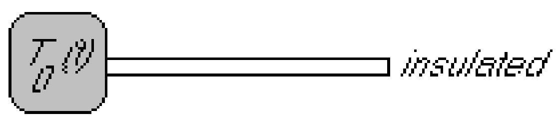
Introduction

1 §1 Examples

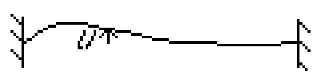
Partial differential equations:

- Standard examples:
  - Steady heat conduction in a plate:

  - Unsteady heat conduction in a bar:



  - Vibrations of a string:

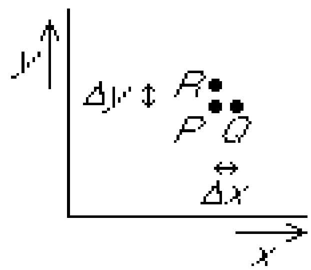


- Fluid mechanics;
- Heat transfer;
- Solid mechanics;
- Dynamics;
- Electro-magnetodynamics;
- Geometry;
- Optics;
- ...

2 §2 Notations

- Ordinary differential equations: one independent variable
- Partial differential equations: more independent variables
- Partial derivative:

$$u_{xP} \equiv \left(\frac{\partial u}{\partial x}\right)_P = \lim_{\Delta x \to 0} \frac{u_Q - u_P}{\Delta x} \qquad u_{yP} \equiv \left(\frac{\partial u}{\partial y}\right)_P = \lim_{\Delta y \to 0} \frac{u_R - u_P}{\Delta y}$$

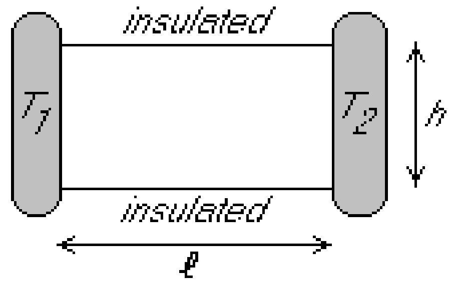


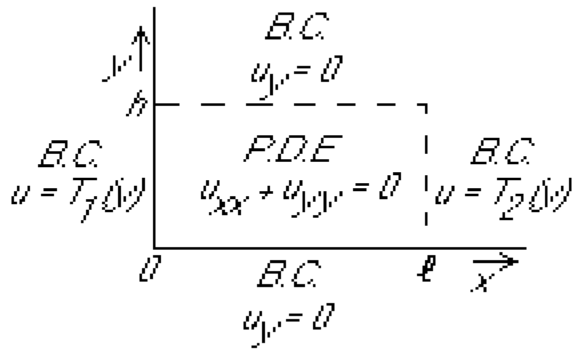
- Order: order of the highest derivative
- Degree: highest degree of the dependent variable
- Linear: first degree
- Domain Ω: the spatial region, i.e.
  - Plate (rectangle in the x, y-plane
  - Bar (line segment $0 < x < \ell$)
  - String (line segment $0 < x < \ell$)
- Boundary $\delta\Omega$: the edges of the domain, i.e.
  - Perimeter of the plate
  - Ends of the bar
  - End points of the string

3 §3 Standard Examples

You must know by heart:

- *The Laplace equation.* Steady heat conduction in a plate:

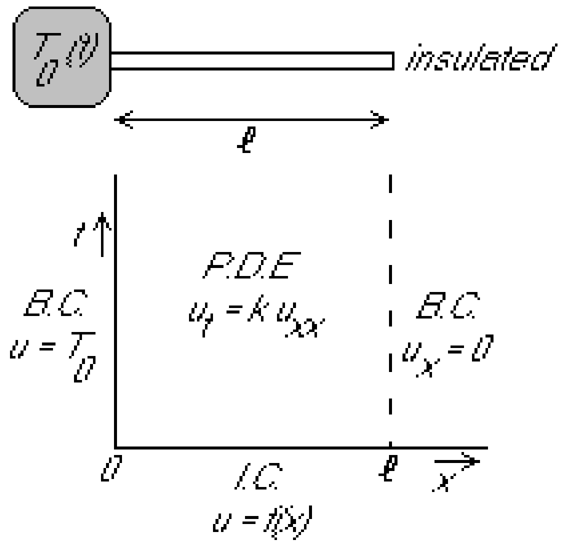




Also describes ideal flows, unidirectional flows, membranes, electro and magnetostatics, complex functions, ...
In any number of dimensions: $\nabla^2 u = 0$.
Properties:

  - Smooth solutions.
  - Boundary-value problems.
  - Maximum property.
  - Unlimited region of influence.
  - A simple example of an *elliptic* equation.

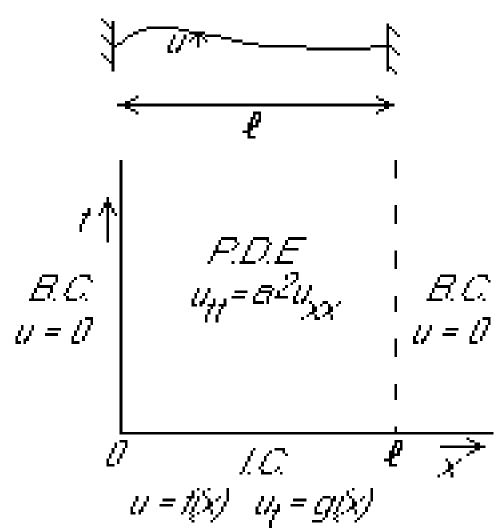
- *The heat equation.* Unsteady heat conduction in a bar:



Also describes unsteady unidirectional flow, ...
In any number of dimensions $u_t = k\nabla^2 u$.

  - Smooth solutions.
  - Initial/boundary value problems.
  - Maximum property.
  - Unlimited region of influence in space.
  - A simple example of a *parabolic* equation.

- Vibrations of a string: the wave equation:



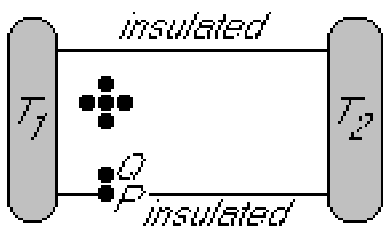
Also describes acoustics in a pipe, steady supersonic flow, water waves, optics, ...
In any number of dimensions $u_{tt} = a^2\nabla^2 u$.

- Propagating waves.
- Propagates singularities.
- Initial/boundary value problems.
- Energy conservation.
- Finite propagation speed.
- A simple example of a *hyperbolic* equation.

4 §4 Boundary Conditions

Boundary condition types:

- Dirichlet: u is given on the boundary



- Neumann: $\partial u/\partial n$ is given on the boundary

$$\frac{\partial u}{\partial n} = \vec{n} \cdot \nabla u$$

- Mixed: a combination of u and $\partial u/\partial n$ is given on the boundary
- ...

5 §5 Properly Posedness

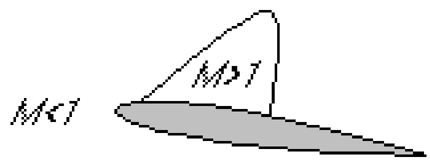
A P.D.E. problem is properly posed if

- a solution exists;
- it is unique
- small changes in the conditions produce correspondingly small changes in the solution.

Introduction

Classification groups P.D.E. with similar properties together.

Example: inviscid fluid flow past a wing cross-section:



Subsonic region $M < 1$: smooth solutions, unlimited region of dependence, iterative solution. These features are indicative of an elliptic PDE.

Supersonic region $M > 1$: singularities, characteristic lines, spatial marching solution. Typical for a hyperbolic PDE.

One set of PDEs that has a unambiguous classification are 2D second order quasilinear equations:

$$au_{xx} + 2bu_{xy} + cu_{yy} + d = 0$$

where $a = a(x, y, u, u_x, u_y)$, $b = b(x, y, u, u_x, u_y)$, $c = c(x, y, u, u_x, u_y)$, and $d = d(x, y, u, u_x, u_y)$.

The classification for these equations is:

- $b^2 - ac > 0$: hyperbolic
- $b^2 - ac = 0$: parabolic
- $b^2 - ac < 0$: elliptic

2.19 (e)

1 2.19 (e), §1 Asked

Classify:

$$yu_{xx} - 2u_{xy} + e^x u_{yy} + u = 3$$

2 2.19 (e), §2 Solution

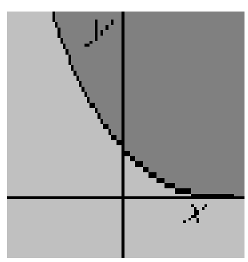
$$yu_{xx} - 2u_{xy} + e^x u_{yy} + u = 3$$

$$a = y, b = -1, c = e^x \quad \Longrightarrow \quad b^2 - ac = 1 - ye^x$$

Parabolic for

$$1 - ye^x = 0 \quad \Longrightarrow \quad y = e^{-x},$$

elliptic if y is greater than this, hyperbolic if y is less.



2nd Order

The general n-dimensional second order quasilinear equations is:

$$
\begin{aligned}
&a_{11}u_{x_1x_1}+ \\
&2a_{21}u_{x_2x_1} + a_{22}u_{x_2x_2}+ \\
&2a_{31}u_{x_3x_1} + 2a_{32}u_{x_3x_2} + 2a_{33}u_{x_3x_3} \\
&+ \ldots + d = 0
\end{aligned}
$$

Its coefficients can be placed into a symmetric matrix A, just like those of a quadratic form can be.

Example:

$$au_{xx} + 2bu_{yx} + cu_{yy} + d = 0$$

The matrix A is here:

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

In index notation, the n-dimensional equation can be written as:

$$\sum_i \sum_j a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + d = 0$$

where $a_{ij} = a_{ij}(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_n})$ is a symmetric matrix and $d = d(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots$

Classification is based on the eigenvalues of A:

- parabolic if any eigenvalues are zero; otherwise:
- elliptic if all eigenvalues are the same sign;
- hyperbolic if all eigenvalues except one are of the same sign;
- ultrahyperbolic, otherwise.

Exercise: Figure out whether that is consistent with what we defined for the two-dimensional case,

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

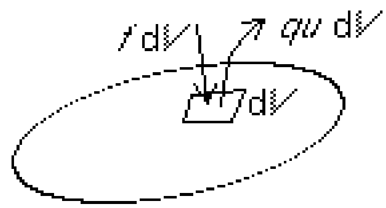
- hyperbolic if $b^2 - ac > 0$.
- parabolic if $b^2 - ac = 0$.
- elliptic if $b^2 - ac < 0$.

