Introduction

Eigenvalues:

- buckling;
- modes of vibration;
- dynamical systems;
- principal axes;
- boundary layer instability;
- heat conduction;
- acoustics;
- electrical circuits;
- stability of numerical methods;
- exam questions;
- ...

Definition

A nonzero vector $\vec{v}$ is an eigenvector of a matrix A if $A\vec{v}$ is a multiple of $\vec{v}$:

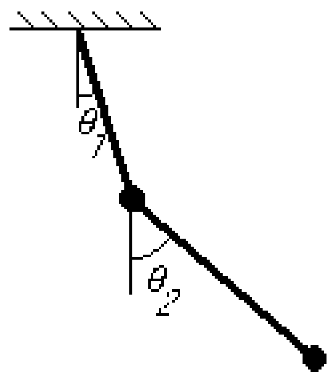
$$A\vec{v} = \lambda\vec{v}$$

The number λ is called the corresponding eigenvalue.

Graphically, if $\vec{v}$ is an eigenvector of A, then the vector $A\vec{v}$ is in the same (or exactly opposite direction) as $\vec{v}$:

An eigenvector is indeterminate by a constant that must be chosen.

Example



Equations of motion:

$$M\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix} + K\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = 0$$

Setting $\vec{\theta} \equiv (\theta_1, \theta_2)$

$$M\ddot{\vec{\theta}} + K\vec{\theta} = 0$$

Premultiplying by M^{-1} and defining $A = M^{-1}K$,

$$\ddot{\vec{\theta}} + A\vec{\theta} = 0$$

Try solutions of the form $\vec{\theta} = \vec{C}e^{i\omega t}$. The constant vector $\vec{C}$ determines the "mode shape:" $\theta_1/\theta_2 = C_1/C_2$. The exponential gives the time-dependent amplitude of this mode shape, with ω the natural frequency.

Plugging the assumed solution into the equations of motion:

$$-\omega^2\vec{C} + A\vec{C} = 0 \qquad \Longrightarrow \qquad A\vec{C} = \omega^2\vec{C}$$

So the mode shape $\vec{C}$ is an eigenvector of A and the corresponding eigenvalue gives the square of the frequency.

There will be two different eigenvectors $\vec{C}$, hence two mode shapes and two corresponding frequencies.

Note: we may lose symmetry in the above procedure. There are better ways to do this.

Procedure

To find the eigenvalues and eigenvectors of a matrix A,

1. Find the zeros of the determinant $|A - \lambda I|$ (i.e. of matrix A with $-\lambda$ added to each main diagonal element.) (The book uses $\lambda I - A$. This is very error-prone, and I do not recommend it.) For an $n \times n$ matrix A, $|A - \lambda I|$ is an n-th degree polynomial in λ. From it, we can find n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, (which do not all need to be distinct, however.)

2. When the eigenvalues are found, for each eigenvalue λ_i the corresponding eigenvector(s) can be found as the basis of the null space of $A - \lambda_i I$. *Note: Do not leave undetermined coefficients in eigenvectors. This is counted as an error.*