7.36

1 7.36, §1 Asked

Asked: Find the horizontal perturbation velocity in a supersonic flow above a membrane overlaying a compressible variable medium.

2 7.36, §2 PDE Model

- Domain $\bar{\Omega}$: $0 \leq x < \infty, 0 \leq y < \infty$
- Unknown horizontal perturbation velocity $u = u(x, y)$
- Hyperbolic
- Two homogeneous initial conditions
- One mixed boundary condition at $y = 0$ and a regularity constraint at $y = \infty$
- Constant $a = \tan \mu$, where μ is the Mach angle.

Try a Laplace transform. The physics and the fact that Laplace transforms like only initial conditions suggest that x is the one to be transformed. Variable x is our "time-like" coordinate.

3 7.36, §3 Transform

Transform the PDE:

$$u_{xx} = a^2 u_{yy} \quad \xrightarrow{\text{Table 6.3, \# 3}} \quad s^2 \hat{u} - \cancel{su(0,y)} - \cancel{u_x(0,y)} = a^2 \hat{u}_{yy}$$

Transform the BC:

$$u_y - pu = f(x) \quad \Longrightarrow \quad \hat{u}_y - p\hat{u} = \hat{f}(s)$$

4 7.36, §4 Solve

Solve the PDE, again effectively a constant coefficient ODE:

$$s^2\hat{u} = a^2\hat{u}_{yy}$$

$$s^2 = a^2k^2 \quad \Longrightarrow \quad k = \pm s/a$$

$$\hat{u} = Ae^{sy/a} + Be^{-sy/a}$$

Apply the BC at $y = \infty$:

$$A = 0$$

Apply the BC at $y = 0$:

$$\hat{u}_y - p\hat{u} = \hat{f} \quad \Longrightarrow \quad -\frac{s}{a}B - pB = \hat{f}$$

Solving for B and plugging it into the expression for $\hat{u}$ gives:

$$\hat{u} = -\frac{a\hat{f}}{s + ap}e^{-sy/a}$$

5 7.36, §5 Back

We need to find the original to

$$\hat{u} = -\frac{a}{s + ap}\hat{f}e^{-sy/a}$$

Looking in the tables:

$$\frac{1}{s + ap} \quad \xrightarrow{\text{Table 6.4, \# 3}} \quad e^{-apx}$$

The other factor is a shifted function f, restricted to the interval that its argument is positive:

$$e^{-sy/a}\hat{f} \quad \xrightarrow{\text{Table 6.3, \# 6}} \quad \bar{f}\left(x - \frac{y}{a}\right)$$

With the bar, I indicate that I only want the part of the function for which the argument is positive. This could be written instead as

$$f\left(x - \frac{y}{a}\right)H\left(x - \frac{y}{a}\right)$$

where the Heaviside step function $H(x) = 0$ if x is negative and 1 if it is positive.

Use convolution, Table 6.3, # 7. again to get the product.

$$u(x,y) = -\int_0^x a\bar{f}\left(\xi - \frac{y}{a}\right) e^{-ap(x-\xi)}\,\mathrm{d}\xi$$

This *must* be cleaned up. I do not want bars or step functions in my answer.

I can do that by restricting the range of integration to only those values for which $\bar{f}$ is nonzero. (Or H is nonzero, if you prefer) Two cases now exist:

$$u(x,y) = -\int_{y/a}^x af\left(\xi - \frac{y}{a}\right) e^{-ap(x-\xi)}\,\mathrm{d}\xi \qquad (x > \frac{y}{a})$$

$$u(x,y) = 0 \qquad (x < \frac{y}{a})$$

It is neater if the integration variable is the argument of f. So, define $\phi = \xi - y/a$ and convert:

$$u(x,y) = -\int_0^{x-y/a} af(\phi)\, e^{-apx+py+ap\phi}\,\mathrm{d}\phi \qquad (x > \frac{y}{a})$$

$$u(x,y) = 0 \qquad (x < \frac{y}{a})$$

This allows me to see which physical f values I actually integrate over when finding the flow at an arbitrary point:

6 7.36, §6 Alternate

An alternate solution procedure is to define a new unknown:

$$v \equiv u_y - pu$$

You must derive the problem for v:

The boundary condition is simply:

$$v(x,0) = f(x)$$

To get the PDE for v, use

$$\frac{\partial[PDE]}{\partial y} - p[PDE] \quad \Longrightarrow \quad v_{tt} = a^2 v_{xx}$$

Similarly, for the initial conditions:

$$\frac{\partial[ICs]}{\partial y} - p[ICs] \quad \Longrightarrow \quad v(0,y) = v_x(0,y) = 0$$

After finding v, I still need to find u from the definition of v:

$$v \equiv u_y - pu$$

Where do you get the integration constant??