2.21 (b)

1 2.21 (b), §1 Asked

Classify:

$$u_t - \nabla \cdot (p\nabla u) + qu = f$$



2 2.21 (b), §2 Solution

$$u_t - \nabla \cdot (p\nabla u) + qu = f$$

Written out:

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\nabla u \equiv \text{grad } u = \hat{\imath}u_x + \hat{\jmath}u_y + \hat{k}u_z$$

$$\nabla \cdot \vec{v} \equiv \text{div } \vec{v} = v_{1x} + v_{2y} + v_{3z}$$

$$u_t - (pu_x)_x - (pu_y)_y - (pu_z)_z + qu = f$$

Highest derivatives of u:

$$-pu_{xx} - pu_{yy} - pu_{zz} + \ldots = f$$

Coefficient matrix A

$$A = \begin{pmatrix} -p & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = -p$ and $\lambda_4 = 0$.

Introduction

1 §1 Motivation

It is possible to simplify many P.D.E.s by using coordinate systems that are special to the problem:

- in unsteady pipe flows, use the lines along which sound waves propagate (characteristic lines) as coordinate lines to simplify the P.D.E.;
- in steady supersonic flows, use the Mach lines along which disturbances propagate (characteristic lines) as coordinate lines to simplify the P.D.E.;
- in problems with anisotropic properties, rotate your coordinate system along the principal or physical directions;
- ...

2 §2 Formulae

- Old independent coordinates $x_1, x_2, \ldots, x_n$
- New independent coordinates $\xi_1, \xi_2, \ldots, \xi_n$

(It may include time as one coordinate). Assume

$$\xi_1 = \xi_1(x_1, x_2, \ldots, x_n) \quad \xi_2 = \xi_2(x_1, x_2, \ldots, x_n) \quad \ldots \quad \xi_n = \xi_n(x_1, x_2, \ldots, x_n)$$

The original n-dimensional second order quasilinear equation:

$$\sum_i \sum_j a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + d = 0$$

where $a_{ij} = a_{ij}(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_n})$ is a symmetric matrix and $d = d(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots$

To convert:

- The new matrix coefficients are

$$\boxed{a'_{kl} = \sum_i \sum_j a_{ij} \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_l}{\partial x_j}}$$

- The new lower order terms are

$$\boxed{d' = d + \sum_k \left(\sum_i \sum_j a_{ij} \frac{\partial^2 \xi_k}{\partial x_i \partial x_j} \right) \frac{\partial u}{\partial \xi_k}}$$

- In the coefficients, express $x_1, x_2, \ldots, x_n$ in terms of $\xi_1, \xi_2, \ldots, \xi_n$ by solving the given relationships for them;
- In the coefficients, rewrite the first order derivatives:

$$\boxed{\frac{\partial u}{\partial x_i} = \sum_k \frac{\partial u}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_i}}$$

Matrix notation: Let B^T be the matrix of new coordinate derivatives:

$$\mathrm{d}\vec{\xi} = B^T \, \mathrm{d}\vec{x} \qquad \Longrightarrow \qquad b^T_{ki} = \frac{\partial \xi_k}{\partial x_i}$$

then

$$A' = B^T A B$$

3 §3 Rotation

For linear, constant coefficient equations, rotate the coordinate system to the principal axes of A:

$$\boxed{\vec{\xi} = B^T \vec{x} \qquad A' = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)}$$

provided that B consists of the orthonormal eigenvectors of A.

Our equation simplifies to

$$\lambda_1 u_{\xi_1\xi_1} + \lambda_2 u_{\xi_2\xi_2} + \ldots + \lambda_n u_{\xi_n\xi_n} + d' = 0$$

Notes:

1. If A is not constant, we must select a point P for which we determine the eigenvectors. The new A' will then only be diagonal at the point P.
2. If A is not constant, trying to set B at every point equal to the eigenvectors of the A at that point will not usually work since it requires n^2 equations to be satisfied by the n components of $\vec{\xi}$.
3. If we do not normalize the eigenvectors,

$$A' = \mathrm{diag}(|\vec{v}_1|^2 \lambda_1, |\vec{v}_2|^2 \lambda_2, \ldots, |\vec{v}_n|^2 \lambda_n)$$

2.24

1 2.24, §1 Asked

Given:

$$3u_{xx} - 2u_{xy} + 2u_{yy} - 2u_{yz} + 3u_{zz} + 12u_y - 8u_z = 0$$

Asked: Classify and put in canonical form.

2 2.24, §2 Solution

$$3u_{xx} - 2u_{xy} + 2u_{yy} - 2u_{yz} + 3u_{zz} + 12u_y - 8u_z = 0$$

Identify the matrix:

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

To find the new coordinates (transformation matrix), find the eigenvalues and eigenvectors of A:

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2(2-\lambda) - (3-\lambda) - (3-\lambda)$$

Hence $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 4$.

For $\lambda_1 = 1$,

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} / \sqrt{6}$$

For $\lambda_2 = 3$,

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} / \sqrt{2}$$

For $\lambda_3 = 4$,

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} / \sqrt{3}$$

The new equation is:

$$u_{\xi\xi} + 3u_{\eta\eta} + 4u_{\theta\theta} + 12u_y - 8u_z = 0$$

However, that still contains the old coordinates in the first order terms. Use the transformation formulae and total differentials to convert the first order derivatives:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \theta \end{pmatrix} \quad \begin{pmatrix} \xi \\ \eta \\ \theta \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$u_y = u_\xi \frac{2}{\sqrt{6}} - u_\theta \frac{1}{\sqrt{3}}$$

$$u_z = u_\xi \frac{1}{\sqrt{6}} - u_\eta \frac{1}{\sqrt{2}} + u_\theta \frac{1}{\sqrt{3}}$$

Hence in the rotated coordinate system, the PDE is:

$$u_{\xi\xi} + 3u_{\eta\eta} + 4u_{\theta\theta} + \frac{16}{\sqrt{6}} u_\xi + \frac{8}{\sqrt{2}} u_\eta - \frac{20}{\sqrt{3}} u_\theta = 0$$

This could be reduced further by stretching the coordinates. If

$$\xi = \bar{\xi} \quad \eta = \sqrt{3}\bar{\eta} \quad \theta = 2\bar{\theta}$$

then

$$u_{\bar{\xi}\bar{\xi}} + u_{\bar{\eta}\bar{\eta}} + u_{\bar{\theta}\bar{\theta}} + \frac{16}{\sqrt{6}} u_{\bar{\xi}} + \frac{8}{\sqrt{6}} u_{\bar{\eta}} - \frac{10}{\sqrt{3}} u_{\bar{\theta}} = 0$$

Note that tall that is left in the second order derivative terms is the *sign* of the eigenvalues. Now you know why we classify based on the sign of the eigenvalues!

You could further set $u = ve^{a\xi+b\eta+c\theta}$ and choose a, b, and c to get rid of the first order derivatives. You need:

$$a = -\frac{8}{\sqrt{6}} \quad b = -\frac{4}{\sqrt{6}} \quad c = \frac{5}{\sqrt{3}}$$

Then

$$v_{\bar{\xi}\bar{\xi}} + v_{\bar{\eta}\bar{\eta}} + v_{\bar{\theta}\bar{\theta}} - \frac{65}{3} v = 0$$

2D Coordinate Changes

More powerful simplifications are possible in 2D.

In the initial independent coordinates x, y:

$$au_{xx} + 2bu_{xy} + cu_{yy} + d = 0$$

In the new independent coordinates ξ, η

$$a'u_{\xi\xi} + 2b'u_{\xi\eta} + c'u_{\eta\eta} + d' = 0$$

The new coefficients may be found by writing out the transformation formulae from the introduction for the two-dimensional case, and are:

$$\begin{aligned}
a' &= a\left(\xi_x\right)^2 + 2b\left(\xi_x\right)\left(\xi_y\right) + c\left(\xi_y\right)^2 \\
b' &= a\left(\xi_x\right)\left(\eta_x\right) + b\left(\xi_x\right)\left(\eta_y\right) + b\left(\xi_y\right)\left(\eta_x\right) + c\left(\xi_y\right)\left(\eta_y\right) \\
c' &= a\left(\eta_x\right)^2 + 2b\left(\eta_y\right)\left(\eta_y\right) + c\left(\eta_y\right)^2 \\
d' &= d + \left(a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy}\right)u_\xi + \left(a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy}\right)u_\eta
\end{aligned}$$

The trick now is to demand that a', b' and c' are as simple as possible and from that compute what the required ξ and η are.

Characteristic Coordinates

Characteristic coordinates are coordinates so that a' and c' vanish:

$$2b' u_{\xi\eta} + d' = 0$$

Finding characteristic coordinates:

Vanishing of a' requires that ξ satisfies

$$a\left(\xi_x\right)^2 + 2b\left(\xi_x\right)\left(\xi_y\right) + c\left(\xi_y\right)^2 = 0$$

while for c' to vanish,

$$a\left(\eta_x\right)^2 + 2b\left(\eta_x\right)\left(\eta_y\right) + c\left(\eta_y\right)^2 = 0$$

Note that ξ and η must satisfy the exact same equation, but they must be different solutions to be valid independent coordinates.

To solve the equation for ξ (η goes the same way), divide by $\left(\xi_y\right)^2$:

$$a\left(-\frac{\xi_x}{\xi_y}\right)^2 - 2b\left(-\frac{\xi_x}{\xi_y}\right) + c = 0$$

and note that, from your calculus or thermo,

$$-\frac{\xi_x}{\xi_y} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{\xi \text{ is constant}}$$

So the lines of constant ξ should satisfy the ODE

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a}}$$

We can achieve this by taking ξ to be the integration constant in the solution of this ODE!

By taking the other sign for the square root, you can get a second independent coordinate η.

Bottom line, to get characteristic coordinates, solve the plus and minus sign ODEs above, and equate the integration constants to ξ and η.

Notes:

1. Since integration constants are not unique, the characteristic coordinates are not. But the lines of constant ξ and η are unique, and are called *characteristic lines* or *characteristics.*

2. Elliptic equations do not have characteristics, and parabolic ones only a single family.

Application to the wave equation:

$$u_{tt} - a^2 u_{xx} = 0$$

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - a^2 = 0 \qquad \Longrightarrow \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \pm a$$

$$x = at + \xi \qquad x = -at + \eta$$

Since d' remains zero:

$$u_{\xi\eta} = 0 \qquad \Longrightarrow \qquad u_\eta = f(\eta) \qquad \Longrightarrow \qquad u = f_1(\xi) + f_2(\eta)$$

Hence the D'Alembert solution:

$$u = f_1(x - at) + f_2(x + at),$$

which is a right travelling 'wave' plus a left travelling one. Example 4.10 figures out what f_1 and f_2 are in terms of given initial displacement u and velocity u_t at the initial time.

