9.47

1 9.47(a), §1 Asked

Given:

$$A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$$

Asked: All eigenvalues and linearly independent eigenvectors.

2 9.47(a), §2 Solution

Eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 6 \\ -2 & -2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = 0$$

There are two roots: $\lambda_1 = 1$ and $\lambda_2 = 2$

The eigenvector corresponding to λ_1 satisfies

$$(A - \lambda_1 I)\vec{v}_1 = 0 = \begin{pmatrix} 5-1 & 6 \\ -2 & -2-1 \end{pmatrix} \vec{v}_1$$

Solving using Gaussian elimination:

$$\left(\begin{array}{cc|c} 4 & 6 & 0 \\ -2 & -3 & 0 \end{array}\right) \quad \begin{array}{l} (1) \\ (2) \end{array} \quad \Longrightarrow \quad \left(\begin{array}{cc|c} \boxed{4} & 6 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} (1) \\ (2') = 2(2) + (1) \end{array}$$

Equation (1) gives $v_{1x} = -\frac{3}{2}v_{1y}$. In order to get a vector, instead of a set of possible vectors, one component must be arbitrarily chosen. *Remember: undetermined constants in eigenvectors are not allowed.* To get simple numbers, take $v_{1y} = -2$, then $v_{1x} = 3$:

$$\vec{v}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Check:

$$A\vec{v}_1 \stackrel{?}{=} \lambda_1 \vec{v}_1 \qquad \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Note: the null space of the matrix above is

$$\begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} v_{1y}$$

so $(-\frac{3}{2}, 1)$ would also have been an acceptable eigenvector, just messier.

The eigenvector corresponding to λ_2 satisfies

$$(A - \lambda_2 I)\vec{v}_2 = 0 = \begin{pmatrix} 5-2 & 6 \\ -2 & -2-2 \end{pmatrix} \vec{v}_1$$

Solving using Gaussian elimination:

$$\left(\begin{array}{cc|} 3 & 6 \\ -2 & -4 \end{array}\right) \quad \begin{array}{l} (1) \\ (2) \end{array} \quad \Longrightarrow \quad \left(\begin{array}{cc|} 3 & 6 \\ 0 & 0 \end{array}\right) \quad \begin{array}{l} (1) \\ (2') = 3(2) + 2(1) \end{array}$$

Choosing $v_{2y} = 1$, then $v_{2x} = -2$:

$$\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

3 9.47(b), §1 Asked

Given:

$$A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$$

with eigenvalues and corresponding eigenvectors

$$\lambda_1 = 1,\ \vec{v}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad \lambda_2 = 2,\ \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Asked: A matrix P such that $A' = P^{-1}AP$ is diagonal.

4 9.47(b), §2 Solution

- The matrix $P^{-1}AP$ is the new matrix A' after a basis transformation described by a transformation matrix P.
- To get A' diagonal, we want to take the new basis to be the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ of A.
- A transformation matrix consists of the new basis vectors expressed in terms of the old basis, so:

$$P = (\vec{v}_1\, \vec{v}_2) = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$$

Check:

$$A' = P^{-1}AP = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{-1} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^T \begin{pmatrix} 3 & -4 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

5 9.47(c), §1 Asked

Given:

$$A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$$

with eigenvalues and corresponding eigenvectors

$$\lambda_1 = 1,\ \vec{v}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad \lambda_2 = 2,\ \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Asked: A^6 and $A^4 - 5A^3 + 7A^2 - 2A + 5$

6 9.47(c), §2 Solution

Do it first in the eigenvector basis!

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \implies A'^6 = \begin{pmatrix} 1^6 & 0 \\ 0 & 2^6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 64 \end{pmatrix}$$

This works for diagonal matrices only.

Now transform back:

$$A^6 = PA'^6P^{-1} = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 64 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}^{-1}$$

$$A^6 = \begin{pmatrix} 3 & -128 \\ -2 & 64 \end{pmatrix} \frac{1}{-1} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^T = \begin{pmatrix} 253 & 378 \\ -126 & -188 \end{pmatrix}$$

Note that this is very different from

$$\begin{pmatrix} 5^6 & 6^6 \\ (-2)^6 & (-2)^6 \end{pmatrix}$$

(Answer in the book is for A^{10}

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \Longrightarrow \quad A'^4 - 5A'^3 + 7A'^2 - 2A' + 5I =$$

$$\begin{pmatrix} 1\,1^4 - 5\,1^3 + 7\,1^2 - 2\,1 + 5 & 0 \\ 0 & 2^4 - 5\,2^3 + 7\,2^2 - 2\,2 + 5 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}$$

$$A^4 - 5A^3 + 7A^2 - 2A + 5I = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 18 & -10 \\ -12 & 5 \end{pmatrix} \frac{1}{-1} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^T = \begin{pmatrix} 2 & -6 \\ 2 & 9 \end{pmatrix}$$

7 9.47(d), §1 Asked

Given:

$$A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$$

with eigenvalues and corresponding eigenvectors

$$\lambda_1 = 1,\ \vec{v}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad \lambda_2 = 2,\ \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Asked: A matrix B so that $B^2 = A$

8 9.47(d), §2 Solution

Do it first in the eigenvector basis!

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \Longrightarrow \quad B' = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

This works for diagonal matrices only.

Now transform back:

$$B = PB'P^{-1} = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}^{-1}$$

$$B = \begin{pmatrix} 3 & -2\sqrt{2} \\ -2 & \sqrt{2} \end{pmatrix} \frac{1}{-1} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^T = \begin{pmatrix} -3+4\sqrt{2} & -6+6\sqrt{2} \\ 2-2\sqrt{2} & 4-3\sqrt{2} \end{pmatrix}$$