

# Some Revised Notes on Ordinary Differential Equations

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## Abstract

This are some revised notes on ordinary differential equations. They supplement the current text book [1].

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## 1 Partial Fractions

Partial fraction expansion simplifies fractions of the form

$$\frac{T(s)}{B(s)}$$

where  $T(s)$  and  $B(s)$  are polynomials in some variable  $s$ . This is of importance, for example, in integration of such fractions and in Laplace transformation. This section explains the procedure.

First, in case  $T(s)$  has a degree equal or higher than  $B(s)$ , you should perform long division to take the ratio apart into powers of  $s$  plus a remainder. That remainder is again a fraction of the form  $T(s)/B(s)$ , but the new  $T(s)$  is of a degree less than  $B(s)$ .

Now the only thing you still have to know is how to deal with a ratio  $T(s)/B(s)$  when the degree of  $T(s)$  is less than that of  $B(s)$ .

The first thing you will need to do is factor  $B(s)$ . Assume it is a polynomial of some degree  $n$ , in other words

$$B(s) = C(s^n + b_{n-1}s^{n-1} + \dots + b_2s^2 + b_1s + b_0)$$

where  $C$  is some nonzero constant and the  $b_i$  for  $i = n-1, \dots, 2, 1, 0$  are  $n$  additional constants. You can simply take  $C$  out of the entire ratio  $T(s)/B(s)$  before proceeding. From now on, it will be assumed that you have done that, so that there is no longer a  $C$  in the above expression to worry about. It is then known from complex variable theory that  $B(s)$  can always be written in the form

$$B(s) = (s - s_1)(s - s_2)(s - s_3) \dots (s - s_n)$$

Here the  $n$  constants  $s_i$ , ( $i = 1, 2, \dots, n$ ), are the locations, or roots, where  $B(s)$  is zero. So  $B(s_i) = 0$  for  $i = 1, 2, \dots, n$ . Note that some of these roots may coincide. In other words some of the  $s_i$  may be equal to each other. For example

$$s^3 + s^2 - 5s + 3 = (s - 1)(s - 1)(s + 3)$$

has  $s_1 = s_2 = 1$  and  $s_3 = -3$ .

There is a further complication. Some of the roots  $s_i$  may be complex. For example, a quadratic with a negative discriminant has complex roots instead of real ones. However, assuming that the polynomial  $B(s)$  is real, (for real  $s$ ), you would probably not want to deal with complex numbers. You can avoid that because for real  $B(s)$ , the complex roots come in “complex-conjugate pairs.” That means that for every complex root  $s_i$  there is a second root  $s_j$  so that  $(s - s_i)(s - s_j)$  is a *real* quadratic with roots  $s_i$  and  $s_j$ . So a real  $B(s)$  can always be written as a product of real factors linear in  $s$  and real factors quadratic in  $s$ .

For example, you might have a  $B(s)$  that can be written as

$$B(s) = (s - 4)(s - 3)^3(s^2 - 2s + 5)(s^2 + 4s + 13)^2$$

That would be a polynomial of degree 10, ( $1 + 3 + 2 + 4 = 10$ ). For this  $B(s)$ , any desired ratio  $T(s)/B(s)$  can be written as

$$\begin{aligned} \frac{T(s)}{B(s)} = & \frac{C_1}{s - 4} + \\ & \frac{C_2}{s - 3} + \frac{C_3}{(s - 3)^2} + \frac{C_4}{(s - 3)^3} + \\ & \frac{C_5s + C_6}{s^2 - 2s + 5} + \\ & \frac{C_7s + C_8}{s^2 + 4s + 13} + \frac{C_9s + C_{10}}{(s^2 + 4s + 13)^2} \end{aligned}$$

The right hand side is the partial fraction expansion of the ratio.