2.22 (d)

1 2.22 (d), §1 Asked

Given:

$$u_{xx} + y u_{yy} = 0$$

Asked: Find the characteristics.

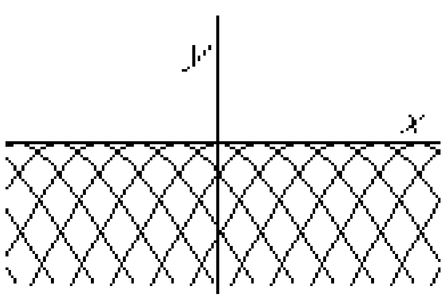
2 2.22 (d), §2 Solution

$$u_{xx} + y u_{yy} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \pm\sqrt{-y}$$

$$\frac{\mathrm{d} - y}{\sqrt{-y}} = \pm\,\mathrm{d}x \quad \Longrightarrow \quad 2\sqrt{-y} = \pm(x - C)$$

$$y = -\tfrac{1}{4}(x - C)^2$$



2.27 (d)

1 2.27 (d), §1 Asked

Given:

$$e^y u_{xx} + 2e^x u_{xy} - e^{2x-y} u_{yy} = 0$$

Asked: Find the coordinates that reduce it to 2D canonical form.

2 2.27 (d), §2 Solution

$$e^y u_{xx} + 2e^x u_{xy} - e^{2x-y} u_{yy} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a} = (1 \pm \sqrt{2})e^{x-y}$$

$$e^y \,\mathrm{d}y = (1 \pm \sqrt{2})e^x \,\mathrm{d}x \quad \Longrightarrow \quad e^y = (1 \pm \sqrt{2})e^x + C$$

$$\xi = (1 + \sqrt{2})e^x - e^y \qquad \eta = (1 - \sqrt{2})e^x - e^y$$

The resulting P.D.E.:

$$b' = a\,(\xi_x)\,(\eta_x) + b\,(\xi_x)\,(\eta_y) + b\,(\xi_y)\,(\eta_x) + c\,(\xi_y)\,(\eta_y) = -4e^{2x+y}$$

$$d' = d + (a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy})\,u_\xi + (a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy})\,u_\eta$$
$$d' = \left[(1 + \sqrt{2})e^{x+y} + e^{2x}\right] u_\xi + \left[(1 - \sqrt{2})e^{x+y} + e^{2x}\right] u_\eta$$

Get rid of x and y:

$$e^x = \frac{1}{2\sqrt{2}}(\xi - \eta) \quad e^y = \frac{1 - \sqrt{2}}{2\sqrt{2}}\xi - \frac{1 + \sqrt{2}}{2\sqrt{2}}\eta$$

$$-(\xi - \eta)[(1 - \sqrt{2})\xi - (1 + \sqrt{2})\eta]u_{\xi\eta} = (1 + \sqrt{2})\eta u_\xi + (1 - \sqrt{2})\xi u_\eta$$

2.28 (n)

1 2.28 (n), §1 Asked

Given:

$$\sin^2(x)u_{xx} + 2\cos(x)u_{xy} - u_{yy} = 0$$

Asked: Reduce it to 2D canonical form.

2 2.28 (n), §2 Solution

$$\sin^2(x)u_{xx} + 2\cos(x)u_{xy} - u_{yy} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \frac{\cos(x) \pm 1}{\sin^2(x)}$$

$$y = -\frac{1}{\sin(x)} \pm \mathrm{cotg}(x) + C$$

$$\xi = y + \frac{1}{\sin(x)} + \mathrm{cotg}(x) \quad \eta = y + \frac{1}{\sin(x)} - \mathrm{cotg}(x)$$

Parabolic Case

In the parabolic case,

$$a\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2b\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + c = 0$$

leads to only one root.

Take η as the single integration constant and ξ as anything else, say $\xi = x$.

Canonical form:

$$a' u_{\xi\xi} + d' = 0$$

2.28 (m)

1 2.28 (m), §1 Asked

Given:

$$xu_{xx} + 2\sqrt{xy}u_{xy} + yu_{yy} - u_y = 0$$

Asked: Reduce it to 2D canonical form.

2 2.28 (m), §2 Solution

$$xu_{xx} + 2\sqrt{xy}u_{xy} + yu_{yy} - u_y = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \sqrt{\frac{y}{x}}$$

Parabolic.

$$\frac{\mathrm{d}y}{\sqrt{y}} = \frac{\mathrm{d}x}{\sqrt{x}} \quad \Longrightarrow \quad \sqrt{y} = \sqrt{x} + C$$

$$\xi = x \qquad \eta = \sqrt{y} - \sqrt{x}$$

Elliptic Case

In the elliptic case,

$$a\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2b\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + c = 0$$

leads to complex roots.

Take ξ^* as the integration constant of either root. Then take $\xi = \Re(\xi^*)$ and $\eta = \Im(\xi^*)$.

Canonical form:

$$a' u_{\xi\xi} + a' u_{\eta\eta} + d' = 0$$

2.28 (l)

1 2.28 (l), §1 Asked

Given:

$$u_{xx} + (1+y)^2 u_{yy} = 0$$

Asked: Reduce it to 2D canonical form.

2 2.28 (l), §2 Solution

$$u_{xx} + (1+y)^2 u_{yy} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \pm \mathrm{i}(1+y)$$

Elliptic.

$$\frac{\mathrm{d}y}{1+y} = \mathrm{i}\,\mathrm{d}x \qquad \ln|1+y| - \mathrm{i}x = \xi^*$$

$$\xi = \ln|1+y| \qquad \eta = -x$$

Introduction

Wave equation:

$$u_{tt} = a^2 u_{xx}$$

Characteristics:

$$x + at = \xi \qquad x - at = \eta$$

General solution:

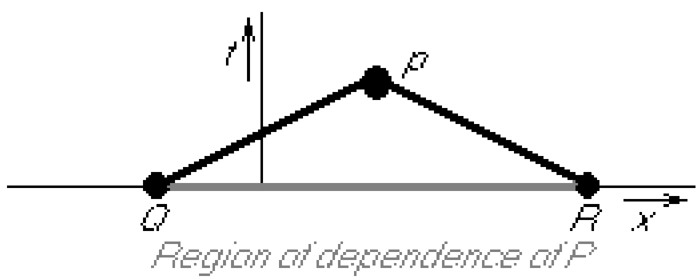
$$u(x,t) = f_1(x - at) + f_2(x + at)$$

Here $f_1(x - at)$ is a function that moves to the right with speed a; a 'right-going wave'. And $f_2(x - at)$ is a function that moves to the left with speed a; a 'left-going wave'.

D'Alembert solution:

Assume no boundaries, $(-\infty < x < \infty)$. Then we can solve for f_1 and f_2 in terms of the given initial string displacement $f(x) = u(x,0)$ and initial velocity $g(x) = u_t(x,0)$ to give:

$$u(x,t) = \frac{f(x - at) + f(x + at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)\,\mathrm{d}\xi$$



$$u_P = \frac{u_Q + u_R}{2} + \frac{1}{2a}\int_Q^R u_t\,\mathrm{d}\xi$$

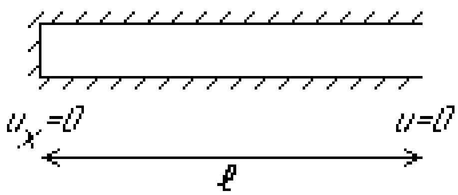
This is derived in example 4.10 in the book.

If x is restricted by finite boundaries, we must somehow extend the problem to doubly infinite x without boundaries. But our solution without boundaries should still satisfy the boundary conditions we are given. That is often possible by clever use of symmetry.

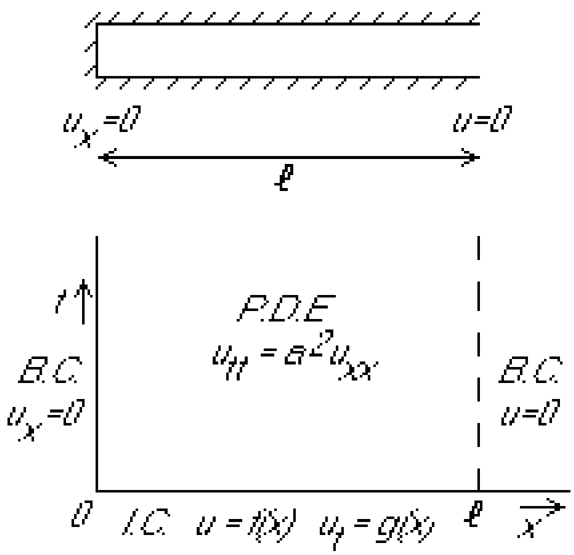
7.28

1 7.28, §1 Asked

Asked: Find the pressure for sound wave propagation in a tube with one end closed and one end open.



2 7.28, §2 P.D.E. Model



- Finite domain $\bar{\Omega}$: $0 \leq x < \ell$
- Unknown pressure $u = u(x,t)$
- Hyperbolic
- Two initial conditions

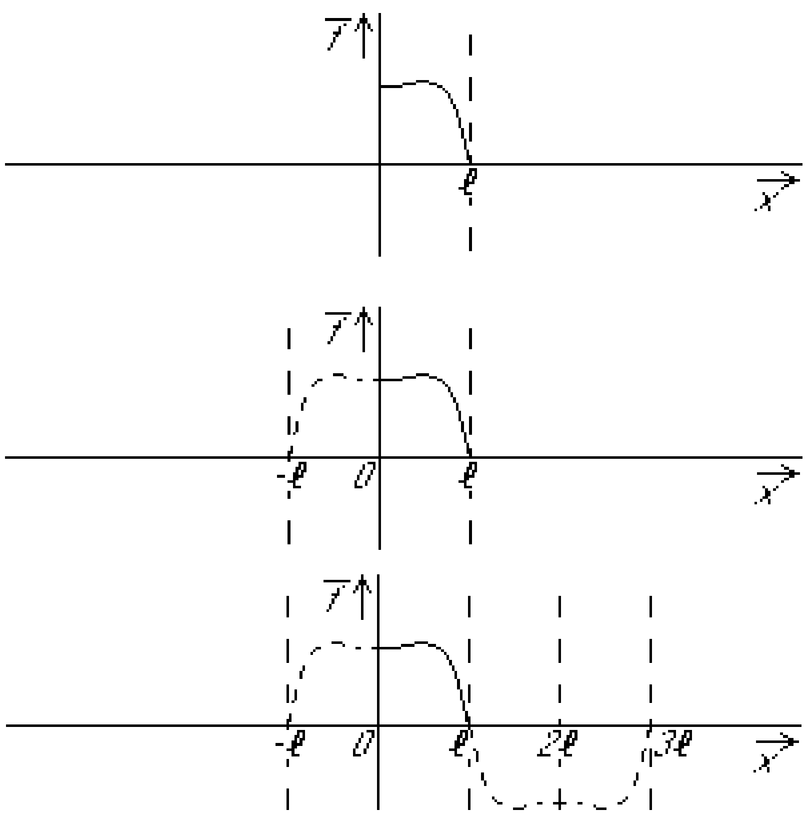
- One homogeneous Neumann boundary condition at $x = 0$ and a homogeneous Dirichlet condition at $x = \ell$.
- Constant speed of sound a and much smaller flow velocities.

