11.25

1 11.25, §1 Asked

Solve:

$$\dddot{r} - 3\ddot{r} + 3\dot{r} - r = \frac{e^t}{t}$$

2 11.25, §2 Solution

$$\dddot{r} - 3\ddot{r} + 3\dot{r} - r = \frac{e^t}{t}$$

Homogeneous equation:

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \quad \Longrightarrow \quad \lambda_1 = \lambda_2 = \lambda_3 = 1$$
$$r_h = C_1 e^t + C_2 t e^t + C_3 t^2 e^t$$

Variation of parameters:

$$r = C_1 e^t + C_2 t e^t + C_3 t^2 e^t$$
$$\dot{C}_1 e^t + \dot{C}_2 t e^t + \dot{C}_3 t^2 e^t = 0 \tag{1}$$
$$\dot{r} = C_1 e^t + C_2 (t+1) e^t + C_3 (t^2 + 2t) e^t$$
$$\dot{C}_1 e^t + \dot{C}_2 (t+1) e^t + \dot{C}_3 (t^2 + 2t) e^t = 0 \tag{2}$$
$$\ddot{r} = C_1 e^t + C_2 (t+2) e^t + C_3 (t^2 + 4t + 2) e^t$$
$$\dddot{r} = \dot{C}_1 e^t + \dot{C}_2 (t+2) e^t + \dot{C}_3 (t^2 + 4t + 2) e^t + \ldots$$

Into the O.D.E.:

$$\dot{C}_1 e^t + \dot{C}_2 (t+2) e^t + \dot{C}_3 (t^2 + 4t + 2) e^t + \ldots = \frac{e^t}{t} \tag{3}$$

Total system of equations for unknowns $\dot{C}_1$,$\dot{C}_2$, and $\dot{C}_3$:

$$\left(\begin{array}{ccc|c} 1 & t & t^2 & 0 \\ 1 & t+1 & t^2 + 2t & 0 \\ t & t^2 + 2t & t^3 + 4t^2 + 2t & 1 \end{array}\right) \qquad \begin{array}{l} (1') = e^{-t}(1) \\ (2') = e^{-t}(2) \\ (3') = te^{-t}(3) \end{array}$$

Forward elimination:

$$\left(\begin{array}{ccc|c} 1 & t & t^2 & 0 \\ 0 & 1 & 2t & 0 \\ 0 & 2t & 4t^2+2t & 1 \end{array}\right) \qquad \begin{array}{l} (1') \\ (2'') = (2') - (1') \\ (3'') = (3') - t(1') \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & t & t^2 & 0 \\ 0 & 1 & 2t & 0 \\ 0 & 0 & 2t & 1 \end{array}\right) \qquad \begin{array}{l} (1') \\ (2'') \\ (3''') = (3'') - 2t(2'') \end{array}$$

Back substitution: $\dot{C}_3 = 1/2t$, $\dot{C}_2 = -1$, $\dot{C}_1 = \frac{1}{2}t$, hence

$$C_1 = \frac{1}{4}t^2 + C_{10} \quad C_2 = -t + C_{20} \quad C_3 = \ln\sqrt{t} + C_{30}$$

Total solution:

$$r = t^2 \ln\sqrt{t}e^t + C_{10}e^t + C_{20}te^t + C_{30}^* t^2 e^t$$