6.48

1 6.48, §1 Asked

Given: A basis $S = \{\vec{u}_1, \vec{u}_2\}$ and another basis $S' = \{\vec{v}_1, \vec{v}_2\}$. The basis vectors of S can be expressed in terms of the Cartesian basis $E = \{\hat{\imath}, \hat{\jmath}\}$ as $\vec{u}_1|_E = (1,2)$ and $\vec{u}_2|_E = (2,3)$ and those of S' as $\vec{v}_1|_E = (1,3)$ and $\vec{v}_2|_E = (1,4)$.

Asked: Find (a) the change of basis matrix P from S to S'; (b) the change of basis matrix Q from S' back to S.

2 6.48, §2 Solution

By definition, for any vector $\vec{w}$,

$$\vec{w}\Big|_S = P\vec{w}\Big|_{S'}$$

where P contains the basis vectors of the S'-system in terms of the S-system. Unfortunately, these basis vectors are given in terms of the E-system, not the S-system.

Trick: go over the E system:

$$\vec{w}\Big|_{S'} \quad \Longrightarrow \quad \vec{w}\Big|_E \quad \Longrightarrow \quad \vec{w}\Big|_S$$

$$\vec{w}\Big|_E = \left(\vec{v}_1\Big|_E \vec{v}_2\Big|_E\right)\vec{w}\Big|_{S'} \qquad \vec{w}\Big|_E = \left(\vec{u}_1\Big|_E \vec{u}_2\Big|_E\right)\vec{w}\Big|_S$$

So

$$\vec{w}\Big|_S = \left(\vec{u}_1\Big|_E \vec{u}_2\Big|_E\right)^{-1}\left(\vec{v}_1\Big|_E \vec{v}_2\Big|_E\right)\vec{w}\Big|_{S'}$$

$$P = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ -1 & -2 \end{pmatrix}$$

$$Q = P^{-1} = \begin{pmatrix} 2 & 5 \\ -1 & -3 \end{pmatrix}$$