Try D'Alembert

3 7.28, §3 Boundaries

- Get rid of the boundaries by imagining that the pipe extends from $-\infty < x < \infty$
- To do so, we must extend the initial conditions $f(x)$ and $g(x)$ to all x. Call the extended functions $\bar{f}(x)$ and $\bar{g}(x)$.
- The extended functions should make the given boundary conditions automatic.

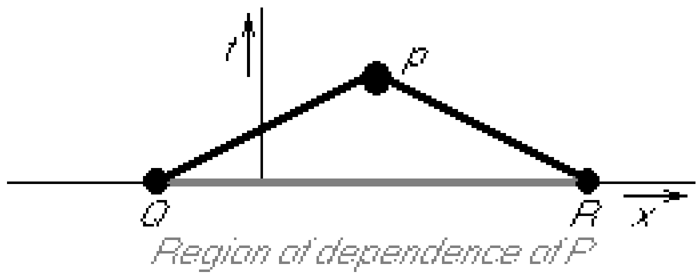
To make the boundary condition $u_x = 0$ at $x = 0$ automatic, create *symmetry* around $x = 0$ To make the boundary condition $u = 0$ at $x = \ell$ automatic, create *antisymmetry* (odd symmetry) around $x = \ell$.



4 7.28, §4 Solution

$$u(x,t) = \frac{\bar{f}(x-at)+\bar{f}(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \bar{g}(\xi)\,\mathrm{d}\xi$$

Probably pretty easy to evaluate.



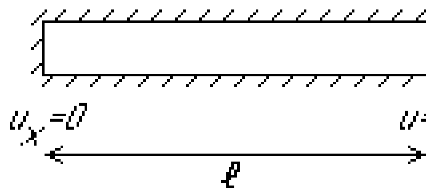
In the range $0 \leq x \leq \ell$, the found solution is exactly the same as for the finite pipe!

Note that if f and/or g does not satisfy the given boundary conditions, $\bar{f}$ and $\bar{g}$ may have kinks or jumps.

7.28

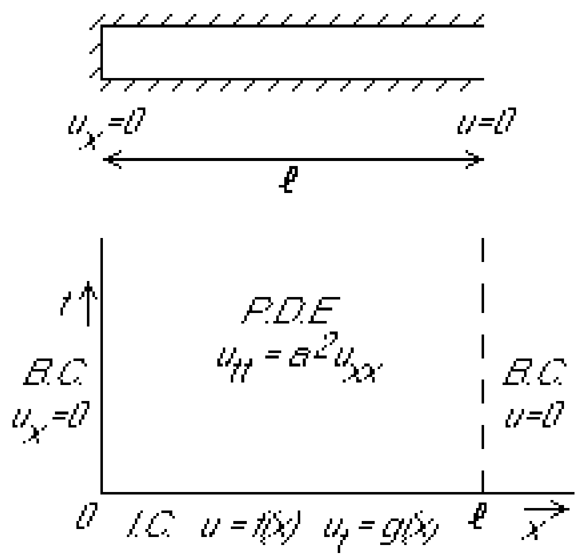
1 7.28, §1 Asked

Asked: Find the unsteady pressure field $u(x, t)$ in a pipe with one end closed and the other open to the athmospere.



2 7.28, §2 P.D.E. Model

Always draw the depend-variable picture first.



- Unknown pressure $u = u(x, t)$.
- Constant speed of sound a.
- Finite domain $\bar{\Omega}$: $0 \leq x \leq \ell$.
- Hyperbolic PDE (waves, singularities, evolution).

- Two initial conditions.
- One homogeneous Neumann boundary condition at $x = 0$ and one homogeneous Dirichlet condition at $x = \ell$.

We will try to find a solution of this problem in the form

$$u = \sum_n u_n(t)X_n(x)$$

i.e. a sum, in which each term is a product of a function of x only times a function of t only.

3 7.28, §3 Eigenfunctions

To find the solution in the form,

$$u(x,t) = \sum_n u_n(t)X_n(x)$$

we try substituting an individual term of this general form into the PDE. In particular, we substitute a trial solution $u = T(t)X(x)$ into the homogeneous P.D.E. $u_{tt} = a^2 u_{xx}$. This gives:

$$T''(t)X(x) = a^2 T(t)X''(x)$$

Now we can take the terms depending on t only to one side of the equation, and the ones depending on x only to the other side:

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}$$

This trick is why this solution procedure is called the "method of separation of variables.'

While the right hand side, $X''(x)/X(x)$, does not depend on t, you would think that it would depend on the position x; both X and X'' change when x changes. But actually, X''/X does *not* change with x; after all, if we change x, it does nothing to t, so the left hand side does not change. And since the right hand side is the same, it too does not change. So the right hand side does not depend on either x or t; it must be a constant. By convention, we call the constant $-\lambda$:

$$\frac{T''}{a^2 T} = \frac{X''}{X} = \text{ constant } = -\lambda$$

If we also require X to satisfy the same homogeneous boundary conditions as u, i.e., that at $x = 0$, its x-derivative is zero, and that at $x = \ell$, X itself is zero, we get the following problem for X:

$$X'' + \lambda X = 0 \qquad X'(0) = 0 \qquad X(\ell) = 0$$

This is an ordinary differential equation boundary value problem.

Note that this problem is completely homogeneous: $X(x) = 0$ satisfies both the PDE *and* the boundary conditions. This is similar to the eigenvalue problem for vectors $A\vec{v} = \lambda\vec{v}$, which is certainly always true when $\vec{v} = 0$. But for the eigenvalue problem, we are interested in *nonzero* vectors $\vec{v}$ so that $A\vec{v} = \lambda v$, which only occurs for special values $\lambda_1, \lambda_2, \dots$ of λ. Similarly, we are interested only in *nonzero solutions X(x)* of the above ODE and boundary conditions.

Eigenvalue problems for functions such as the one above are called "Sturm-Liouville problems." The biggest differences from matrix eigenvalue problems are:

- There are infinitely many eigenvalues $\lambda_1, \lambda_2, \dots$ and corresponding eigenfunctions $X_1(x)$, $X_2(x)$, ... rather than just n eigenvalues and eigenvectors.
- We cannot write a determinant to find the eigenvalues. Instead we must solve the problem using our methods for solving ODE.

Fortunately, the above ODE is simple: it is a constant coefficient one, so we write its characteristic polynomial:

$$k^2 + \lambda = 0 \quad\Longrightarrow\quad k = \pm\sqrt{-\lambda} = \pm \mathrm{i}\sqrt{\lambda}$$

We must now find *all* possible eigenvalues λ and corresponding eigenfunctions that satisfy the required boundary conditions. We must look at all possibilities, one at a time.

Case $\lambda < 0$*:*

Since $k = \pm\sqrt{-\lambda}$

$$X = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$$

We try to satisfy the boundary conditions:

$$X'(0) = 0 = A\sqrt{-\lambda} - B\sqrt{-\lambda} \quad\Longrightarrow\quad B = A$$

$$X(\ell) = 0 = A\left(e^{\sqrt{-\lambda}\ell} + e^{-\sqrt{-\lambda}\ell}\right) \quad\Longrightarrow\quad A = 0$$

So $A = B = 0$; there are *no* nontrivial solutions for $\lambda < 0$.

Case $\lambda = 0$*:*

Since $k_1 = k_2 = 0$ we have a multiple root of the characteristic equation, and the solution is

$$X = Ae^{0x} + Bxe^{0x} = A + Bx$$

We try to satisfy the boundary conditions again:

$$X'(0) = 0 = B \qquad X(\ell) = 0 = A$$

So $A = B = 0$; there are again no nontrivial solutions.

Case $\lambda > 0$:

Since $k = \pm\sqrt{-\lambda} = \pm \mathrm{i}\sqrt{\lambda}$, the solution of the ODE is after cleanup:

$$X = A \sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right)$$

We try to satisfy the first boundary condition:

$$X'(0) = 0 = A\sqrt{\lambda}$$

Since we are looking at the case $\lambda > 0$, this can only be true if $A = 0$. So, we need

$$X = B\cos\left(\sqrt{\lambda}x\right)$$

We now try to also satisfy the second boundary condition:

$$X(\ell) = 0 = B\cos\left(\sqrt{\lambda}\ell\right) = 0$$

For a nonzero solution, B may not be zero, so the cosine must be zero. For positive argument, a cosine is zero at $\frac{1}{2}\pi, \frac{3}{2}\pi, \ldots$, so that our eigenvalues are

$$\sqrt{\lambda_1} = \frac{\pi}{2\ell}, \sqrt{\lambda_2} = \frac{3\pi}{2\ell}, \sqrt{\lambda_3} = \frac{5\pi}{2\ell}, \ldots$$

The same as for eigenvectors, for our eigenfunctions we *must choose* the one undetermined parameter B. Choosing each $B = 1$, we get the eigenfunctions:

$$X_1 = \cos\left(\frac{\pi x}{2\ell}\right), X_2 = \cos\left(\frac{3\pi x}{2\ell}\right), X_3 = \cos\left(\frac{5\pi x}{2\ell}\right), \ldots$$

Total:

The only eigenvalues for this problem are the positive ones above, with the corresponding eigenfunctions. If we want to evaluate them on a computer, we need a general formula for them. You can check that it is:

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4\ell^2} \qquad X_n = \cos\left(\frac{(2n-1)\pi x}{2\ell}\right) \qquad (n = 1, 2, 3, \ldots)$$

Just try a few values for n. We have finished finding the eigenfunctions.

