22.12

1 22.12, §1 Asked

Solve as a System:

$$\ddot{x} + 2\dot{x} - 8x = 4 \qquad x(0) = 1, \dot{x}(0) = 2$$

2 22.12, §2 Solution

Solve as a System:

$$\ddot{x} + 2\dot{x} - 8x = 4 \qquad x(0) = 1, \dot{x}(0) = 2$$

Define new dependent variables $x_1 = x$ and $x_2 = \dot{x}$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 8x_1 - 2x_2 + 4 \end{aligned}$$

Matrix form $\dot{\vec{x}} = A\vec{x} + b$:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 8 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

Homogeneous equation:

$$\vec{x}_h = C_1 \vec{v}_1 e^{\lambda_1 t} + C_2 \vec{v}_2 e^{\lambda_2 t}$$

where λ_1 and λ_2 are the eigenvalues of A and $\vec{v}_1$ and $\vec{v}_2$ the eigenvectors:

$$\begin{vmatrix} -\lambda & 1 \\ 8 & -2-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 8 = 0$$

$$\lambda_1 = 2, \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \lambda_2 = -4, \vec{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

Particular solution $\dot{\vec{x}}_p = A\vec{x}_p + b$: guess that $\vec{x}_p$ is constant, then $A\vec{x}_p = -\vec{b}$. Solve:

$$\vec{x}_p = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$$

Total solution:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} + C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4t}$$

Put in the initial conditions:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} C_1 \\ 2C_1 \end{pmatrix} + \begin{pmatrix} C_2 \\ -4C_2 \end{pmatrix}$$

which gives $C_2 = \frac{1}{6}$, $C_1 = \frac{4}{3}$.

Final solution:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{4}{3} \\ \frac{8}{3} \end{pmatrix} e^{2t} + \begin{pmatrix} \frac{1}{6} \\ -\frac{2}{3} \end{pmatrix} e^{-4t}$$