2.27 (d)

1 2.27 (d), §1 Asked

Given:

$$e^y u_{xx} + 2e^x u_{xy} - e^{2x-y} u_{yy} = 0$$

Asked: Find the coordinates that reduce it to 2D canonical form.

2 2.27 (d), §2 Solution

$$e^y u_{xx} + 2e^x u_{xy} - e^{2x-y} u_{yy} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a} = (1 \pm \sqrt{2})e^{x-y}$$

$$e^y \,\mathrm{d}y = (1 \pm \sqrt{2})e^x \,\mathrm{d}x \quad \Longrightarrow \quad e^y = (1 \pm \sqrt{2})e^x + C$$

$$\xi = (1+\sqrt{2})e^x - e^y \qquad \eta = (1-\sqrt{2})e^x - e^y$$

The resulting P.D.E.:

$$b' = a\left(\xi_x\right)\left(\eta_x\right) + b\left(\xi_x\right)\left(\eta_y\right) + b\left(\xi_y\right)\left(\eta_x\right) + c\left(\xi_y\right)\left(\eta_y\right) = -4e^{2x+y}$$

$$d' = d + \left(a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy}\right)u_\xi + \left(a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy}\right)u_\eta$$
$$d' = \left[(1+\sqrt{2})e^{x+y} + e^{2x}\right]u_\xi + \left[(1-\sqrt{2})e^{x+y} + e^{2x}\right]u_\eta$$

Get rid of x and y:

$$e^x = \frac{1}{2\sqrt{2}}(\xi - \eta) \quad e^y = \frac{1-\sqrt{2}}{2\sqrt{2}}\xi - \frac{1+\sqrt{2}}{2\sqrt{2}}\eta$$

$$-(\xi - \eta)[(1-\sqrt{2})\xi - (1+\sqrt{2})\eta]u_{\xi\eta} = (1+\sqrt{2})\eta u_\xi + (1-\sqrt{2})\xi u_\eta$$