If you look way back to the beginning of this section, you may wonder about the function $T(t)$. It satisfied

$$\frac{T''}{a^2T} = -\lambda$$

Now that we have found the values for λ from the X-problem, we could solve this ODE too, and find functions $T_1(t), T_2(t), \ldots$. Many people do exactly that. However, if you want to mindlessly follow the crowd, please keep in mind the following:

1. The values of λ can only be found from the Sturm-Liouville problem for X. The problem for T is *not* a Sturm-Liouville problem and *can never* produce the correct values for λ.
2. The functions $T(t)$ do *not* satisfy the same initial conditions at time $t = 0$ as u does.
3. Finding T is useless if the PDE is inhomogeneous; it simply does not work. (Unless you add still more artificial tricks to the mix, as the book does.)

We will just ignore the entire T. Instead in the next section we will systematically solve the problem for u without tricks using our found eigenfunctions. What we do there will always work. If you want to try to take a shortcut for an homogeneous PDE, well, the responsibility and risk are yours alone. Someday I will stop seeing students getting themselves in major trouble this way at the final, but it may not be this year.

4 7.28, §4 Solve

So what is the procedure for solving the original problem for the pressure u having found the eigenfunctions X_n? It is "to switch to the basis of eigenfunctions." Or in plain English, it is to write everything in terms of eigenfunctions. And if I say everything, I mean *everything.*

We first write our solution $u(x,t)$ in terms of the eigenfunctions:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x)$$

The coefficients $u_n(t)$ are called the "Fourier coefficients" of u. They are really just the coordinates of u in the basis of eigenfunctions. The sum is called the "Fourier series" for u.

We know our eigenfunctions $X_n(x)$, but not our Fourier coefficients $u_n(t)$. In fact, the $u_n(t)$ are exactly what we want to find out: if we know $u_n(t)$, we can find the pressure u we are looking for by doing the sum above. On a computer probably, if we want to get high accuracy. Or just the first few terms by hand, if we accept some numerical error.

Second, we also write our PDE, $u_{tt} = a^2 u_{xx}$, in terms of the eigenfunctions:

$$\sum_{n=1}^{\infty} \ddot{u}_n(t) X_n(x) = a^2 \sum_{n=1}^{\infty} u_n(t) X_n''(x)$$

This PDE will *always* simplify; that is how the method of separation of variables works. Look up the Sturm-Liouville problem for X_n in the previous section; it was $X_n''(x) = -\lambda_n X_n(x)$. So we can get rid of the x-derivatives in the PDE:

$$\sum_{n=1}^{\infty} \ddot{u}_n(t) X_n(x) = a^2 \sum_{n=1}^{\infty} \left(-\lambda_n u_n(t)\right) X_n(x)$$

Now if two functions are equal, all their Fourier coefficients must be equal, so we have, for any value of n,

$$\ddot{u}_n(t) = -a^2\lambda_n u_n(t) \qquad (n = 1, 2, 3, \ldots)$$

That no longer contains x at all: *the PDE has become a set of ODE in t only.* And we (hopefully) know how to solve those! Getting rid of x is really what the method of separation variables does for us.

We can solve the ODE above easily. It is a constant coefficient one, with a characteristic equation $k^2 = -a^2\lambda_n$, hence $k = \pm ia\sqrt{\lambda_n}$, giving

$$u_n(t) = C_{1n}e^{ia\sqrt{\lambda_n}t} + C_{2n}e^{-ia\sqrt{\lambda_n}t}$$

or after cleaning up,

$$u_n(t) = D_{1n}\cos\left(a\sqrt{\lambda_n}t\right) + D_{2n}\sin\left(a\sqrt{\lambda_n}t\right)$$

So, we have already found our pressure a bit more precisely:

$$u(x,t) = \sum_{n=1}^{\infty}\left[D_{1n}\cos\left(a\sqrt{\lambda_n}t\right) + D_{2n}\sin\left(a\sqrt{\lambda_n}t\right)\right]X_n(x)$$

but we still need to figure out what the integration constants D_{1n} and D_{2n} are.

Third, we write our initial condition $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ in terms of the eigenfunctions. Writing the Fourier series for the two functions as

$$f(x) = \sum_{n=1}^{\infty} f_n X_n(x) \qquad g(x) = \sum_{n=1}^{\infty} g_n X_n(x).$$

and using the Fourier series for u above, the two initial conditions become

$$\sum_{n=1}^{\infty} D_{1n}X_n(x) = \sum_{n=1}^{\infty} f_n X_n(x)$$

$$\sum_{n=1}^{\infty} a\sqrt{\lambda_n}D_{2n}X_n(x) = \sum_{n=1}^{\infty} g_n X_n(x).$$

The Fourier coefficients must again be equal, so we conclude that the coefficients we are looking for are

$$D_{1n} = f_n \qquad D_{2n} = \frac{g_n}{a\sqrt{\lambda_n}}$$

The Fourier series for u becomes now

$$u(x,t) = \sum_{n=1}^{\infty}\left[f_n\cos\left(a\sqrt{\lambda_n}t\right) + \frac{g_n}{a\sqrt{\lambda_n}}\sin\left(a\sqrt{\lambda_n}t\right)\right]X_n(x)$$

where

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4\ell^2} \qquad X_n = \cos\left(\frac{(2n-1)\pi x}{2\ell}\right)$$

So, if we can find the Fourier coefficients f_n and g_n of functions $f(x)$ and $g(x)$, we are done.

Now $f(x)$ and $g(x)$ are, supposedly, given functions, but how do we find their Fourier coefficients? We need a transformation matrix P^{-1} to get coefficients in a new basis for vectors. The equivalent for functions is the following important formula:

$$\boxed{f_n = \frac{\int_0^l f(x)X_n(x)\mathrm{d}x}{\int_0^l X_n(x)^2\mathrm{d}x}}$$

which is called the "orthogonality relation". Even if $f(x) = 1$, say, we still need to do those integrals. The same for g of course:

$$g_n = \frac{\int_0^l g(x)X_n(x)\mathrm{d}x}{\int_0^l X_n(x)^2\mathrm{d}x}$$

We are done! Or at least, we have done as much as we can do until someone tells us the actual functions $f(x)$ and $g(x)$. If they do, we just do the integrals above to find all the f_n and g_n, (maybe analytically or on a computer), and then we can sum the expression for $u(x,t)$ for any x and t that strikes our fancy.

Note that we did not have to do anything with the boundary conditions $u_x(0,t) = 0$ and $u(\ell,t) = 0$; since every eigenfunction X_n satisfies them, the expression for u above automatically also satisfies these homogeneous boundary conditions.

5 7.28, §5 Comparison

Separation of variables solution found as:

$$u = \sum_{n=1}^{\infty}\left[f_n\cos\left(\frac{(2n-1)\pi at}{2\ell}\right) + \frac{2\ell g_n}{(2n-1)\pi a}\sin\left(\frac{(2n-1)\pi at}{2\ell}\right)\right]\cos\left(\frac{(2n-1)\pi x}{2\ell}\right)$$

- Shows the natural frequencies (tones) to be $\pi a/2\ell$, $3\pi a/2\ell$,...
- Shows the energy in each harmonic.
- Not restricted to the 1D wave equation.

D'Alembert:

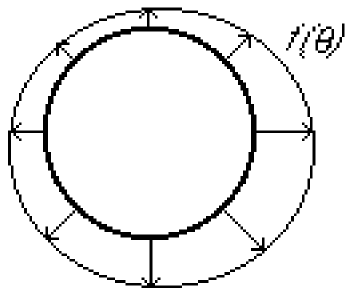
$$u(x,t) = \frac{\bar{f}(x-at) + \bar{f}(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\bar{g}(\xi)\,\mathrm{d}\xi$$

- I can evaluate the pressure at any point without doing big sums.
- Shows how wave fronts propagate.
- Shows regions of influence and dependence.

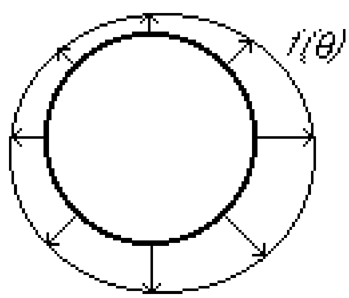
7.38

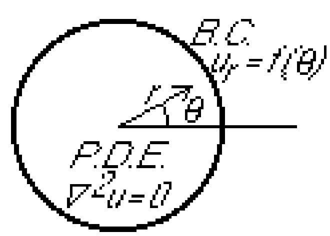
1 7.38, §1 Asked

Asked: Find the ideal flow in a cylinder if the normal (radial) velocity at the perimeter is known.



2 7.38, §2 P.D.E. Model





- Finite domain $\bar{\Omega}$: $0 \leq r \leq 1, 0 \leq \vartheta < 2\pi$
- Unknown velocity potential $u = u(r, \vartheta)$

- Elliptic equation

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\vartheta\vartheta} = 0$$

- One Neumann boundary condition at $r = 1$.

We will try separation of variables.

3 7.38, §3 Eigenfunctions

If we substitute a trial solution $u = R(r)\Theta(\vartheta)$ into the homogeneous P.D.E. $u_{rr} + u_r/r + u_{\vartheta\vartheta}/r^2 = 0$, we get:

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

which separates into

$$r^2\frac{T''}{T} + r\frac{T'}{T} = -\frac{\Theta''}{\Theta} = \text{ constant } = \lambda$$

Make sure that *all* r terms are at the same side of the equation!

Now which ODE gives us the Sturm-Liouville problem, and thus the eigenvalues? Not the one for $R(r)$; u has an *inhomogeneous* boundary condition on the perimeter $r = 1$. Eigenvalue problems must be homogeneous; they simply don't work if anything is inhomogeneous.

We are in luck with $\Theta(\vartheta)$ however. The unknown $u(r,\vartheta)$ has "periodic" boundary conditions in the ϑ-direction. If ϑ increases by an amount 2π, $u(r,\vartheta)$ returns to exactly the same values as before: it is a "periodic function" of ϑ. Periodic boundary conditions are homogeneous: the zero solution satisfies them. After all, zero remains zero however many times you go around the circle.

The Sturm-Liouville problem for Θ is:

$$-\Theta'' = \lambda\Theta$$

$$\Theta(0) = \Theta(2\pi) \qquad \Theta'(0) = \Theta'(2\pi)$$

Note that for a second order ODE, we need two boundary conditions. So we wrote down that both Θ, as well as its derivative are exactly the same at $\vartheta = 0$ and 2π.

