2.24

1 2.24, §1 Asked

Given:

$$3u_{xx} - 2u_{xy} + 2u_{yy} - 2u_{yz} + 3u_{zz} + 12u_y - 8u_z = 0$$

Asked: Classify and put in canonical form.

2 2.24, §2 Solution

$$3u_{xx} - 2u_{xy} + 2u_{yy} - 2u_{yz} + 3u_{zz} + 12u_y - 8u_z = 0$$

Identify the matrix:

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

To find the new coordinates (transformation matrix), find the eigenvalues and eigenvectors of A:

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2(2-\lambda) - (3-\lambda) - (3-\lambda)$$

Hence $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 4$.

For $\lambda_1 = 1$,

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} / \sqrt{6}$$

For $\lambda_2 = 3$,

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} / \sqrt{2}$$

For $\lambda_3 = 4$,

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} / \sqrt{3}$$

The new equation is:

$$u_{\xi\xi} + 3u_{\eta\eta} + 4u_{\theta\theta} + 12u_y - 8u_z = 0$$

However, that still contains the old coordinates in the first order terms. Use the transformation formulae and total differentials to convert the first order derivatives:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \theta \end{pmatrix} \quad \begin{pmatrix} \xi \\ \eta \\ \theta \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$u_y = u_\xi \frac{2}{\sqrt{6}} - u_\theta \frac{1}{\sqrt{3}}$$

$$u_z = u_\xi \frac{1}{\sqrt{6}} - u_\eta \frac{1}{\sqrt{2}} + u_\theta \frac{1}{\sqrt{3}}$$

Hence in the rotated coordinate system, the PDE is:

$$u_{\xi\xi} + 3u_{\eta\eta} + 4u_{\theta\theta} + \frac{16}{\sqrt{6}} u_\xi + \frac{8}{\sqrt{2}} u_\eta - \frac{20}{\sqrt{3}} u_\theta = 0$$

This could be reduced further by stretching the coordinates. If

$$\xi = \bar{\xi} \quad \eta = \sqrt{3}\bar{\eta} \quad \theta = 2\bar{\theta}$$

then

$$u_{\bar{\xi}\bar{\xi}} + u_{\bar{\eta}\bar{\eta}} + u_{\bar{\theta}\bar{\theta}} + \frac{16}{\sqrt{6}} u_{\bar{\xi}} + \frac{8}{\sqrt{6}} u_{\bar{\eta}} - \frac{10}{\sqrt{3}} u_{\bar{\theta}} = 0$$

Note that tall that is left in the second order derivative terms is the *sign* of the eigenvalues. Now you know why we classify based on the sign of the eigenvalues!

You could further set $u = ve^{a\bar{\xi}+b\bar{\eta}+c\bar{\theta}}$ and choose a, b, and c to get rid of the first order derivatives. You need:

$$a = -\frac{8}{\sqrt{6}} \quad b = -\frac{4}{\sqrt{6}} \quad c = \frac{5}{\sqrt{3}}$$

Then

$$v_{\bar{\xi}\bar{\xi}} + v_{\bar{\eta}\bar{\eta}} + v_{\bar{\theta}\bar{\theta}} - \frac{65}{3} v = 0$$