Analysis II Spring 2007

Homework Problems

Do not print out this page. Keep checking for changes. Complete assignment will normally be available the day after the last lecture whose material is included in the test (Saturday, normally).

01/17 W

1. p13, q31
2. p13, q32
3. p14, q48
4. p32, q66
5. p32, q69
6. p32, q87
7. p53, q32

01/24 W

1. p54, q47
2. p78, q54
3. p78, q62
4. p78, q64
5. p79, q70
6. p79, q72
7. p80, q96

02/31 W

1. p78, q46
2. p78, q60
3. p80, q84
4. p80, q102
5. p102, q32
6. p103, q44

02/07 W

1. p104, q63
2. p104, q62 (use Cartesian coordinates except in the actual integration)
3. p132, q39 (use Cartesian coordinates except in the actual integration)
4. p132, q42
5. p132, q44
6. p133, q56
7. p160, q37
8. p160, q38

02/14 W

1. p160, q43

2. p160, q44
3. p160, q47 (finish)
4. Find the scale factors h_r and h_θ of polar coordinates in two dimensions.
5. Use them to find the Laplacian in polar coordinates.
6. The PDE problem $\nabla^2 u = 0$ in the first quadrant, with boundary conditions $u(x, 0) = 0$, $u(0, y) = 1$ has two possible solutions:
   (a) $u = 0$ for $y < x$ and $u = 1$ for $y > x$ in Cartesian coordinates.
   (b) $u = 2\theta/\pi$ in polar coordinates.

   Show that both solutions satisfy the PDE and boundary conditions. So, which solution is correct, and why?
7. Consider the following steady heat conduction problem for a triangular plate of a material with unit heat conduction coefficient k:
   (a) the temperature is zero on the side $y = 0$ of the plate;
   (b) the heat flux coming out of the side $x = 1$ of the plate is zero;
   (c) the heat flux entering the third side $x = y$ is constant and equal to one.

   Write the partial differential equation and all three boundary conditions as mathematical equations for the temperature $T(x, y)$.
8. Which of the following possibilities is the correct temperature distribution for the previous problem: (a) $T = \sqrt{2}y$; (b) $T = 2\theta/\pi$; (c) $T = \sin\theta$? Explain why.
9. If steady heat conduction satisfies div grad$T = 0$, what does the divergence theorem tell us about Neumann boundary conditions? Explain this physically. Does the same apply to the unsteady heat equation?
10. Which of the following situations is described by the inhomogeneous Laplace equation (the Poisson equation):
    (a) the membrane of a drum when it is being played;
    (b) the membrane of a drum when the drum sticks and various other stuff are resting on top of it;
    (c) the membrane of a drum when you drop it.

    Explain why.

02/23 F

1. Show that the following solution satisfies both the heat equation $u_t = u_{xx}$ and the wave equation $u_{tt} = u_{xx}$ for $0 \le t \le 1$ and $0 \le x \le 2$:

   $$u = 0 \text{ for } x < t, \qquad u = 1 \text{ for } t < x < 1 + t, \qquad u = 0 \text{ for } 1 + t < x$$

   It is a valid solution to only one equation, however. Which one, and why?
2. Show that the following solution satisfies both the heat equation $u_t = u_{xx}$ and the wave equation $u_{tt} = u_{xx}$ for $0 \le t \le 2$ and $0 \le x \le 2$:

   $$u = 0 \text{ for } x < \tfrac{1}{2}t, \qquad u = 1 \text{ for } \tfrac{1}{2}t < x < 1 + \tfrac{1}{2}t, \qquad u = 0 \text{ for } 1 + \tfrac{1}{2}t < x$$

   It is a valid solution to neither equation, however. Why not?
3. Write the physical problem of heat conduction in a iron bar (iron heat conduction coefficient $k = 80.2$ W/m-K, density $\rho = 7,840$ kg/m^3, specific heat $C_p = 0.45$ kJ/kg-K) of length 2 m and diameter 10 cm in terms of the temperature $T(x, t)$ if one end is at 25°C and 20 W of heat is entering the bar at the other end. Also, at time zero the bar is at a uniform 25°C throughout. List PDE, BCs, and IC in the xt-plane with all numerical constants filled in, but do not try to solve. We will learn how to solve later, Hint: use units!