Pretend that we do not know the solution of this Sturm-Liouville problem! Write the characteristic equation of the ODE:

$$k^2 + \lambda = 0 \quad \Longrightarrow \quad k = \pm\mathrm{i}\sqrt{\lambda}$$

Lets look at all possibilities:

Case $\lambda = 0$:

Since $k_1 = k_2 = 0$:

$$\Theta = A + B\vartheta$$

Boundary conditions:

$$\Theta(0) = \Theta(2\pi) \qquad \Longrightarrow \qquad A = A + B2\pi$$

That can only be true if $B = 0$. Then the second boundary condition is

$$\Theta'(0) = \Theta'(2\pi) \qquad \Longrightarrow \qquad 0 = 0$$

hence $\Theta = A$. No undetermined constants in eigenfunctions! Simplest is to choose $A = 1$:

$$\Theta_0(\vartheta) = 1$$

Case $\lambda \neq 0$:

We will be lazy and try to do the cases of positive and negative λat the same time. For positive λ, the cleaned-up solution is

$$\Theta = A\cos\left(\sqrt{\lambda}\vartheta\right) + B\sin\left(\sqrt{\lambda}\vartheta\right)$$

This also applies for negative λ, except that the square roots are then imaginary.

Lets write down the boundary conditions first:

$$\Theta(0) = \Theta(2\pi) \qquad \Longrightarrow \qquad A = A\cos\left(\sqrt{\lambda}2\pi\right) + B\sin\left(\sqrt{\lambda}2\pi\right)$$

$$\Theta'(0) = \Theta'(2\pi) \qquad \Longrightarrow \qquad B\sqrt{\lambda} = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}2\pi\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}2\pi\right)$$

These two equations are a bit less simple than the ones we saw so far. Rather than directly trying to solve them and make mistakes, this time let us write out the augmented matrix of the system of equations for A and B:

$$\left(\begin{array}{cc|c} 1-\cos\left(\sqrt{\lambda}2\pi\right) & -\sin\left(\sqrt{\lambda}2\pi\right) & 0 \\ \sin\left(\sqrt{\lambda}2\pi\right) & 1-\cos\left(\sqrt{\lambda}2\pi\right) & 0 \end{array}\right)$$

Any nontrivial solution must be nonunique (since zero is also a solution). So the determinant of the matrix must be zero, which is:

$$1 - 2\cos\left(\sqrt{\lambda}2\pi\right) + \cos^2\left(\sqrt{\lambda}2\pi\right) + \sin^2\left(\sqrt{\lambda}2\pi\right) = 0$$

or

$$\cos\left(\sqrt{\lambda}2\pi\right) = 1$$

A cosine is only equal to 1 when its argument is an integer multiple of 2π. Hence the only possible eigenvalues are

$$\sqrt{\lambda_1} = 1 \quad \sqrt{\lambda_2} = 2 \quad \sqrt{\lambda_3} = 3 \quad \ldots$$

If λ is negative, $\cos\left(i\sqrt{-\lambda}2\pi\right) = \cosh\left(\sqrt{-\lambda}2\pi\right)$ which is always greater than one for nonzero λ.

For the found eigenvalues, the system of equations for A and B becomes:

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Hence we can find *neither* A or B; there are *two* undetermined constants in the solution:

$$\Theta_n = A\cos(n\vartheta) + B\sin(n\vartheta)$$

We had this situation before with eigenvector in the case of double eigenvalues, where an eigenvalue gave rise two linearly independent eigenvectors. Basically we have the same situation here: each eigenvalue is double. Similar to the case of eigenvectors of symmetric matrices, here we want two linearly independent, and more specifically, orthogonal eigenfunctions. A suitable pair is

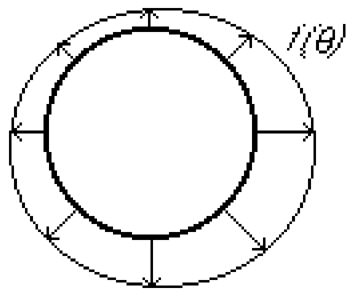
$$\Theta_n^1(\vartheta) = \cos(n\vartheta)$$
$$\Theta_n^2(\vartheta) = \sin(n\vartheta)$$

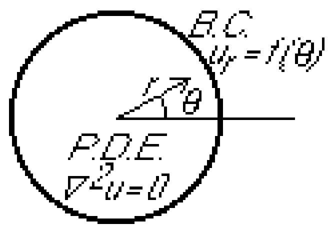
Total:

We can tabulate the complete set of eigenvalues and eigenfunctions now as:

$$\begin{array}{rcc} \lambda_0 = 0 & \multicolumn{2}{c}{\Theta_0 = 1} \\ \lambda_1 = 1 & \Theta_1^1 = \cos(\vartheta) & \Theta_1^2 = \sin(\vartheta) \\ \lambda_2 = 4 & \Theta_2^1 = \cos(2\vartheta) & \Theta_2^2 = \sin(2\vartheta) \\ \lambda_3 = 9 & \Theta_3^1 = \cos(3\vartheta) & \Theta_3^2 = \sin(3\vartheta) \\ \lambda_4 = 16 & \Theta_4^1 = \cos(4\vartheta) & \Theta_4^2 = \sin(4\vartheta) \\ \vdots & \vdots & \vdots \end{array}$$

4 7.38, §4 Solve





We will again expand all variables in the problem in a Fourier series. Let's start with the function $f(\vartheta)$ giving the outflow through the perimeter.

$$f(\vartheta) = f_0 + \sum_{n=1}^{\infty} f_n^1 \cos(n\vartheta) + \sum_{n=1}^{\infty} f_n^2 \sin(n\vartheta)$$

This is the way a Fourier series of a periodic function with period 2π always looks.

Since $f(\vartheta)$ is supposedly known, we should again be able to find its Fourier coeficients using orthogonality. The formulae are as before.

$$f_0 = \frac{\int_{\vartheta=0}^{2\pi} f(\vartheta) 1 \,\mathrm{d}\vartheta}{\int_{\vartheta=0}^{2\pi} 1^2 \,\mathrm{d}\vartheta}$$

(the bottom is of course equal to 2π,)

$$f_n^1 = \frac{\int_{\vartheta=0}^{2\pi} f(\vartheta) \cos(n\vartheta) \,\mathrm{d}\vartheta}{\int_{\vartheta=0}^{2\pi} \cos^2(n\vartheta) \,\mathrm{d}\vartheta} \quad (n = 1, 2, \ldots)$$

$$f_n^2 = \frac{\int_{\vartheta=0}^{2\pi} f(\vartheta) \sin(n\vartheta) \,\mathrm{d}\vartheta}{\int_{\vartheta=0}^{2\pi} \sin^2(n\vartheta) \,\mathrm{d}\vartheta} \quad (n = 1, 2, \ldots)$$

(the bottoms are equal to π.)

Since I hate typing big formulae, allow me to write the Fourier series for $f(\vartheta)$ much more compactly as

$$f(\vartheta) = \sum_{n,i}^{\infty} f_n^i \Theta_n^i(\vartheta)$$

where $\Theta_n^1 = \cos(n\vartheta)$ and $\Theta_n^2 = \sin(n\vartheta)$. Also, all three formulae for the Fourier coefficients can be summarized as

$$f_n^i = \frac{\int_{\vartheta=0}^{2\pi} f(\vartheta)\Theta_n^i(\vartheta)\,\mathrm{d}\vartheta}{\int_{\vartheta=0}^{2\pi} \Theta_n^{i2}(\vartheta)\,\mathrm{d}\vartheta} \qquad (n = 1, 2, 3, \ldots; i = 1, 2)$$

For $n = 0$, only the value $i = 1$ is relevant, of course; $\Theta_0^1 = \cos(0\vartheta) = 1 = \Theta_0$. There is no $\Theta_0^2 = \sin(0\vartheta) = 0$.

Next, let's write the unknown $u(r, \vartheta)$ as a compact Fourier series:

$$u(r, \vartheta) = \sum_{n,i} u_n^i(r)\Theta_n^i(\vartheta)$$

We put this into P.D.E. $u_{rr} + u_r/r + u_{\vartheta\vartheta}/r^2 = 0$:

$$\sum_{n,i} u_n^i(r)''\Theta_n^i(\vartheta) + \frac{1}{r}\sum_{n,i} u_n^i(r)'\Theta_n^i(\vartheta) + \frac{1}{r^2}\sum_{n,i} u_n^i(r)\Theta_n^i(\vartheta)'' = 0$$

Using the Sturm-Liouville equation $\Theta_n^i(\vartheta)'' = -\lambda\Theta_n^i(\vartheta)$, where λ was found to be n^2, this simplifies to

$$\sum_{n,i} u_n^i(r)''\Theta_n^i(\vartheta) + \frac{1}{r}\sum_{n,i} u_n^i(r)'\Theta_n^i(\vartheta) - \frac{1}{r^2}\sum_{n,i} n^2 u_n^i(r)\Theta_n^i(\vartheta) = 0$$

We get the following ODE for $u_n^i(r)$:

$$u_n^i(r) + \frac{1}{r}u_n^i(r)' - \frac{n^2}{r^2}u_n^i(r) = 0$$

or multiplying by r^2:

$$r^2 u_n^i(r) + r u_n^i(r)' - n^2 u_n^i(r) = 0$$

This is *not* a constant coefficient equation. Writing down a characteristic equation is no good.

Fortunately, we have seen this one before: it is the Euler equation. You solved that one by changing to the logaritm of the independent variable, in other words, by rewriting the equation in terms of

$$\rho \equiv \ln r$$

instead of r. The r-derivatives can be converted as in:

$$\frac{\mathrm{d}u_n^i}{\mathrm{d}r} = \frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{1}{r}$$

$$\frac{\mathrm{d}^2 u_n^i}{\mathrm{d}r^2} = \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{1}{r}\right] = \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\right]\frac{1}{r} - \frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{1}{r^2}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\right]\frac{\mathrm{d}\rho}{\mathrm{d}r}\frac{1}{r} - \frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{1}{r^2} = \frac{\mathrm{d}^2u_n^i}{\mathrm{d}\rho^2}\frac{1}{r^2} - \frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{1}{r^2}$$

The ODE becomes in terms of ρ:

$$\frac{\mathrm{d}^2u_n^i}{\mathrm{d}\rho^2} - n^2u_n^i = 0$$

This is now a constant coefficient equation, so we can write the characteristic polynomial, $k^2 - n^2 = 0$, or $k = \pm n$, which has a double root when $n = 0$. So we get for $n = 0$:

$$u_0^1 = A_0^1 + B_0^1\rho = A_0^1 + B_0^1 \ln r$$

while for $n \neq 0$:

$$u_n^i = A_n^i e^{n\rho} + B_n^i e^{-n\rho} = A_n^i r^n + B_n^i r^{-n}$$

Now both $\ln r$ as well as r^{-n} are infinite when $r = 0$. But that is in the middle of our flow region, and the flow is obviously not infinite there. So from the 'boundary condition' at $r = 0$ that the flow is not singular, we conclude that all the B-coefficients must be zero. Since $r^0 = 1$, all coefficients are of the form $A_n^i r^n$, including the one for $n = 0$.