Note that the individual terms in the partial fraction expansion are *much* simpler than the original ratio. The original ratio has a polynomial of degree 10 in the bottom. And a polynomial of a degree up to 9 in the top. So if you want to do anything with the given ratio, it will become much easier if you use the right hand side above to do it. That is why you want to do partial fraction expansions.

That leaves two key questions to be answered:

1. In general, how do you know what terms there are in the right hand side?
2. How do you find the values of all these coefficients  $C_1, C_2, C_3, \dots$ ?

The answer to the first question is as follows;

1. For every factor  $(s - s_i)$  that appears  $k$  times in  $B(s)$ , there are terms in the right hand side of the form

$$\frac{C_{i,1}}{s - s_i} + \frac{C_{i,2}}{(s - s_i)^2} + \dots + \frac{C_{i,k}}{(s - s_i)^k}$$

Check it out for the factors  $(s - 4)$  (single) and  $(s - 3)$  (triple) in the example given earlier. (What you want to call the constants is of course up to you, as long as each has a unique name.)

2. For every factor  $(s^2 + a_i s + b_i)$  that appears  $k$  times in  $B(s)$ , there are terms in the right hand side of the form

$$\frac{C_{i,1}s + C_{i,2}}{s^2 + a_i s + b_i} + \frac{C_{i,3}s + C_{i,4}}{(s^2 + a_i s + b_i)^2} + \dots + \frac{C_{i,2k-1}s + C_{i,2k}}{(s^2 + a_i s + b_i)^k}$$

Check it out for the factors  $(s^2 - 2s + 5)$  (single) and  $(s^2 + 4s + 13)$  (double) in the example above.

The second question was how to find all these constants. The method you must use in this class is to crunch it out:

1. Bring all the terms in the partial fraction expansion over the common denominator  $B(s)$ .
2. Multiply out the top. This must equal the given  $T(s)$ . So the net coefficient of each power of  $s$  must match the corresponding coefficient in  $T(s)$ . That gives you your equations for your unknown coefficients. Use Gaussian elimination to solve them.

For the example, its partial fraction expansion becomes, when brought over the common denominator  $B(s)$ :

$$\frac{\begin{aligned} &C_1(s-3)^3(s^2-2s+5)(s^2+4s+13)^2 \\ &+C_2(s-4)(s-3)^2(s^2-2s+5)(s^2+4s+13)^2 \\ &+C_3(s-4)(s-3)(s^2-2s+5)(s^2+4s+13)^2 \\ &+C_4(s-4)(s^2-2s+5)(s^2+4s+13)^2 \\ &+(C_5s+C_6)(s-4)(s-3)^3(s^2+4s+13)^2 \\ &+(C_7s+C_8)(s-4)(s-3)^3(s^2-2s+5)(s^2+4s+13) \\ &+(C_9s+C_{10})(s-4)(s-3)^3(s^2-2s+5) \end{aligned}}{(s-4)(s-3)^3(s^2-2s+5)(s^2+4s+13)^2}$$

Multiply out the top, (don't make any mistakes, of course), then equate the net coefficients of the  $s^0, s, s^2, \dots, s^8$  and  $s^9$  powers to the corresponding coefficients in the given  $T(s)$ . That gives 10 equations for the 10 unknowns  $C_1, C_2, \dots, C_{10}$ . Solve using Gaussian elimination. (Cramer's rule is *not* recommended.) That will be fun!

Note: Of course, if  $B(s)$  is not real, there is no point in using quadratics. Just expand in linear factors of the form  $s - s_i$  (with  $s_i$  now complex) only.

