will give much better results when using small numbers of wide "spikes."

19.1.1.2 Solution gf1d-b

Question:

Show that

$$\tilde{u}(x) = \int_{\xi=-\infty}^{\infty} \tfrac{1}{2}|x-\xi| f(\xi)\,\mathrm{d}\xi$$

is a solution to

$$u_{xx} = f(x) \qquad -\infty < x < \infty$$

You can assume that function $f(\xi)$ becomes zero rapidly at large ξ. (If you want, you can assume it is zero beyond some value $\xi_{\max}$ of $|\xi|$.) Find out what function $\tilde{u}$ is relative to some given second anti-derivative u_0 of f.

Answer:

First of all, define a second anti-derivative of f to be u_0. That allows you from now on to write f as u_0''. Note also that u_0 is *one* possible solution to the Poisson equation. The most general solution is this particular solution plus the general solution of the homogeneous equation. Use your knowledge of ordinary differential equations to show that the most general solution is

$$u_0 + A + Bx$$

It must therefore be shown that the Green's fuction solution $\tilde{u}$ is of this form.

Now restrict the region of integration of $\tilde{u}$ to $-R \leqslant \xi \leqslant R$ where R is some large number. Take $R > \xi_{\max}$ so that the integral is zero beyond R. Also, take R large enough that the

[1] `gf1d_run.m`
[2] `gf1d_f.m`
[3] `gf1d_sumgf.m`
[4] `gf1d_u_exa.m`

particular x at which you want to find u is in the range $-R < x < R$. Show that that since f is zero for $\xi > \xi_{\max}$,

$$u_0(\xi) = C_1 + C_2\xi \qquad \text{for} \qquad \xi > \xi_{\max}$$

Also, since f is zero for $\xi < -\xi_{\max}$,

$$u_0(\xi) = D_1 + D_2\xi \qquad \text{for} \qquad \xi < -\xi_{\max}$$

According to the above, your value of R is large enough that R is in the first range and $-R$ is in the second.

Split the integral into two parts $\xi < x$ and $\xi > x$ because the absolute value in the integral is different in these two cases. You get

$$\tilde{u} = \int_{\xi=-R}^{x} \tfrac{1}{2}(x-\xi)u_0''(\xi)\,\mathrm{d}\xi + \int_{\xi=x}^{R} \tfrac{1}{2}(\xi-x)u_0''(\xi)\,\mathrm{d}\xi$$

Integrate each term by parts and clean up. Use the fact that $\pm R$ fall in the mentioned ranges.

You then find that $\tilde{u}$ is indeed a solution to the Poisson equation.

19.1.2 More on delta and Green's functions

19.2 The Poisson equation in infinite space

19.2.1 Overview

19.2.2 Loose derivation

19.2.2.1 Solution pninfl-a

Question:

Do an analysis similar to either this subsection, or the next one, to derive the Green's function of the Poisson equation in three dimensional infinite space.

Answer:

The answer is the result stated in the overview section.

19.2.3 Rigorous derivation

19.3 The Poisson or Laplace equation in a finite region

19.3.1 Overview

19.3.2 Intro to the solution procedure

19.3.3 Derivation of the integral solution

19.3.3.1 Solution pnfd-a

Question:

Perform the equivalent analysis in the three dimensional case.

Answer:

One difference is that the Green's function is different.

Another difference is in the behavior of the solution at large distances. In three dimensions:

$$u_{\text{out}} \sim C_1 + \frac{C_2}{\rho} + \ldots$$

However, the final answer turns out to be very similar. (Of course, area integrals become volume integrals and contour integrals become surface integrals.)

19.3.4 Boundary integral (panel) methods

19.3.5 Poisson's integral formulae

19.3.6 Derivation

19.3.6.1 Solution pnifd-a

Question:

Find a suitable solution u_{out} outside the sphere in three dimensions. Show that it satisfies the Laplace equation.

Answer:

In three dimensions, you will have to take

$$u_{\text{out}}(r,\vartheta,\varphi) = A\frac{1}{r}u(\bar{r},\vartheta,\varphi) \qquad \text{where} \qquad \bar{r} = \frac{1}{r}.$$

The Laplacian in spherical coordinates is readily available in table books. Taking derivatives goes in the same way as in two dimensions. However, note that you will need to use the product rule immediately.

19.3.6.2 Solution pnifd-b

Question:

Derive the Poisson integral formula in three dimensions as given in the previous subsection.

Answer:

In two dimensions the source distribution drops out completely. In three dimensions, both the source and dipole distributions stay.

In particular, you cannot get $\partial(u - u_{\text{out}})/\partial r$ to be zero. You can however take A so that it only involves the given u on the boundary, not the unknown radial derivative of u.

Then you will need to combine

$$2\frac{\partial G}{\partial n_{\vec{\xi}}} + G$$

and clean up that combination using similar procedures as in three dimensions.

If you cannot find the unit vector $\hat{\imath}_r$ in spherical coordinates, evaluate it as $\vec{r}/r$ with $\vec{r} = (x,y,z)$ expressed in spherical coordinates. Use trig to clean up the dot product $\hat{\imath}_r \cdot \hat{\imath}_\rho$ a bit.

