Introduction

1 §1 Motivation

It is possible to simplify many P.D.E.s by using coordinate systems that are special to the problem:

- in unsteady pipe flows, use the lines along which sound waves propagate (characteristic lines) as coordinate lines to simplify the P.D.E.;
- in steady supersonic flows, use the Mach lines along which disturbances propagate (characteristic lines) as coordinate lines to simplify the P.D.E.;
- in problems with anisotropic properties, rotate your coordinate system along the principal or physical directions;
- ...

2 §2 Formulae

- Old independent coordinates $x_1, x_2, \ldots, x_n$
- New independent coordinates $\xi_1, \xi_2, \ldots, \xi_n$

(It may include time as one coordinate). Assume

$$\xi_1 = \xi_1(x_1, x_2, \ldots, x_n) \quad \xi_2 = \xi_2(x_1, x_2, \ldots, x_n) \quad \ldots \quad \xi_n = \xi_n(x_1, x_2, \ldots, x_n)$$

The original n-dimensional second order quasilinear equation:

$$\sum_i \sum_j a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + d = 0$$

where $a_{ij} = a_{ij}(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_n})$ is a symmetric matrix and $d = d(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots$

To convert:

- The new matrix coefficients are

$$\boxed{a'_{kl} = \sum_i \sum_j a_{ij} \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_l}{\partial x_j}}$$

- The new lower order terms are

$$\boxed{d' = d + \sum_k \left(\sum_i \sum_j a_{ij} \frac{\partial^2 \xi_k}{\partial x_i \partial x_j} \right) \frac{\partial u}{\partial \xi_k}}$$

- In the coefficients, express $x_1, x_2, \ldots, x_n$ in terms of $\xi_1, \xi_2, \ldots, \xi_n$ by solving the given relationships for them;

- In the coefficients, rewrite the first order derivatives:

$$\boxed{\frac{\partial u}{\partial x_i} = \sum_k \frac{\partial u}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_i}}$$

Matrix notation: Let B^T be the matrix of new coordinate derivatives:

$$\mathrm{d}\vec{\xi} = B^T \,\mathrm{d}\vec{x} \qquad \Longrightarrow \qquad b^T_{ki} = \frac{\partial \xi_k}{\partial x_i}$$

then

$$A' = B^T A B$$

3 §3 Rotation

For linear, constant coefficient equations, rotate the coordinate system to the principal axes of A:

$$\boxed{\vec{\xi} = B^T \vec{x} \qquad A' = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)}$$

provided that B consists of the orthonormal eigenvectors of A.

Our equation simplifies to

$$\lambda_1 u_{\xi_1\xi_1} + \lambda_2 u_{\xi_2\xi_2} + \ldots + \lambda_n u_{\xi_n\xi_n} + d' = 0$$

Notes:

1. If A is not constant, we must select a point P for which we determine the eigenvectors. The new A' will then only be diagonal at the point P.

2. If A is not constant, trying to set B at every point equal to the eigenvectors of the A at that point will not usually work since it requires n^2 equations to be satisfied by the n components of $\vec{\xi}$.

3. If we do not normalize the eigenvectors,

$$A' = \mathrm{diag}(|\vec{v}_1|^2\lambda_1, |\vec{v}_2|^2\lambda_2, \ldots, |\vec{v}_n|^2\lambda_n)$$