Note: there are more intelligent ways of finding the coefficients than crunching it out as we must do in this class. For example, consider once more the example:

$$\begin{aligned} \frac{T(s)}{B(s)} &= \frac{C_1}{s-4} + \\ &\frac{C_2}{s-3} + \frac{C_3}{(s-3)^2} + \frac{C_4}{(s-3)^3} + \\ &\frac{C_5s+C_6}{s^2-2s+5} + \\ &\frac{C_7s+C_8}{s^2+4s+13} + \frac{C_9s+C_{10}}{(s^2+4s+13)^2} \end{aligned}$$

If you multiply the expansion in the right hand side by the single factor  $(s-4)$  and then evaluate it at  $s=4$ , you get  $C_1$ . (All the other terms are zero at  $s=4$  because of the multiplication by  $(s-4)$ .) The bottom line is therefore that if you multiply the *left* hand side by  $(s-4)$  and then evaluate it at  $s=4$ , you get  $C_1$  too. And that is doable because the left hand side is given. (To evaluate at  $s=4$ , you either need to divide out the common factor  $(s-4)$  from top and bottom or use l'Hopital once. Dividing out the common factor from top and bottom is a simple matter of a long division of the bottom if you have it in unfactored form.) To find  $C_4$ , multiply by  $(s-3)^3$  and evaluate at  $s=3$ . To find  $C_3$ , multiply by  $(s-3)^3$ , differentiate the result once, and evaluate at  $s=3$ . Etcetera. Especially if you want just a single coefficient, this can be much more convenient. Or you can use it to test the correctness of a few sample coefficients.

## 2 Completing the square

Completing the square simply means that you write a quadratic

$$ax^2 + bx + c$$

as

$$a \left[ \underbrace{\left( x + \frac{b}{2a} \right)^2}_{\substack{\text{shifted} \\ x}} + \underbrace{\frac{c}{a} - \frac{b^2}{4a^2}}_{\substack{\text{new} \\ \text{constant}}} \right]$$

If you multiply out, you see that it is the same.

One place where you often need this is in Laplace transforms. Laplace transforms are often given in terms of a quadratic  $s^2 + K$  where  $K$  is an arbitrary constant. But you might have  $as^2 + bs + c$  instead of something of the form  $s^2 + K$ . However, you can write the part inside the square brackets above as  $s^2 + K$  if you use the shift theorem to account for the  $b/2a$  inside the parentheses. And the additional factor  $a$  is trivial to account for.

## 3 Solution of systems using diagonalization

You should know by now how to solve a system of ordinary differential equations of the form

$$\dot{\vec{u}} = A\vec{u} + \vec{f} \quad \vec{u}(0) = \vec{g}$$

where unknown vector  $\vec{u}$  and given vector  $\vec{g}$  depend on time, but  $\vec{f}$  is a given constant vector and  $A$  a given constant matrix. (Yes, I will use  $\vec{u}$  instead of  $\vec{x}$  here.) The dot of course indicates the time derivative.

However, suppose that  $\dot{\vec{u}}$  would be replaced by the second order derivative  $\ddot{\vec{u}}$ ? Like in

$$\ddot{\vec{u}} = A\vec{u} + \vec{f} \quad \vec{u}(0) = \vec{g} \quad \dot{\vec{u}}(0) = \vec{h}$$

That happens in mechanics when the forces only depend on position (no friction). Note that for this second order system we also need the initial velocities  $\dot{\vec{u}}(0)$ .

Of course, you can convert the above system to a double-size first order one. But suppose you want to keep the system size the same? Well, you can solve the system directly using the basis of eigenvectors of matrix  $A$ , assuming it is not defective. And the relevant matrix  $A$  is typically a real symmetric one in these applications, so never defective.

I will now show how the solution procedure works. First of course you must find the eigenvalues and eigenvectors of  $A$ :

$$\begin{array}{cccc} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \vec{e}_1 & \vec{e}_2 & \dots & \vec{e}_n \end{array}$$

