Eigenvector Basis

Examples:

- decomposing motion along the fundamental modes;
- writing solid body motion along the principal axes;
- separation of variables;
- improving numerical schemes;
- ...

Diagonalization:

If we use the eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ of a matrix A as a new basis, so that the transformation matrix P contains the eigenvectors:

$$P = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)\,,$$

then the transformed matrix A' is much simpler than the original A. In particular, it is diagonal:

$$A' = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Reason: for any arbitrary vector

$$\vec{w} = w_1\Big|_S \vec{v}_1 + w_2\Big|_S \vec{v}_2 + \ldots + w_n\Big|_S \vec{v}_n$$

then

$$A\vec{w} = w_1\Big|_S \lambda_1 \vec{v}_1 + w_2\Big|_S \lambda_2 \vec{v}_2 + \ldots + w_n\Big|_S \lambda_n \vec{v}_n$$

So A increases the first coordinate in the eigenvector basis by λ_1, the second by λ_2, etcetera. That is exactly what the diagonal matrix A' does with the vector of coefficients $(w_1\big|_S, w_2\big|_S, \ldots, w_n\big|_S)$.

Remember that the relationship between A and A' is

$$A = PA'P^{-1} \text{ or } A' = P^{-1}AP$$

Note: If an $n \times n$ matrix A has less than n independent eigenvectors, it is *not* diagonalizable. It is called *defective.* Most matrices are however diagonalizable:

- As long as all n-eigenvalues are distinct, the matrix is diagonalizable.
- *Normal* matrices, which commute with their transpose, $AA^H = A^H A$, always have a complete set of *orthonormal* eigenvectors anyway.
- Even if the matrix has less than n different eigenvalues and is not normal, it *might* still be diagonalizable.