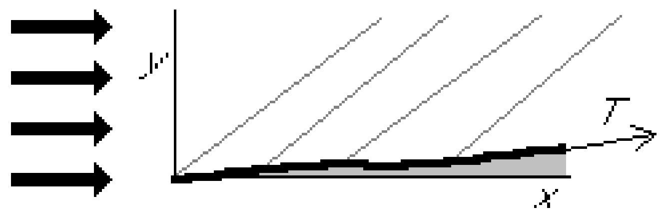

Figure 1: Supersonic flow over a membrane.

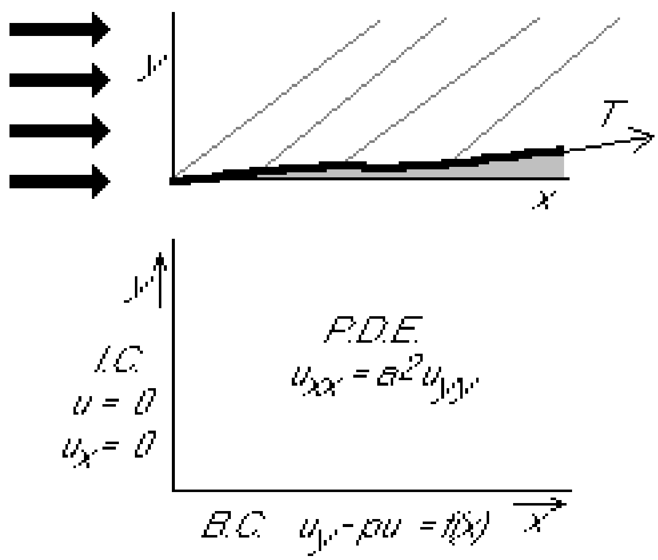

Figure 2: Supersonic flow over a membrane.

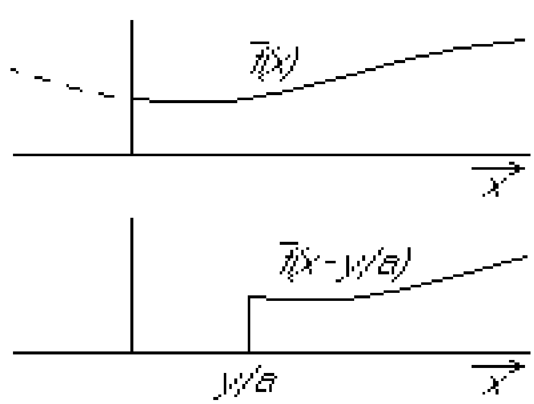

Figure 3: Function $\bar{f}$.

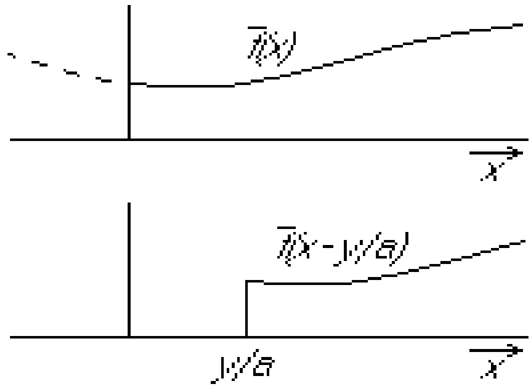

Figure 4: Function $\bar{f}$.

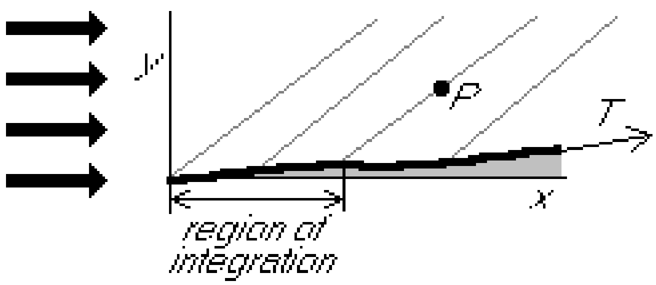

Figure 5: Supersonic flow over a membrane.

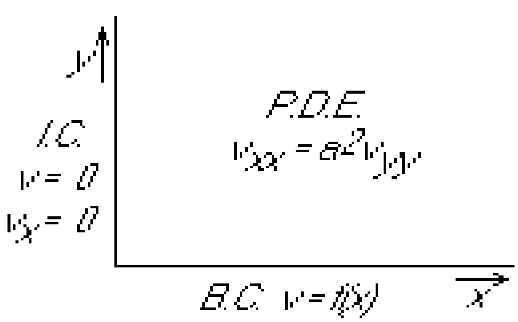

Figure 6: Problem for v.