Properties of the Laplace Transform		
Property	$f(t)$	$\widehat{f}(s)$
<b>P1: Inversion</b>	$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widehat{f}(s)e^{st} ds$	$\int_0^\infty f(t)e^{-st} dt$
<b>P2: Linearity</b>	$C_1 f_1(t) + C_2 f_2(t)$	$C_1 \widehat{f}_1(s) + C_2 \widehat{f}_2(s)$
<b>P3: Dilation</b>	$f(\omega t)$	$\omega^{-1} \widehat{f}(s/\omega)$
<b>P4: Differentiation</b>	$f^{(n)}(t)$	$s^n \widehat{f}(s) - s^{n-1} f(0^+) - \dots - f^{(n-1)}(0^+)$
<b>P5: Differentiation</b>	$t^n f(t)$	$(-1)^n \widehat{f}^{(n)}(s)$
<b>P6: Shift</b>	$H(t - \tau) f(t - \tau)$ $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$	$e^{-\tau s} \widehat{f}(s)$
<b>P7: Shift</b>	$e^{\sigma t} f(t)$	$\widehat{f}(s - \sigma)$
<b>P8: Convolution</b>	$\int_0^t f(t - \tau) g(\tau) d\tau$ Do <i>not</i> write as $f * g$	$\widehat{f}(s) \widehat{g}(s)$

Special Laplace Transform Pairs					
	$f(t)$	$\widehat{f}(s)$	$f(t)$	$\widehat{f}(s)$	
<b>S1:</b>	1	$\frac{1}{s}$	<b>S8:</b>	$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
<b>S2:</b>	$t^n$	$\frac{n!}{s^{n+1}}$	<b>S9:</b>	$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
<b>S3:</b>	$e^{\sigma t}$	$\frac{1}{s - \sigma}$	<b>S10:</b>	$t \sin(\omega t)$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$
<b>S4:</b>	$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$	<b>S11:</b>	$t \cos(\omega t)$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
<b>S5:</b>	$\frac{1}{\sqrt{\pi t}} e^{-k^2/4t}$	$\frac{1}{\sqrt{s}} e^{-k\sqrt{s}}$	<b>S12:</b>	$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$
<b>S6:</b>	$\frac{k}{\sqrt{4\pi t^3}} e^{-k^2/4t}$	$e^{-k\sqrt{s}}$	<b>S13:</b>	$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$
<b>S7:</b>	$\operatorname{erfc}\left(k/2\sqrt{t}\right)$	$\frac{1}{s} e^{-k\sqrt{s}}$	<b>S14:</b>	$\delta(t - \tau)$	$e^{-\tau s}$

Table 1: Properties of the Laplace Transform. ( $k, \tau, \omega > 0, n = 1, 2, \dots$ )

But you always needed to do that.

Next you write *every* vector in the problem in terms of the eigenvectors:

$$\begin{aligned}\vec{u} &= u'_1 \vec{e}_1 + u'_2 \vec{e}_2 + \dots + u'_n \vec{e}_n \\ \vec{f} &= f'_1 \vec{e}_1 + f'_2 \vec{e}_2 + \dots + f'_n \vec{e}_n \\ \vec{g} &= g'_1 \vec{e}_1 + g'_2 \vec{e}_2 + \dots + g'_n \vec{e}_n \\ \vec{h} &= h'_1 \vec{e}_1 + h'_2 \vec{e}_2 + \dots + h'_n \vec{e}_n\end{aligned}$$

Here the primes indicate coefficients of the vectors in the basis of eigenvectors. Note that the  $u'_i$  and  $f'_i$  in general depend on time but the  $g'_i$  and  $h'_i$  are constants, for any  $i$  from 1 to  $n$ .

You will need to figure out what the coefficients of the given vectors  $\vec{f}$  and  $\vec{g}$  are now. Note that the above equations can be written in matrix form as

$$E \begin{pmatrix} f'_1 \\ f'_2 \\ \vdots \\ f'_n \end{pmatrix} = \vec{f} \quad E \begin{pmatrix} g'_1 \\ g'_2 \\ \vdots \\ g'_n \end{pmatrix} = \vec{g} \quad E \begin{pmatrix} h'_1 \\ h'_2 \\ \vdots \\ h'_n \end{pmatrix} = \vec{h} \quad E \equiv (\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n)$$

Matrix  $E$ , of course, is our transformation matrix to the basis of eigenvectors. In any case, the above equations must be solved to find the  $f'_i$ ,  $g'_i$ , and  $h'_i$ . (In doing that, remember that for a real symmetric matrix, you take the eigenvectors orthonormal, after which the inverse matrix  $E^{-1}$  is just the transpose one,  $E^T$ .)