Hence our solution can be more precisely written

$$u(r, \vartheta) = \sum_{n,i} A_n^i r^n \Theta_n^i(\vartheta)$$

Next we expand the boundary condition $u_r(1, \vartheta) = f(\vartheta)$ at $r = 1$ in a Fourier series:

$$\sum_{n,i} nA_n^i \Theta_n^i(\vartheta) = \sum_{n,i} f_n^i \Theta_n^i(\vartheta)$$

producing

$$nA_n^i = f_n^i$$

For $n = 0$, we see immediately that A_0 can be anything, but we need $f_0 = 0$ for a solution to exist! According to the orthogonality relationship for f_0, this requires:

$$\int_0^{2\pi} f(\vartheta)\,\mathrm{d}\vartheta = 0$$

Are you surprised that the net outflow through the perimeter must be zero for this steady flow?

For nonzero n:

$$A_n^i = \frac{f_n^i}{n}$$

and our solution becomes

$$u = A_0 + \sum_{n,i} f_n^i \frac{r^n}{n}\Theta_n(\vartheta)$$

where A_0 can be anything.

5 7.38, §5 Total

Let's summarize our results, and write the eigenfunctions out in terms of the individual sines and cosines.

Required for a solution is that:

$$\int_0^{2\pi} f(\vartheta)\,\mathrm{d}\vartheta = 0$$

Then:

$$f_n^1 = \frac{1}{\pi}\int_{\vartheta=0}^{2\pi} f(\vartheta)\cos(n\vartheta)\,\mathrm{d}\vartheta \quad (n = 1, 2, \ldots)$$
$$f_n^2 = \frac{1}{\pi}\int_{\vartheta=0}^{2\pi} f(\vartheta)\sin(n\vartheta)\,\mathrm{d}\vartheta \quad (n = 1, 2, \ldots)$$

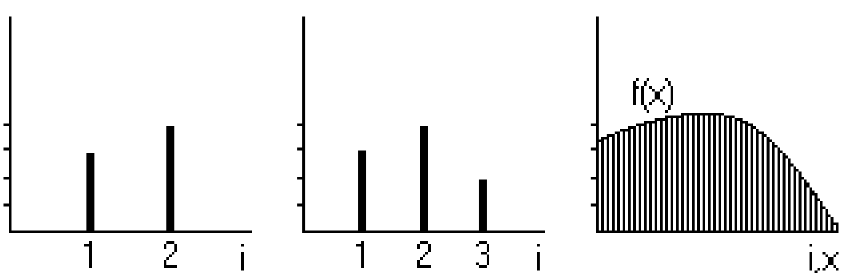
$$u = A_0 + \sum_{n=1}^{\infty}\left\{f_n^1\frac{r^n}{n}\cos(n\vartheta) + f_n^2\frac{r^n}{n}\sin(n\vartheta)\right\}$$

where A_0 can be anything.

6 7.38, §6 Notes

(The material in this section is elective).

It may be interesting to see exactly how functions are similar to vectors. Let's start with a vector in two dimensions, like the vector $\vec{v} = (3, 4)$. I can represent this vector graphically as a point in a plane, but I can also represent it as the 'spike function' in the first figure below:



The first coefficient, v_1, is 3, giving a spike of height of 3 when the subscript, call it i, is 1. The second coefficient $v_2 = 4$, so we have a spike of height 4 at $i = 2$. Similarly, the three-dimensional vector $\vec{v} = (3, 4, 2)$ can be graphed as the three-spike function in the second figure. If I keep adding more dimensions, going to the limit of infinite-dimensional space, my spike graph v_i becomes a function graph f of a continuous coordinate x instead of i. You can think of function $f(x)$ as a column vector of numbers, with the numbers being the successive values of $f(x)$. In this way, vectors become functions. And vector analysis turns into functional analysis.

To take the dot product of two vectors $\vec{v}$ and $\vec{w}$, we multiply corresponding coefficients and sum:

$$\vec{v} \cdot \vec{w} \equiv \sum_{i=0}^{n} v_i w_i$$

For functions $f(x)$ and $g(x)$, the sum over i becomes an integral over x:

$$(f, g) \equiv \int_{x=0}^{\ell} f(x) g(x)\, \mathrm{d}x$$

Since we now have a dot product, or inner product, for functions we can define the "norm" of a function, $||f|| \equiv \sqrt{(f, f)}$, corresponding to length for vectors. More importantly, we can define orthogonality for functions. Functions f and g are orthogonal if the integral above is zero.

For vectors we have matrices that turn vectors into other vectors: a matrix A turns a vector $\vec{v}$ into another vector $A\vec{v}$. For functions we have "operators" that turn functions into other functions. For example, the operator $\partial^2/\partial x^2$ turns a function $f(x)$ into another function $f''(x)$. Among the functions of period 2π, a function such as $\cos(nx)$ is an eigenfunction of this operator:

$$\frac{\partial^2}{\partial x^2} \cos(nx) = -n^2 \cos(nx)$$

The eigenvalue is n^2.

Symmetry for matrices can be expressed as $\vec{v} \cdot (A\vec{w}) = (A\vec{v}) \cdot \vec{w}$ because this can be written using matrix multiplication as $\vec{v}^T A \vec{w} = \vec{v}^T A^T \vec{w}$, which can only be true for all vectors $\vec{v}$ and $\vec{w}$ if $A = A^T$. And since $(f, \partial^2 g/\partial x^2) = (\partial^2 f/\partial x^2, g)$, as can be seen from integration by parts, $\partial^2/\partial x^2$ is a symmetric, or "self-adjoint", operator, with orthogonal eigenfunctions.

How about orthogonality relations? Given eigenfunctions X_n, we have seen that you get the Fourier coefficients of an arbitrary function $f(x)$ by the following formula:

$$f_n = \frac{\int f X_n \,\mathrm{d}x}{\int X_n^2 \,\mathrm{d}x} \equiv \frac{(X_n, f)}{(X_n, X_n)}$$

But where does this come from? Remember that we get the coordinates in the new coordinate system, (here, the Fourier coefficients f_n), by multiplying the original vector, (here $f(x)$), by the inverse transformation matrix P^1.

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{pmatrix} = P^{-1} f(x)$$

Now the transformation matrix P has the eigenfunctions as columns:

$$P = (X_1\ X_2\ X_3\ \ldots)$$

Since the eigenfunctions are orthogonal, you get P^{-1} by simply taking the transpose, so:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} X_1^T \\ X_2^T \\ X_3^T \\ \vdots \end{pmatrix} f(x) = \begin{pmatrix} (X_1, f) \\ (X_2, f) \\ (X_3, f) \\ \vdots \end{pmatrix}$$

giving $f_n = (X_n, f)$. The difference from $f_n = (X_n, f)/(X_n, X_n)$ above is simply due to the fact that we usually do not normalize eigenfunctions to norm 1.

Note: I simply told you that the proper orthogonal eigenfunctions for the double eigenvalues in 7.38 are $\cos(nx)$ and $\sin(nx)$, but I could actually have derived it from Gram-Schmidt! There is really nothing new for PDE, if you think of it this way.

7 7.38, §7 More Fun

Our final result was

$$u = A_0 + \sum_{n=1}^{\infty} \left\{ f_n^1 \frac{r^n}{n} \cos(n\vartheta) + f_n^2 \frac{r^n}{n} \sin(n\vartheta) \right\}$$

We can write it directly in terms of the given $f(x)$ if we substitute in the expressions for the Fourier coefficients:

$$u = A_0 + \sum_{n=1}^{\infty} \int_0^{2\pi} f(\phi) \cos(n\phi)\, \mathrm{d}\phi \frac{r^n}{n\pi} \cos(n\vartheta) + \int_0^{2\pi} f(\phi) \sin(n\phi)\, \mathrm{d}\phi \frac{r^n}{n\pi} \sin(n\vartheta)$$

We can clean it up by combining terms and interchanging integration and summation:

$$u = A_0 + \int_0^{2\pi} \sum_{n=1}^{\infty} \left\{ \frac{r^n}{n\pi} [\cos(n\phi)\cos(n\vartheta) + \sin(n\phi)\sin(n\vartheta)] \right\} f(\phi)\, \mathrm{d}\phi$$

$$u = A_0 + \int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} \frac{r^n}{n\pi} \cos(n[\vartheta - \phi]) \right\} f(\phi)\, \mathrm{d}\phi$$

This we can clean up even more by giving a name to the function within the curly brackets:

$$u = A_0 + \int_0^{2\pi} G(r, \vartheta - \phi) f(\phi)\, \mathrm{d}\phi$$

Nice, not? We can even simplify G by converting to complex exponentials and differentiating:

$$G(r,\vartheta) = \sum_{n=1}^{\infty} \frac{r^n}{n\pi} \cos(n\vartheta) = \sum_{n=1}^{\infty} \left\{ \frac{r^n}{2n\pi} e^{in\vartheta} + \frac{r^n}{2n\pi} e^{-in\vartheta} \right\}$$

$$2\pi \frac{\partial G}{\partial r} = \sum_{n=1}^{\infty} \left\{ r^{-1} \left(re^{i\vartheta}\right)^n + r^{-1} \left(re^{-i\vartheta}\right)^n \right\} = \frac{e^{i\vartheta}}{1 - re^{i\vartheta}} + \frac{e^{-i\vartheta}}{1 - re^{-i\vartheta}}$$

The last because the sums are geometric series.

