6.47(a)

1 6.47(a), §1 Asked

Given: A new basis $S = \{\vec{u}_1, \vec{u}_2\}$. The new basis vectors can be expressed in terms of the original Cartesian basis $E = \{\hat{\imath}, \hat{\jmath}\}$ as $\vec{u}_1|_E = (1, 2)$ and $\vec{u}_2|_E = (3, 5)$.

Asked: Find (1) the change of basis matrix P from E to S; (2) the change of basis matrix Q from S to E; (3) the components (a', b') of an arbitrary vector $\vec{v}$ in the S basis if $\vec{v}$ has components (a, b) in the Cartesian coordinate system ($\vec{v} = a\hat{\imath} + b\hat{\jmath}$.)

2 6.47(a), §2 Solution

The basis vectors of the S-system are given in terms of the E-system, so, to get the transformation matrix, we simply put them as columns of the matrix:

$$P \equiv \left(\vec{u}_1\Big|_E \vec{u}_2\Big|_E\right) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$$

Let's check this, just to be sure. The arbitrary vector $\vec{v}$ can be expressed as

$$\vec{v} = a\hat{\imath} + b\hat{\jmath} = a'\vec{u}_1 + b'\vec{u}_2$$

so

$$\vec{v}\Big|_E \equiv \begin{pmatrix} a \\ b \end{pmatrix} = a'\vec{u}_1\Big|_E + b'\vec{u}_2\Big|_E = \left(\vec{u}_1\Big|_E \vec{u}_2\Big|_E\right)\begin{pmatrix} a' \\ b' \end{pmatrix} = P\vec{v}\Big|_S$$

Since

$$\vec{v}\Big|_S = P^{-1}\vec{v}\Big|_E$$

the transformation matrix Q from S to E is

$$Q = P^{-1} = \frac{1}{-1}\begin{pmatrix} 5 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$$

So

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

So $a' = -5a + 3b$ and $b' = 2a - b$.