Next remember that in the basis of the eigenvectors, matrix  $A$  becomes a diagonal one, with diagonal values equal to the eigenvalues. Therefore the original system of ordinary differential equations simplifies to decoupled equations:

$$\begin{aligned}\ddot{u}'_1 &= \lambda_1 u'_1 + f'_1 & u'_1(0) &= g'_1 & \dot{u}'_1(0) &= h'_1 \\ \ddot{u}'_2 &= \lambda_2 u'_2 + f'_2 & u'_2(0) &= g'_2 & \dot{u}'_2(0) &= h'_2 \\ &\vdots & & & & \\ \ddot{u}'_n &= \lambda_n u'_n + f'_n & u'_n(0) &= g'_n & \dot{u}'_n(0) &= h'_n\end{aligned}$$

You should be able to solve each of these scalar second order equations easily.

Finally you can find the solution vector  $\vec{u}$  at any desired time by summing:

$$\vec{u} = u'_1 \vec{e}_1 + u'_2 \vec{e}_2 + \dots + u'_n \vec{e}_n \equiv \sum_{i=1}^n u'_i \vec{e}_i$$

Of course, you could also solve the *first* order system that way. Compared to the class procedure, that has one big advantage. In the class procedure, we solved a system of equations for the variation of parameters, and one for the initial conditions. In the above method, the matrix of the two systems of equations to solve is the same,  $E$ , so you can use a single augmented matrix with two right hand sides (being  $\vec{f}$  and  $\vec{g}$ ). (And if  $A$  is symmetric, it is easier still, because you only need to multiply by  $E^T$ .)

## 4 An example partial differential equation

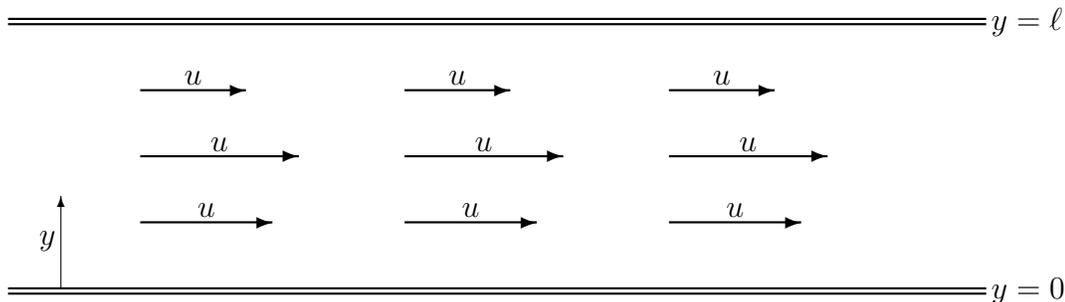


Figure 1: Laminar viscous flow in a long duct.

Consider unsteady viscous laminar flow of, say, water, in a long and thin horizontal two-dimensional duct. The velocity  $u$  depends on the time and the vertical position  $y$ , so  $u = u(t; y)$ . However, for a very long duct, it does not depend on the streamwise coordinate  $x$ .

According to fluid mechanics, the velocity develops according to the equation

$$\dot{u} = \nu \frac{\partial^2}{\partial y^2} u + f$$

Here  $f$  is some given function of  $t$  and  $y$  accounting for forces like gravity or electromagnetic ones. The equation above is called a *partial differential equation* because there are derivatives with respect to two variables; not just  $t$  but also  $y$ . To solve it, you also need an initial condition:

$$u(0; y) = g$$

where  $g$  is some given function.

