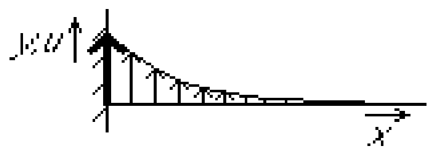
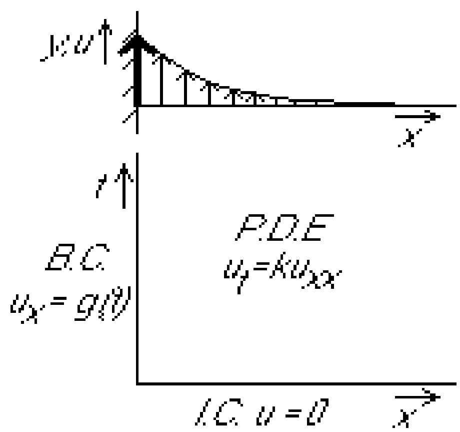
7.24

1 7.24, §1 Asked

Asked: Find the flow velocity in a viscous fluid being dragged along by an accelerating plate.



2 7.24, §2 PDE Model



- Semi-infinite domain $\bar{\Omega}$: $0 \leq x < \infty$
- Unknown vertical velocity $u = u(x,t)$
- Parabolic
- One homogeneous initial condition
- One Neumann boundary condition at $x = 0$ and a regularity constraint at $x = \infty$
- Constant kinematic viscosity κ

Try a Laplace transform in t.

3 7.24, §3 Transform

Transform the PDE:

$$u_t = \kappa u_{xx} \quad \xRightarrow{\text{Table 6.3, \# 3}} \quad s\hat{u} - \cancel{u(x,0)} = \kappa \hat{u}_{xx}$$

Transform the BC:

$$u_x = g(t) \quad \Longrightarrow \quad \hat{u}_x = \hat{g}(s)$$

4 7.24, §4 Solve

Solve the PDE:

$$s\hat{u} = \kappa \hat{u}_{xx}$$

This is a constant coefficient ODE in x, with s simply a parameter. Solve from the characteristic equation:

$$s = \kappa k^2 \quad \Longrightarrow \quad k = \pm\sqrt{s/\kappa}$$

$$\hat{u} = Ae^{\sqrt{s/\kappa}\,x} + Be^{-\sqrt{s/\kappa}\,x}$$

Apply the BC at $x = \infty$ that u must be regular there:

$$A = 0$$

Apply the given BC at $x = 0$:

$$\hat{u}_x = \hat{g}(s) \quad \Longrightarrow \quad -B\sqrt{\frac{s}{\kappa}} = \hat{g}$$

Solving for B and plugging it into the solution of the ODE, $\hat{u}$ has been found:

$$\hat{u} = -\sqrt{\frac{\kappa}{s}} e^{-\sqrt{s/\kappa}\,x} \hat{g}$$

5 7.24, §5 Back

We need to find the original function u corresponding to the transformed

$$\hat{u} = -\sqrt{\frac{\kappa}{s}} e^{-\sqrt{s/\kappa}\,x} \hat{g}$$

We do not really know what $\hat{g}$ is, just that it transforms back to g. However, we can find the other part of $\hat{u}$ in the tables.

$$-\sqrt{\frac{\kappa}{s}}e^{-\sqrt{s/\kappa}\,x} \quad \xRightarrow{\text{Table 6.4, \# 7}} \quad -\sqrt{\frac{\kappa}{\pi t}}e^{-x^2/4\kappa t}$$

How does $\hat{g}$ times this function transform back? The product of two functions, say $\hat{f}(s)\hat{g}(s)$, does *not* transform back to $f(t)g(t)$. The convolution theorem Table 6.3 # 7 is needed:

$$u(x,t) = -\int_0^t \sqrt{\frac{\kappa}{\pi(t-\tau)}}e^{-x^2/4\kappa(t-\tau)}g(\tau)\,\mathrm{d}\tau$$