Some Other Equations

1 §1 Introduction

Generally speaking, equations become more difficult when the order goes up.

For a first order equation, even if you cannot solve, you can always draw little line segments with the right slope in the x, y plane and then draw trajectories following those directions.

For some equations there are tricks that allow you to reduce the order.

Nonlinear equations are generally more difficult than linear ones.

2 §2 Handbooks

Look it up in a mathematical handbook. Schaum's Mathematical Handbook has some. Abramowitz and Stegun has a large collection of equations solvable by Bessel functions and other standard functions, and the properties of these function.

Avoid exact equations in Schaum's Mathematical Handbook.

3 §3 Power Series

Expand the solution in a power series, equating all powers in the O.D.E. to zero.

4 §4 Euler

$$a_0y + a_1xy' + a_2x^2y'' + a_3x^3y^{(3)} + \ldots + a_nx^ny^{(n)} = q$$

where $a_0, a_1, \ldots, a_n$ are all constants but q can be any function of x.

The substitution $\xi = \ln x$ turns this into a constant coefficient equation for $y(\xi)$. Reason:

$$y' = \frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi}y = \frac{1}{x}y_\xi$$

$$y'' = -\frac{1}{x^2}y_\xi + \frac{1}{x}\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi}y_\xi = -\frac{1}{x^2}y_\xi + \frac{1}{x^2}y_{\xi\xi}$$
$$y''' = \ldots$$

5 §5 No y

If the derivatives of y, but not y itself appear, simply take y' instead of y as the unknown. A second order equation for y becomes first order for y'.

6 §6 No x

If derivatives with respect to x appear, but not x itself, use y as the new *independent* variable and y' as the new dependent variable.

$$y'' = \frac{\mathrm{d}y'}{\mathrm{d}y}y'$$

$$y''' = \ldots$$

The order of the equation for $y'(y)$ is one less than that of the equation for $y(x)$.

7 §7 Linear

If the equation is linear and homogeneous, setting $y = e^f$ gives an equation not involving f itself:

$$y = e^f \qquad y' = e^f f' \qquad y'' = e^f f'^2 + e^f f'' \qquad \ldots$$

If the equation is linear and homogeneous, and you know a solution $y_1(x)$, setting $y = C(x)y_1(x)$ gives a *linear* equation for C not involving C itself.