Note that so far, the above system looks almost exactly like the first order system of ordinary differential equations in the previous section. However, where the system of ordinary differential equations has vectors, the scalar partial differential equation above has functions of  $y$ . The only other difference is that where the system of ordinary differential equations had some matrix  $A$ , the partial differential equation above has an “operator”

$$L \equiv \nu \frac{\partial^2}{\partial y^2}$$

But that is no big difference: when you apply a matrix  $A$  on a vector  $\vec{v}$ , you get a new vector  $A\vec{v}$ . In exactly the same way, if you apply  $L$  above on a function  $F(y)$ , you get a new function of  $y$  equal to  $\nu F''(y)$ . It is the same thing.

There is however one thing really different for the partial differential equation; it has *boundary conditions in  $y$* . The fluid must be at rest at the walls of

the duct. With the walls at  $y = 0$  and  $y = \ell$ , (with  $\ell$  the height of the duct), that means

$$y = 0: \quad u(t; 0) = 0 \quad y = \ell: \quad u(t; \ell) = 0$$

(It is like the first and the last component of vector  $\vec{u}$  would have to be zero.)

Still, you can solve the partial differential equation much like the system of ordinary differential equations in the previous section. I will now show you how.

First, we need the eigenfunctions of the operator  $L$ . Now a simple second-order derivative operator has eigenfunctions that are sines and cosines. So here the eigenfunctions could be sines or cosines of  $y$ . But the eigenfunctions must satisfy the above boundary conditions for  $u$  too. And these boundary conditions better be homogenous! (I will tell you in the next section what to do if the boundary conditions for  $u$  at  $y = 0$  and  $y = \ell$  are not homogeneous.) Fortunately, the ones above are homogeneous; there are no terms independent of  $u$ . So we can proceed. The cosines of  $y$  are out: cosines are 1 at  $y = 0$ , not 0. The sines are always 0 at zero, so that is OK. But they must also be 0 at  $y = \ell$ , and that only happens for

$$\begin{aligned} Y_1 &= \sin(\pi y/\ell) & Y_2 &= \sin(2\pi y/\ell) & Y_3 &= \sin(3\pi y/\ell) & \dots \\ \lambda_1 &= -\nu\pi^2/\ell^2 & \lambda_2 &= -\nu 2^2\pi^2/\ell^2 & \lambda_3 &= -\nu 3^2\pi^2/\ell^2 & \dots \end{aligned}$$

You find the eigenvalues by simply computing  $LY_i$  for  $i = 1, 2, 3, \dots$ . That also verifies that the  $Y_i$  are really eigenfunctions like I told you.

(Note that  $\sin(-\pi x/\ell) = -\sin(\pi x/\ell)$ , so that is not an additional independent eigenfunction. That is just like  $-\vec{e}_1$  would not be an additional eigenvector in the previous section.)

Next you write everything in terms of these eigenfunctions:

$$\begin{aligned} u &= u_1 Y_1 + u_2 Y_2 + u_3 Y_3 + \dots \\ f &= f_1 Y_1 + f_2 Y_2 + f_3 Y_3 + \dots \\ g &= g_1 Y_1 + g_2 Y_2 + g_3 Y_3 + \dots \end{aligned}$$

I can leave out the primes of the previous section, because nobody ever writes the components of a function of  $y$ . Allow me to write the first of the expansions above out showing the arguments of the functions:

$$u = u_1(t)Y_1(y) + u_2(t)Y_2(y) + u_3(t)Y_3(y) + \dots$$

I do that so that you can see why the solution method we are using is called “separation of variables” (the one for a partial differential equation, not for an ordinary one.) Each term is separated in a function of  $t$  times a function of  $y$ .