4. For acoustics in a pipe, the (small) axial air velocity u satisfies Newton's second law

$$\rho u_t = -p_x$$

where ρ is the mass per unit volume, which can be approximated as constant and equal to 1.225 kg/m^3 and p is the pressure in Pa. The pressure variations are caused by motion-induced fluid compression according to the "continuity equation"

$$\rho u_x + \frac{1}{a^2} p_t = 0$$

where a is the speed of sound, 340 m/s. Reduce these two equation to a wave equation for the pressure $p(x,t)$. Then write the problem for the gage pressure in a pipe of length 0.5 m if one end of the pipe is closed (zero fluid velocity at that end) and the other end of the pipe is open. Assume the initial pressure in the pipe is uniform at 5 Pa gage, and the air velocity is zero, but then at time zero we take our thumb of the open end and from then on the pressure at the open end is ambient at zero gage. List PDE, BCs, and IC in the xt-plane with all numerical constants filled in, but do not try to solve. We will learn how to solve later, Hint: use units!

5. Identify the wave propagation speed a in a string under a tension force of 30 N with a mass per unit length of 0.002 kg/m. Hint; use units!

6. Solve the one-way wave equation in the infinite domain $-\infty \leq x \leq \infty$ if the initial entropy $s(x,0)$ is a given function $f(x)$. Note that the flow velocity u simply translates the entropy distribution along the x-axis.

7. 2.19(b)

02/28 W

1. 2.19(d)
2. 2.19(h)
3. 2.20
4. 2.21 (a) and (b) in 3 spatial dimensions

03/14 W

1. Use the 2D PDE transformation formulae derived in class to transform the Laplace equation $u_{xx} + u_{yy} = 0$ to polar coordinates and see whether you get the correct result.
2. 2.24
3. 2.25
4. Get rid of the first order derivatives in the problem 2.25 by defining a new unknown $v = u/e^{\alpha\xi+\beta\eta+\gamma\theta}$ where ξ, η, θ are the rescaled coordinates and α, β, and γ are constants to be found.

03/21 W

1. 2.22 b, g (sketch the characteristics)
2. 2.27 a, b
3. 2.28 d (solve the PDE)
4. 2.28 f (solve the PDE)
5. 2.28 c (solve the PDE)
6. 2.28 k
7. 2.28 b Check that indeed $a' = c'$ and $b' = 0$ and find d'. We learned previously that constant coefficient elliptic problems can be reduced to the Laplacian by rotating the coordinates to x', y', and then stretching these to orthogonal coordinates x'', y''. But the coordinates ξ and η are not orthogonal (the lines of constant ξ and η are under an oblique angle.) Should not the transformation that reduces the PDE to the Laplace equation be unique??

8. Show that one of the below Laplace equation problems has a unique solution, $u = 0$, but the other has multiple solutions:

$$\text{PDE: } \nabla^2 u = 0 \qquad \text{BC: } u(0,y) = u_y(x,0) = u_y(x,1) = u(1,y) + u_x(1,y) = 0$$

$$\text{PDE: } \nabla^2 u = 0 \qquad \text{BC: } u(0,y) = u_y(x,0) = u_y(x,1) = u(1,y) - u_x(1,y) = 0$$

   Hint: to show uniqueness, use the notes. To show nonuniqueness of the other problem, guess a simple nonzero solution.

9. Solve the heat conduction problem in infinite two-dimensional space $-\infty < x < \infty$, $-\infty < y < \infty$ if heat is being added to the point at the origin at a unit rate.

10. Find the Green's function of the Poisson equation in infinite two-dimensional space. Comment on its behavior at large distances compared to the three-dimensional Green's function.

11. Solve the Poisson equation in infinite two-dimensional space. Rigor is not required.

03/23 F

1. Consider the Poisson or Laplace problem in a two-dimensional finite region Ω:

$$\nabla^2 u = f \text{ in } \Omega \qquad Au + B\frac{\partial u}{\partial n} = g \text{ in } \delta\Omega$$

   with A and B given constants and f and g given functions (and $f = 0$ in the Laplace case.)

   Suppose we define u outside the region Ω to be a chosen function u_{out} that satisfies the Laplace equation, like for example $u_{\text{out}} = 0$. Explain why the thus defined u in all of space is *not* the infinite space solution

$$u_{\text{inf}}(x,y) = \int\int_{\Omega} G(x-\xi, y-\eta) f(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta$$

   that you derived for the infinite space problem. After all, u_{inf} has the same Laplacian everywhere: $\nabla^2 u_{\text{inf}} = f = \nabla^2 u$ inside Ω and $\nabla^2 u_{\text{inf}} = 0 = \nabla^2 u_{\text{out}}$ outside Ω. Should not solutions to the infinite space Poisson equation be unique?

