Introduction

Description:

The *Laplace transform* pairs a function of a real coordinate, call it t, with $0 < t < \infty$, with a different function of a complex coordinate s:

$$u(t,\cdot) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \hat{u}(s,\cdot)$$

The pairing is designed to get rid of derivatives with respect to t in equations for the function u. This works as long as the coefficients do not depend on t (or at the very most are low degree powers of t.) The transformation is convenient since pairings can be looked up in tables.

Typical procedure:

Use tables to find the equations satisfied by $\hat{u}$ from these satisfied by u. Solve for $\hat{u}$ and look up the corresponding u in the tables.

About coordinate t:

In many cases, t is physically time, since time is most likely to satisfy the constraints $0 < t < \infty$ and coefficients independent of t. Also, the Laplace transform likes initial conditions at $t = 0$, not boundary conditions at both $t = 0$ and $t = \infty$.

Table 6.3: Properties of the Laplace transform:

$u(t)$	$\hat{u}(s)$
0. $\dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{-st}\hat{u}(s)\,\mathrm{d}s$	$\displaystyle\int_0^\infty u(t)e^{-st}\,\mathrm{d}t$
1. $C_1u_1(t)+C_2u_2(t)$	$C_1\hat{u}_1(s)+C_2\hat{u}_2(s)$
2. $u(at)$	$a^{-1}\hat{u}(s/a)$
3. $\dfrac{\partial^n u}{\partial t^n}(t)$	$s^n\hat{u}(s)-s^{n-1}u(0)-\ldots-\dfrac{\partial^{n-1}u}{\partial t^{n-1}}(0)$
4. $t^nu(t)$	$(-1)^n\dfrac{\partial^n\hat{u}}{\partial s^n}$
5. $e^{ct}u(t)$	$\hat{u}(s-c)$
6. $\bar{u}(t-b)\equiv H(t-b)u(t-b) = \begin{cases} u(t-b) & (t-b>0) \\ 0 & (t-b<0)\end{cases}$	$e^{-bs}\hat{u}(s)$
7. $(f*g)(t)\equiv\displaystyle\int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau$	$\hat{f}(s)\hat{g}(s)$

Here $a>0$, $b>0$, c are constants, and n is a natural number.

Table 6.4: Laplace transform pairs:

$u(t)$	$\hat{u}(s)$
1. 1	$\frac{1}{s}$
2. t^n	$\frac{n!}{s^{n+1}}$
3. e^{bt}	$\frac{1}{s-b}$
4. $\sin(at)$	$\frac{a}{s^2+a^2}$
5. $\cos(at)$	$\frac{s}{s^2+a^2}$
6. $\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$
7. $\frac{1}{\sqrt{\pi t}}e^{-k^2/(4t)}$	$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}}$
8. $\frac{k}{\sqrt{4\pi t^3}}e^{-k^2/(4t)}$	$e^{-k\sqrt{s}}$
9. $\operatorname{erfc}\left(\frac{k}{2\sqrt{t}}\right)$	$\frac{1}{s}e^{-k\sqrt{s}}$

Here $k > 0$, a and b are constants, n is a natural number, and

$$\operatorname{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-\xi^2}\,\mathrm{d}\xi$$