Once again, you have to compute these coefficients  $f_1, f_2, \dots$  and  $g_1, g_2, \dots$ . But how do you do that? You can hardly invert an infinite “matrix”  $E = (Y_1, Y_2, Y_3, \dots)$  of eigenfunctions like in the previous section. Well, for an operator

like  $L$ , just a constant multiple of the second derivative, there is a trick: you can integrate to find them. In particular,

$$f_i = \frac{\int_{y=0}^{\ell} Y_i(y) f(t; y) \, dy}{\int_{y=0}^{\ell} Y_i^2(y) \, dy} \quad g_i = \frac{\int_{y=0}^{\ell} Y_i(y) g(y) \, dy}{\int_{y=0}^{\ell} Y_i^2(y) \, dy} \quad \text{for all } i = 1, 2, 3, \dots$$

If you are astonished by that, don't be. The second derivative operator is a real symmetric one, so in the vector case you would find the  $\vec{f}' = E^T \vec{f}$ . So you would find the  $f'_i$  as the dot product of the rows of  $E^T$ , the eigenvectors, times the given column vector  $\vec{f}$ . In the eigenfunction case, the dot-product summation becomes integration over  $y$ . And the bottom factors in the ratios above are just correction factors for the fact that I did not normalize the eigenfunctions in any way. You can see why the justification for the equations above is called the *orthogonality* property.

Much like in the previous section, the basis of eigenfunctions makes  $L$  diagonal, with the eigenvalues on the main diagonal. So the partial differential equation becomes a system of independent equations for the coefficients of  $u$ :

$$\begin{aligned} \dot{u}_1 &= \lambda_1 u_1 + f_1 & u_1(0) &= g_1 \\ \dot{u}_2 &= \lambda_2 u_2 + f_2 & u_2(0) &= g_2 \\ \dot{u}_3 &= \lambda_3 u_3 + f_3 & u_3(0) &= g_3 \\ &\vdots & &\vdots \end{aligned}$$

These equations are no more difficult to solve than for the case of ordinary differential equations.

Afterwards, you can find  $u$  at any time  $t$  and position  $y$  you want by summing:

$$u(t; y) = \sum_{i=1}^{\infty} u_i(t) Y_i(y)$$

Of course, you cannot normally sum infinitely many terms, even on a computer. You will need to instruct the computer to stop at some large value of  $i$ , call it  $I$ . The same holds in case you cannot do the earlier integrals for the  $f_i$  and  $g_i$  analytically; then you will need to do them numerically, up to some large  $I$ . And you may even have to solve the ordinary differential equations numerically. (Note that a first order linear equation can be reduced to an integral, so you would not need to use an ODE solver from your library.)

## 5 Some more details

If your advisor actually asks you to do the above thing, your problem might of course be more complicated than my one.

One problem would be if the boundary conditions on  $y$  are not homogeneous. For example, if the top wall of the duct in the previous section moved horizontally with a given speed  $U(t)$ , you would have the boundary conditions

$$y = 0: \quad u(t; 0) = 0 \quad y = \ell: \quad u(t; \ell) = U(t)$$

and the second one is not homogeneous. The trick then is to write  $u$  as something (anything) that satisfies the boundary conditions, and a remainder  $\tilde{u}$ . In this case, a good choice would be

$$u = U \frac{y}{\ell} + \tilde{u}$$

If you replace  $u$  everywhere in the PDE and its initial and boundary conditions by the expression above, you get a problem for  $\tilde{u}$ . That problem will have *homogeneous* boundary conditions, so you are back in business, now for solving for  $\tilde{u}$ .

The next thing is finding those eigenfunctions. If  $L$  is a constant times the second derivative, the eigenfunctions are sines and cosines. Then you look at the boundary conditions to figure out just which ones. But suppose you have something like

$$L \equiv a \frac{\partial^2}{\partial y^2} + b \frac{\partial}{\partial y} + c$$

What then? (Note that the coefficients  $a$ ,  $b$  and  $c$  cannot depend on  $t$ ; otherwise the usual method of separation of variables does not work. But they could and often do, depend on  $y$ )

In the simplest case that  $b = 0$  and  $a$  and  $c$  are constants, the eigenfunctions are still sines and cosines. The constant  $c$  will just change the eigenvalues. So that is relatively trivial.