Also, in an earlier homework you, hopefully, showed that in spherical coordinates

$$\vec{n}\mathrm{d}S = \frac{\nabla F}{F_r}r^2 \sin\theta \, \mathrm{d}\theta\phi$$

where $F = 0$ describes the surface, in this case $r = 1$. The gradient in spherical coordinates is in table books.

19.3.7 The integral formula for the Neumann problem

19.3.8 Smoothness of the solution

Chapter 20

First Order Equations

20.1 Classification and characteristics

20.2 Numerical solution

20.3 Analytical solution

20.4 Using the boundary or initial condition

20.5 The inviscid Burgers' equation

20.5.1 Wave steepening

20.5.2 Shocks

20.5.3 Conservation laws

20.5.4 Shock relation

20.5.5 The entropy condition

20.6 First order equations in more dimensions

20.7 Systems of First Order Equations (None)

Chapter 21

D'Alembert Solution of the Wave equation

21.1 Introduction

21.2 Extension to finite regions

21.2.1 The physical problem

21.2.2 The mathematical problem

21.2.3 Dealing with the boundary conditions

21.2.4 The final solution

Chapter 22

Separation of Variables

22.1 A simple example

22.1.1 The physical problem

22.1.2 The mathematical problem

22.1.3 Outline of the procedure

22.1.4 Step 1: Find the eigenfunctions

22.1.5 Should we solve the other equation?

22.1.6 Step 2: Solve the problem

22.2 Comparison with D'Alembert

22.3 Understanding the Procedure

22.3.1 An ordinary differential equation as a model

22.3.2 Vectors versus functions

22.3.3 The inner product

22.3.4 Matrices versus operators

22.3.5 Some limitations

22.4 Handling Periodic Boundary Conditions

22.4.1 The physical problem

22.4.2 The mathematical problem

22.4.3 Outline of the procedure

22.4.4 Step 1: Find the eigenfunctions

22.4.5 Step 2: Solve the problem

22.4.6 Summary of the solution

22.5 Finding the Green's function

22.6 Inhomogeneous boundary conditions

22.6.1 The physical problem

22.6.2 The mathematical problem

22.6.3 Outline of the procedure

22.6.4 Step 0: Fix the boundary conditions

22.6.5 Step 1: Find the eigenfunctions

22.6.6 Step 2: Solve the problem

22.6.7 Summary of the solution

22.7 Finding the Green's functions

22.8 An alternate procedure

22.8.1 The physical problem

22.8.2 The mathematical problem

22.8.3 Step 0: Fix the boundary conditions

22.8.4 Step 1: Find the eigenfunctions

22.8.5 Step 2: Solve the problem

22.8.6 Summary of the solution

22.9 A Summary of Separation of Variables

22.9.1 The form of the solution

22.9.2 Limitations of the method

22.9.3 The procedure

22.9.4 More general eigenvalue problems

22.10 More general eigenfunctions

22.10.1 The physical problem

22.10.2 The mathematical problem

22.10.3 Step 0: Fix the boundary conditions

22.10.4 Step 1: Find the eigenfunctions

22.10.5 Step 2: Solve the problem

22.10.6 Summary of the solution

22.10.7 An alternative procedure

22.11 A Problem in Three Independent Variables

22.11.1 The physical problem

22.11.2 The mathematical problem

22.11.3 Step 1: Find the eigenfunctions

22.11.4 Step 2: Solve the problem

22.11.5 Summary of the solution

Chapter 23

Fourier Transforms [None]

Chapter 24

Laplace Transforms

24.1 Overview of the Procedure

24.1.1 Typical procedure

24.1.2 About the coordinate to be transformed

24.2 A parabolic example

24.2.1 The physical problem

24.2.2 The mathematical problem

24.2.3 Transform the problem

24.2.4 Solve the transformed problem

24.2.5 Transform back

24.3 A hyperbolic example

24.3.1 The physical problem

24.3.2 The mathematical problem

24.3.3 Transform the problem

24.3.4 Solve the transformed problem

24.3.5 Transform back

24.3.6 An alternate procedure

Appendix A

Addenda

A.1 Distributions

Appendix B

Derivations

B.1 Orthogonal coordinate derivatives

B.2 Harmonic functions are analytic

B.3 Some properties of harmonic functions

B.4 Coordinate transformation derivation

B.5 2D coordinate transformation derivation

B.6 2D elliptical transformation

Appendix C

Notes

C.1 Why this book?

C.2 History and wish list

Web Pages

Below is a list of relevant web pages.

1. Wikipedia[1]

   A valuable source source of information on about every loose end, though somewhat uneven. Some great, some confusing, some overly technical.

[1] `http://wikipedia.org`

References

[1] P. DuChateau and D.W. Zachmann. *Partial Differential Equations*. Schaum's Outline Series. McGraw-Hill, 1986.

[2] M.R. Spiegel and J. Liu. *Mathematical Handbook of Formulas and Tables*. Schaum's Outline Series. McGraw-Hill, second edition, 1999. 40