Characteristic Coordinates

Characteristic coordinates are coordinates so that a' and c' vanish:

$$2b'u_{\xi\eta} + d' = 0$$

Finding characteristic coordinates:

Vanishing of a' requires that ξ satisfies

$$a\left(\xi_x\right)^2 + 2b\left(\xi_x\right)\left(\xi_y\right) + c\left(\xi_y\right)^2 = 0$$

while for c' to vanish,

$$a\left(\eta_x\right)^2 + 2b\left(\eta_x\right)\left(\eta_y\right) + c\left(\eta_y\right)^2 = 0$$

Note that ξ and η must satisfy the exact same equation, but they must be different solutions to be valid independent coordinates.

To solve the equation for ξ (η goes the same way), divide by $\left(\xi_y\right)^2$:

$$a\left(-\frac{\xi_x}{\xi_y}\right)^2 - 2b\left(-\frac{\xi_x}{\xi_y}\right) + c = 0$$

and note that, from your calculus or thermo,

$$-\frac{\xi_x}{\xi_y} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{\xi \text{ is constant}}$$

So the lines of constant ξ should satisfy the ODE

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a}}$$

We can achieve this by taking ξ to be the integration constant in the solution of this ODE!

By taking the other sign for the square root, you can get a second independent coordinate η.

Bottom line, to get characteristic coordinates, solve the plus and minus sign ODE above, and equate the integration constants to ξ and η.

Notes:

1. Since integration constants are not unique, the characteristic coordinates are not. But the lines of constant ξ and η are unique, and are called *characteristic lines* or *characteristics.*

2. Elliptic equations do not have characteristics, and parabolic ones only a single family.

Application to the wave equation:

$$u_{tt} - a^2 u_{xx} = 0$$

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - a^2 = 0 \qquad \Longrightarrow \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \pm a$$

$$x = at + \xi \qquad x = -at + \eta$$

Since d' remains zero:

$$u_{\xi\eta} = 0 \qquad \Longrightarrow \qquad u_\eta = f(\eta) \qquad \Longrightarrow \qquad u = f_1(\xi) + f_2(\eta)$$

Hence the D'Alembert solution:

$$u = f_1(x - at) + f_2(x + at),$$

which is a right travelling 'wave' plus a left travelling one. Example 4.10 figures out what f_1 and f_2 are in terms of given initial displacement u and velocity u_t at the initial time.