In any other case, you will need to solve the basic eigenvalue problem  $LY = \lambda Y$ , an ordinary differential equation, symbolically. But if  $a$ ,  $b$  and/or  $c$  depend on  $y$ , I never taught you how to do that! Then you will need to search through mathematical handbooks. Look under Bessel functions, Gamma function, error function, orthogonal polynomials such as those of Legendre and Hermite, etcetera. Note that you often need to play around a bit with your equation to get it in a form that you can find in the tables. Or look a bit deeper; common conversions are often mentioned somewhere.

There is now also another problem. The orthogonality property no longer applies in the form used in the previous section. There is a theorem, called the ‘‘Sturm-Liouville’’ theorem, that says that you have to find a positive solution  $r$  to the differential equation

$$\frac{d}{dy} ar = br$$

Then you have to push this  $r$ , a function of  $y$ , inside each of the orthogonality

integrals in the previous section as an additional factor:

$$f_i = \frac{\int_{y=0}^{\ell} Y_i(y)f(t; y)r(y) dy}{\int_{y=0}^{\ell} Y_i^2(y)r(y) dy} \quad g_i = \frac{\int_{y=0}^{\ell} Y_i(y)g(y)r(y) dy}{\int_{y=0}^{\ell} Y_i^2(y)r(y) dy}$$

If you are in two spatial dimensions and time, you will have separate  $L_y$  and  $L_z$  operators. Taking  $L_y$  as the simpler operator, after you switch to the basis of eigenfunctions of  $L_y$ , the equations for the  $u_i$  will still contain both  $t$  and  $z$ . You will now need to find the eigenfunctions of the  $L_z$  operator. Note that you may be forced to include the  $L_y$  eigenvalue  $\lambda_i$  inside the definition of the  $L_z$  operator. For example, that happens in polar coordinates, (flow in a pipe), where “ $y$ ” is the angular coordinate and “ $z$ ” the radial one. The net description of  $u$  then involves terms of the form  $u_{ij}(t)Y_i(y)Z_{ij}(z)$  that must be summed over both  $i$  and  $j$ . And the orthogonality integrals become double integrals over both  $y$  and  $z$ . All a whole lot of work, but not fundamentally different from what I told you.

As far as I can think of right now, the above covers all there is to say about the method of separation of variables. Not extremely difficult, but it sure requires a graduate student with a lot of time to carefully get all the details right.

Let me finally warn you about some common mistakes. One mistake that I see a lot is where the student leaves out an eigenfunction with eigenvalue 0. You need *all* the eigenfunctions. Remember that say an eigenfunction 1 is indeed a function: it is the function that is 1 at every position  $y$ . Another mistake that I see a lot is that a student tries to treat  $f = 1$  or  $g = 1$  as a number. It is a *function*, and you still need to write it as  $f_1Y_1 + f_2Y_2 + \dots$  or  $g_1Y_1 + g_2Y_2 + \dots$ . And do the integrals. Then there are the boundary conditions. If the original problem has a boundary condition at some  $y$ -boundary that  $Au + B\partial u/\partial y = C$ , then you should subtract a function that satisfies that boundary condition as described above. And then you should discover that the remainder  $\tilde{u}$  satisfies the boundary condition  $A\tilde{u} + B\partial\tilde{u}/\partial y = 0$ . Your eigenfunctions better satisfies that *homogeneous* boundary condition too, or forget it. Don’t try to define an eigenfunction expansion for a time-like variable that has initial conditions. If you are tempted to do that, instead try a Laplace transform in time. That is another way to solve a lot of simple partial differential equations. For separation of variables as explained here, you really want boundary conditions for the eigenfunctions.

## References

- [1] P.V. O’Neil. *Advanced Engineering Mathematics*. Thomson-Engineering, 6th edition, 2007.