2nd Order

The general n-dimensional second order quasilinear equations is:

$$
\begin{aligned}
&a_{11}u_{x_1x_1}+\\
&2a_{21}u_{x_2x_1}+a_{22}u_{x_2x_2}+\\
&2a_{31}u_{x_3x_1}+2a_{32}u_{x_3x_2}+2a_{33}u_{x_3x_3}\\
&+\ldots+d=0
\end{aligned}
$$

Its coefficients can be placed into a symmetric matrix A, just like those of a quadratic form can be.

Example:

$$au_{xx}+2bu_{yx}+cu_{yy}+d=0$$

The matrix A is here:

$$A=\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

In index notation, the n-dimensional equation can be written as:

$$\sum_i\sum_j a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}+d=0$$

where $a_{ij}=a_{ij}(x_1,x_2,\ldots,x_n,u,u_{x_1},u_{x_2},\ldots,u_{x_n})$ is a symmetric matrix and $d=d(x_1,x_2,\ldots,x_n,u,u_{x_1},u_{x_2},\ldots$

Classification is based on the eigenvalues of A:

- parabolic if any eigenvalues are zero; otherwise:
- elliptic if all eigenvalues are the same sign;
- hyperbolic if all eigenvalues except one are of the same sign;
- ultrahyperbolic, otherwise.

Exercise: Figure out whether that is consistent with what we defined for the two-dimensional case,

$$A=\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

- hyperbolic if $b^2-ac>0$.
- parabolic if $b^2-ac=0$.
- elliptic if $b^2-ac<0$.