Row-canonical form

1 Row Canonical

Row canonical form:

- also zeros *above* the pivots;
- the pivots are normalized to one.

Example echelon form

$$\begin{pmatrix} \boxed{1} & 2 & -1 & 3 \\ 0 & 0 & \boxed{6} & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} (1) \\ (2') \\ (3'') \end{array}$$

Example canonical form

$$\begin{pmatrix} \boxed{1} & 2 & 0 & 2\frac{1}{2} \\ 0 & 0 & \boxed{1} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} (1') = (1) + \frac{1}{6}(2') \\ (2'') = \frac{1}{6}(2') \\ (3'') \end{array}$$

Note that

$$\left(\begin{array}{cccc|c} \boxed{1} & 2 & 0 & 2\frac{1}{2} & 3\frac{1}{2} \\ 0 & 0 & \boxed{1} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} (1') \\ (2'') \\ (3'') \end{array}$$

directly solves to $x = 3\frac{1}{2} - 2y - 2\frac{1}{2}t$ and $z = \frac{1}{2} + \frac{1}{2}t$.

To obtain a row-canonical matrix, first reduce to echelon form. Next eliminate all nonzero elements above the pivots, starting from the last one, and divide each equation by its pivot.

2 Operations Counts

To solve a reasonably sized system of n equations in n unknowns, the amount of work the computer must do varies with algorithm. The operations the computer must do are mostly additions and multiplications, and the number that must be done is roughly:

- Cramer's rule: $n!$ operations (prohibitive);

- Gaussian Elimination/LU-decomposition: reduce to echelon form at $\frac{1}{3}n^3$ operations;
- Gauss-Jordan: reduce to row canonical form at $\frac{1}{2}n^3$ operations;
- Matrix inversion: n^3 operations.

Note that Gausian elimination can be much more efficient still for sparse matrices (i.e. matrices with a lot of zeros.) Always use the most specific algorithm for your matrix.