   The integral

$$u_{\text{inf}}(x,y) = \int\int_{\Omega} G(x-\xi, y-\eta) f(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta$$

   is not quite proper. For one, G is infinite at the point $(\xi,\eta) = (x,y)$ so we would be integrating infinity. Also, integrals must be done over finite regions, not infinite ones. Define $u_{\text{inf}}(x,y)$ more carefully to avoid these problems by using limits of integrals, sketching the region of integration.

2. Work out the more careful expression for $u_{\text{inf}}(x,y)$ to get a relation showing the difference between $u(x,y)$ and $u_{\text{inf}}(x,y)$. Hint: use Green's second identity. You may assume that at a large distance ρ from the point (x,y), u_{out} can be approximated as $c + d/\rho$ where c and d are constants.

3. Except for the constant, graphically explain the physical meaning of the effect of the boundary conditions, i.e. explain the difference between u and u_{inf} in physical terms.

03/28 W

1. Show that if $u(r,\theta)$ satisfies the Laplace equation $\nabla^2 u = 0$ *inside* the unit circle around the origin, then $u_{\text{out}}(r,\theta) = Au(1/r,\theta)$ satisfies the Laplace equation *outside* the unit circle. Hint: write $u_{\text{out}}(r,\theta) = Au(\rho,\theta)$ where $\rho = 1/r$ then work out $\nabla^2 u_{\text{out}}$ in terms of the ρ derivatives of $u(\rho,\theta)$ and show it is a multiple of the Laplacian of $u(\rho,\theta)$.

2. For the Dirichlet problem for the Laplace equation in a unit circle,

$$\nabla^2 u = 0 \text{ for } r \le 1, \qquad u(1,\theta) = g(\theta)$$

   choose A in Green's integral expression for u so that the unknown boundary heat flux disappears.

3. To get rid of the constant c, show that it is equal to

$$c = -\int g \frac{\mathrm{d}S}{2\pi}$$

by using the mean value theorem 3.1, proved in problem 3.1.

4. Show that Green's formula then reduces to the two-dimensional case of (3.5) in the book.
5. Work out the integral to obtain the familiar form of Poisson's integral formula in two dimensions, given in problem 3.37.

04/04 W

1. 3.38 Think of it as a problem with negative temperatures at one side of the circle and positive at the other side. The temperature at the center is asked.
2. 3.39 Hint: the solution inside the circle is easy to guess.
3. 3.41 Remember Gauss.
4. 4.20 A number T is rational if it can be written as the ratio of a pair of integers, eg $1.5 = 3/2 = 6/4 = 9/6 = \ldots$ It is irrational if it cannot, like $\sqrt{2}$. Near any rational number, irrational numbers can be found infinitely closely nearby, and vice versa. For nonzero solutions, try $u = \sin(n\pi x)\sin(n\pi t)$, which satisfies the wave equation and the boundary conditions at $x = 0$ and $x = 1$ and the initial condition at $t = 0$. See when it satisfies the end condition at $t = T$.

   This is the boundary value problem for the wave equation, and would be perfectly OK for if it would have been the Laplace equation. (For the Laplace equation, the $\sin(n\pi t)$ becomes $\sinh(n\pi t)$ and only a unique, zero, solution is possible.) The wave equation needs two initial conditions at $t = 0$, not one condition at $t = 0$ and one at $t = T$.
5. Suppose the initial conditions to the wave equation $u_{tt} = a^2 u_{xx}$ in infinite space $-\infty < x < \infty$ are nonzero only from $x = -1$ to $x = 2$. Identify the regions in the x,t-plane where u will be zero and where it will in general be nonzero.
6. Plot the solution to the wave equation in an infinite acoustic pipe,

   $$u_{tt} = u_{xx} \qquad -\infty < x < \infty,\ t \geq 0$$

   if the initial conditions are

   $$u(x,0) = F(x) \qquad u_t(x,0) = G(x)$$

   where $F(x) = 0$ for all x while $G(x) = 1$ for $-1 < x < 1$ and $G(x) = 0$ everywhere else. In particular, plot u versus x at times $t = 0$, $t = 0.25$, $t = 0.5$, $t = 1$, $t = 2$, and $t = \infty$.

