3.50

1 3.50, §1 Asked

Solve:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + y^2}{2xy}$$

2 3.50, §2 Solution

Note that the equation is homogeneous

$$\begin{array}{ll} \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2 + y^2}{2xy} & \leftarrow \text{ degree } 2 \\ & \leftarrow \text{ degree } 2 \end{array}$$

or alternatively,

$$\frac{(tx)^2 + (ty)^2}{2txty} = \frac{x^2 + y^2}{2xy}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + \left(\dfrac{y}{x}\right)^2}{2\left(\dfrac{y}{x}\right)}$$

Use new unknown $u = y/x$, i.e., replace y by xu:

$$\frac{\mathrm{d}xu}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u = \frac{1 + u^2}{2u}$$

$$x\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1 - u^2}{2u}$$

Separable:

$$-\frac{2u\,\mathrm{d}u}{1 - u^2} = -\frac{\mathrm{d}x}{x}$$

$$\ln|1 - u^2| = -\ln|x| + C$$

$$|1 - u^2| = \frac{e^C}{|x|}$$

$$u^2 = 1 \pm \frac{e^C}{x} = 1 + \frac{D}{x}$$

Get rid of u in favor of y/x:

$$y^2 = x^2 + Dx$$