9.56(a)

1 9.56(a), §1 Asked

Given:

$$A = \begin{pmatrix} 5 & 4 \\ 4 & -1 \end{pmatrix}$$

Asked: The orthonormal transformation matrix P so that $A' = P^{-1}AP$ is diagonal.

2 9.56(a), §2 Solution

Given:

$$A = \begin{pmatrix} 5 & 4 \\ 4 & -1 \end{pmatrix}$$

Eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 4 \\ 4 & -1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 21 = 0$$

There are two roots: $\lambda_1 = -3$ and $\lambda_2 = 7$

The eigenvector corresponding to $\lambda_1 = -3$ satisfies

$$\left(\begin{array}{cc|} 8 & 4 \\ 4 & 2 \end{array}\right) \quad \begin{array}{l} (1) \\ (2) \end{array} \quad \Longrightarrow \quad \left(\begin{array}{cc|} 8 & 4 \\ 0 & 0 \end{array}\right) \quad \begin{array}{l} (1) \\ (2') = 2(2) - (1) \end{array}$$

Taking $v_{1y} = -2$, then $v_{1x} = 1$, giving an eigenvector (1,-2). Normalizing this vector to length one gives:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Big/ \sqrt{1^2 + (-2)^2} = \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$$

The eigenvector corresponding to $\lambda_2 = 7$ satisfies

$$\left(\begin{array}{cc|} -2 & 4 \\ 4 & -8 \end{array}\right) \quad \begin{array}{l} (1) \\ (2) \end{array} \quad \Longrightarrow \quad \left(\begin{array}{cc|} -2 & 4 \\ 0 & 0 \end{array}\right) \quad \begin{array}{l} (1) \\ (2') = (2) + 2(1) \end{array}$$

Taking $v_{2y} = 1$, then $v_{2x} = 2$, giving after normalization:

$$\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Big/ \sqrt{2^2 + 1^2} = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

Finally:

$$P = (\vec{v}_1\, \vec{v}_2) = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

Check:

$$P^{-1}AP = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 5 & 4 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 7 \end{pmatrix}$$

The diagonal form is what matrix A looks like in the coordinate system x', y' shown below: