9.48(c)

1 9.48(c), §1 Asked

Given:

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$$

Asked: All eigenvalues and linearly independent eigenvectors.

2 9.48(c), §2 Solution

Eigenvalues:

$$0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 1 & 2-\lambda & -1 \\ -1 & 1 & 4-\lambda \end{vmatrix}$$

$$= -\lambda^3 + 7\lambda^2 - 15\lambda + 9 = -(\lambda - 1)(\lambda - 3)^2 = 0$$

There is a single root: $\lambda_1 = 1$ and a double root $\lambda_2 = \lambda_3 = 3$

Eigenvectors corresponding to $\lambda_1 = 1$ satisfy

$$(A - \lambda_1 I)\vec{v}_1 = 0 = \begin{pmatrix} 1-1 & 2 & 2 \\ 1 & 2-1 & -1 \\ -1 & 1 & 4-1 \end{pmatrix} \begin{pmatrix} v_{1x} \\ v_{1y} \\ v_{1z} \end{pmatrix}$$

Solving using Gaussian elimination:

$$\left(\begin{array}{rrr|r} 0 & 2 & 2 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 3 & 0 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

$$\Longrightarrow \left(\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ -1 & 1 & 3 & 0 \end{array}\right) \qquad \begin{array}{l} (1') = (2) \\ (2') = (1) \\ (3) \end{array}$$

$$\Longrightarrow \left(\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right) \qquad \begin{array}{l} (1') \\ (2') \\ (3') = (3) + (1') \end{array}$$

$$\Longrightarrow \quad \left(\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 0 \\ 0 & \boxed{2} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} (1') \\ (2') \\ (3'') = (3') - (2') \end{array}$$

Equation (2') gives $v_{1y} = -v_{1z}$ and then (1') gives $v_{1x} = 2v_{1z}$.

The general solution space is:

$$\begin{pmatrix} v_{1x} \\ v_{1y} \\ v_{1z} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} v_{1z}$$

We *choose* $v_{1z} = 1$ to get

$$\vec{v}_1 = \begin{pmatrix} v_{1x} \\ v_{1y} \\ v_{1z} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Eigenvectors corresponding to $\lambda_2 = \lambda_3 = 3$ satisfy

$$(A - \lambda_2 I)\vec{v}_2 = 0 = \begin{pmatrix} 1-3 & 2 & 2 \\ 1 & 2-3 & -1 \\ -1 & 1 & 4-3 \end{pmatrix} \begin{pmatrix} v_{2x} \\ v_{2y} \\ v_{2z} \end{pmatrix}$$

Solving using Gaussian elimination:

$$\left(\begin{array}{ccc|c} -2 & 2 & 2 & 0 \\ 1 & -1 & -1 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

$$\Longrightarrow \quad \left(\begin{array}{ccc|c} \boxed{-2} & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} (1) \\ (2') = 2(2) + (1) \\ (3') = 2(3) - (1) \end{array}$$

Equation (1') gives $v_{2x} = v_{2y} + v_{2z}$. There are two unknown parameters.

The general solution space is:

$$\begin{pmatrix} v_{2x} \\ v_{2y} \\ v_{2z} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} v_{2y} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} v_{2z}$$

We need *two* independent eigenvectors to span the space corresponding to this multiple root.

We can use the two vectors above, which means choosing $v_{2y} = 1$ and $v_{2z} = 0$ for one, and $v_{2y} = 0$ and $v_{2z} = 1$ for the other. That gives

$$\vec{v}_{2a} = \begin{pmatrix} v_{2ax} \\ v_{2ay} \\ v_{2az} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{v}_{2b} = \begin{pmatrix} v_{2bx} \\ v_{2by} \\ v_{2bz} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

If the three vectors $\vec{v}_1$, $\vec{v}_{2a}$, and $\vec{v}_{2b}$ are used as basis, A becomes diagonal. So despite the multiple root, this A is still diagonalizable. But if the solution space for the second eigenvalue would have been one-dimensional, the matrix would not have been diagonalizable.

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