   Hint: First plot the anti-derivative of $G(x)$, call it $H(x)$, against x. The integration constant is not important, you can take $H(x) = \int_{\xi=0}^{x} G(\xi)\mathrm{d}\xi$. For example, $H(2)$ will be one then. Write the D'Alembert solution in terms of function H, then graphically evaluate and sum the relevant parts.
7. 5.25
8. 5.27 (b)
9. Refer to problem 5.34. Find the general solution to $u_t + uu_x = 0$ using the method of characteristics. Write down both possible forms in which the solution can be written. ($C_1 = C_1(C_2)$ or $C_2 = C_2(C_1)$.)
10. Verify that both of the solutions v and w given in 5.34 are in each region of at least one of the two forms and identify what the arbitrary function in the general solution is then.
11. Neatly draw the characteristic lines of each of the solutions v and w as well as the shocks at $x = t/2$ and $x = 1 + t/2$ of v, and the single shock at $x = 1 + t/2$ of w. Put several characteristics in each region (including in $0 < x < t/2$ for v and $0 < x < t$ for w.) Draw up to $t = 3$, say.
12. Convert the PDE $u_t + uu_x = 0$ to a conservation form $u_t + f_x = 0$ and thus determine what functions $f(u)$ is. What is the conserved integral?

13. Verify that all three shocks satisfy the shock conservation law $v_s = (f_2 - f_1)/(u_2 - u_1)$ assuming $t \leq 2$. Here 1 is the point immediately to the left of the shock and 2 the point immediately to the right of it, and v_s is the shock velocity.
14. Show that one of the three shocks does not satisfy the "entropy condition" $f'_1 > v_s > f'_2$. Such a shock represents an unphysical "expansion shock," and the corresponding solution must be rejected. For physical solutions the velocity f'_1 of the characteristics before the shock must be greater than the shock velocity v_s for the characteristics to catch up with the shock. The shock velocity v_s must be greater than the velocity f'_2 of the characteristics behind the shock for the shock to be able to catch up with those characteristics.

04/11 W

1. Solve 7.27 using D'Alembert, graphically identifying all functions involved in terms of the given f and g.
2. Using the solution of the previous problem, and taking $\ell = 1$, $a = 1$, $f(x) = x$, and $g(x) = 1$, draw $u(x, 0.5)$ as a function of x. Note that the boundary conditions are now satisfied, though the initial condition did not.
3. Using the solution of the previous problems, find $u(0.5, 3)$.
4. Find the separation of variables eigenfunctions of 7.27. Make very sure you do not miss one.
5. Write the solution to $u(x, t)$ in terms of the derived eigenfunctions and plug into the PDE to find expressions for the coefficients. Be very careful to solve special cases separately to get them right.
6. Find the undetermined coefficients in the solution for u by setting $u(x, 0)$ and $u_t(x, 0)$ equal to the given functions $f(x)$ and $g(x)$. Assume again that $a = \ell = 1$, $f(x) = x$ and $g(x) = 1$. Hint: the coefficients corresponding to $g(x)$ can easily be guessed.
7. Make a computer plot of u versus x at times $t = 0$, $t = 0.25$, $t = 0.5$, $t = 0.75$, $t = 1$, and $t = 1.25$. Use whatever software you like. Note: the error is of order $1/3n$, if n is the eigenfunction number, so you want to sum at least about 15 terms for one percent error. Compare the solution at $t = 0.5$ with the one you got from D'Alembert, and the solution at $t = 0$ with the given initial condition. Since $u_t = 1$ initially, you would expect $u(x, 0.25) = u(x, 0) + 0.25$. Is it?
8. Replot summing only one, two, and three eigenfunctions and examine how the solution approaches the exact one if you sum more and more terms. Comment in particularly on the slope of the profiles at time $t = 0$ and $t = 1$ at the boundaries. What should the slope be at those times? What is it?

04/18 W

1. Refer to problem 7.19. Find a function $u_0(x, t)$ that satisfies the inhomogeneous boundary conditions.
2. Continuing the previous problem, define $v = u - u_0$. Find the PDE, BC and IC satisfied by v.
3. Find eigenfunctions in terms of which v may be written, and that satisfy the homogeneous boundary conditions.
4. Solve for v using separation of variables in terms of expressions in terms of the known functions $f(x)$, $g_0(t)$, and $g_1(t)$.
5. 7.26. Note: you can simplify the problem to a standard heat conduction one by defining $v = e^{-\alpha x - \beta t} u$ for suitable constants α and β. Alternatively, you can just solve it directly. Either way, the answer in the book is wrong.
6. 7.35. Minor error in the book's answer.