Integrating and cleaning up produces

$$G(r,\vartheta) = -\frac{1}{2\pi} \ln\left(1 - 2r\cos(\vartheta) + r^2\right)$$

So, we finally have the following Poisson-type integral expression giving u directly in terms of the given $f(\vartheta)$, with no sums:

$$u(r,\vartheta) = A_0 - \frac{1}{2\pi} \int_0^{2\pi} \ln\left(1 - 2r\cos(\vartheta - \phi) + r^2\right) f(\phi)\, \mathrm{d}\phi$$

Neat!

Introduction

Description:

The *Laplace transform* pairs a function of a real coordinate, call it t, with $0 < t < \infty$, with a different function of a complex coordinate s:

$$u(t,\cdot) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \hat{u}(s,\cdot)$$

The pairing is designed to get rid of derivatives with respect to t in equations for the function u. This works as long as the coefficients do not depend on t (or at the very most are low degree powers of t.) The transformation is convenient since pairings can be looked up in tables.

Typical procedure:

Use tables to find the equations satisfied by $\hat{u}$ from these satisfied by u. Solve for $\hat{u}$ and look up the corresponding u in the tables.

About coordinate t:

In many cases, t is physically time, since time is most likely to satisfy the constraints $0 < t < \infty$ and coefficients independent of t. Also, the Laplace transform likes initial conditions at $t = 0$, not boundary conditions at both $t = 0$ and $t = \infty$.

Table 6.3: Properties of the Laplace transform:

$u(t)$	$\hat{u}(s)$
0. $\dfrac{1}{2\pi \mathrm{i}}\displaystyle\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{-st}\hat{u}(s)\,\mathrm{d}s$	$\displaystyle\int_0^\infty u(t)e^{-st}\,\mathrm{d}t$
1. $C_1u_1(t) + C_2u_2(t)$	$C_1\hat{u}_1(s) + C_2\hat{u}_2(s)$
2. $u(at)$	$a^{-1}\hat{u}(s/a)$
3. $\dfrac{\partial^n u}{\partial t^n}(t)$	$s^n\hat{u}(s) - s^{n-1}u(0) - \ldots - \dfrac{\partial^{n-1}u}{\partial t^{n-1}}(0)$
4. $t^n u(t)$	$(-1)^n\dfrac{\partial^n \hat{u}}{\partial s^n}$
5. $e^{ct}u(t)$	$\hat{u}(s-c)$
6. $\bar{u}(t-b) \equiv H(t-b)u(t-b) = \begin{cases} u(t-b) & (t-b>0) \\ 0 & (t-b<0) \end{cases}$	$e^{-bs}\hat{u}(s)$
7. $(f*g)(t) \equiv \displaystyle\int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau$	$\hat{f}(s)\hat{g}(s)$

Here $a > 0$, $b > 0$, c are constants, and n is a natural number.

Table 6.4: Laplace transform pairs:

$u(t)$	$\hat{u}(s)$
1. 1	$\frac{1}{s}$
2. t^n	$\frac{n!}{s^{n+1}}$
3. e^{bt}	$\frac{1}{s-b}$
4. $\sin(at)$	$\frac{a}{s^2+a^2}$
5. $\cos(at)$	$\frac{s}{s^2+a^2}$
6. $\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$
7. $\frac{1}{\sqrt{\pi t}}e^{-k^2/(4t)}$	$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}}$
8. $\frac{k}{\sqrt{4\pi t^3}}e^{-k^2/(4t)}$	$e^{-k\sqrt{s}}$
9. $\operatorname{erfc}\left(\frac{k}{2\sqrt{t}}\right)$	$\frac{1}{s}e^{-k\sqrt{s}}$

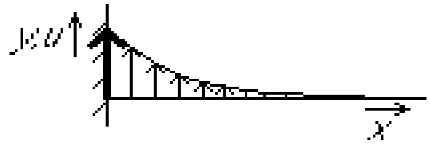
Here $k > 0$, a and b are constants, n is a natural number, and

$$\operatorname{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\xi^2}\, \mathrm{d}\xi$$

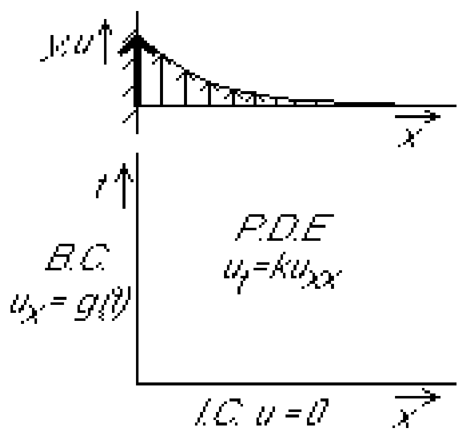
7.24

1 7.24, §1 Asked

Asked: Find the flow velocity in a viscous fluid being dragged along by an accelerating plate.



2 7.24, §2 PDE Model



- Semi-infinite domain $\bar{\Omega}$: $0 \leq x < \infty$
- Unknown vertical velocity $u = u(x,t)$
- Parabolic
- One homogeneous initial condition
- One Neumann boundary condition at $x = 0$ and a regularity constraint at $x = \infty$
- Constant kinematic viscosity κ

Try a Laplace transform in t.

3 7.24, §3 Transform

Transform the PDE:

$$u_t = \kappa u_{xx} \quad \xrightarrow{\text{Table 6.3, \# 3}} \quad s\hat{u} - \cancel{u(x,0)} = \kappa \hat{u}_{xx}$$

Transform the BC:

$$u_x = g(t) \quad \Longrightarrow \quad \hat{u}_x = \hat{g}(s)$$

4 7.24, §4 Solve

Solve the PDE:

$$s\hat{u} = \kappa \hat{u}_{xx}$$

This is a constant coefficient ODE in x, with s simply a parameter. Solve from the characteristic equation:

$$s = \kappa k^2 \quad \Longrightarrow \quad k = \pm\sqrt{s/\kappa}$$

$$\hat{u} = Ae^{\sqrt{s/\kappa}\,x} + Be^{-\sqrt{s/\kappa}\,x}$$

Apply the BC at $x = \infty$ that u must be regular there:

$$A = 0$$

Apply the given BC at $x = 0$:

$$\hat{u}_x = \hat{g}(s) \quad \Longrightarrow \quad -B\sqrt{\frac{s}{\kappa}} = \hat{g}$$

Solving for B and plugging it into the solution of the ODE, $\hat{u}$ has been found:

$$\hat{u} = -\sqrt{\frac{\kappa}{s}} e^{-\sqrt{s/\kappa}\,x} \hat{g}$$

5 7.24, §5 Back

We need to find the original function u corresponding to the transformed

$$\hat{u} = -\sqrt{\frac{\kappa}{s}} e^{-\sqrt{s/\kappa}\,x} \hat{g}$$

We do not really know what $\hat{g}$ is, just that it transforms back to g. However, we can find the other part of $\hat{u}$ in the tables.

$$-\sqrt{\frac{\kappa}{s}}e^{-\sqrt{s/\kappa}\,x} \quad \xrightarrow{\text{Table 6.4, \# 7}} \quad -\sqrt{\frac{\kappa}{\pi t}}e^{-x^2/4\kappa t}$$

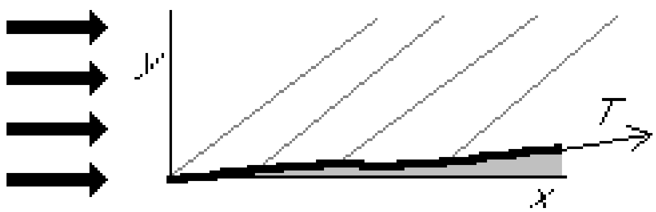
How does $\hat{g}$ times this function transform back? The product of two functions, say $\hat{f}(s)\hat{g}(s)$, does *not* transform back to $f(t)g(t)$. The convolution theorem Table 6.3 # 7 is needed:

$$u(x,t) = -\int_0^t \sqrt{\frac{\kappa}{\pi(t-\tau)}}e^{-x^2/4\kappa(t-\tau)}g(\tau)\,\mathrm{d}\tau$$

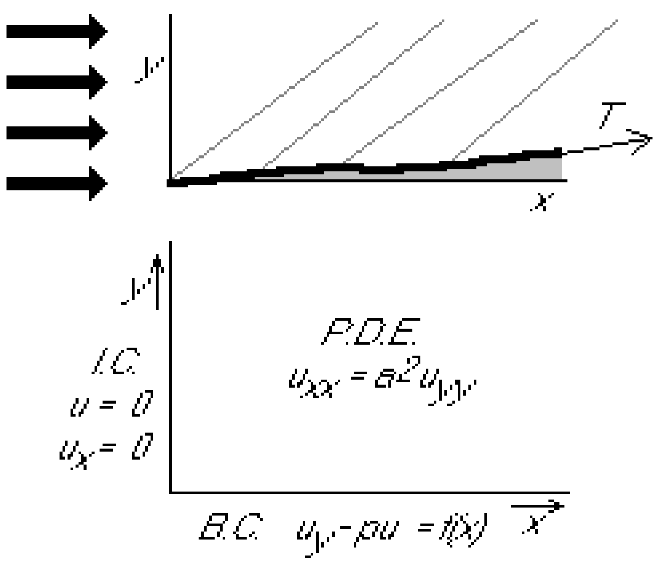
7.36

1 7.36, §1 Asked

Asked: Find the horizontal perturbation velocity in a supersonic flow above a membrane overlaying a compressible variable medium.



2 7.36, §2 PDE Model



- Domain $\bar{\Omega}$: $0 \leq x < \infty, 0 \leq y < \infty$
- Unknown horizontal perturbation velocity $u = u(x, y)$
- Hyperbolic
- Two homogeneous initial conditions

- One mixed boundary condition at $y = 0$ and a regularity constraint at $y = \infty$
- Constant $a = \tan\mu$, where μ is the Mach angle.

Try a Laplace transform. The physics and the fact that Laplace transforms like only initial conditions suggest that x is the one to be transformed. Variable x is our "time-like" coordinate.

3 7.36, §3 Transform

Transform the PDE:

$$u_{xx} = a^2 u_{yy} \quad \xrightarrow{\text{Table 6.3, \# 3}} \quad s^2\hat{u} - \cancel{su(0,y)} - \cancel{u_x(0,y)} = a^2 \hat{u}_{yy}$$

Transform the BC:

$$u_y - pu = f(x) \quad \Longrightarrow \quad \hat{u}_y - p\hat{u} = \hat{f}(s)$$

4 7.36, §4 Solve

Solve the PDE, again effectively a constant coefficient ODE:

$$s^2\hat{u} = a^2\hat{u}_{yy}$$

$$s^2 = a^2k^2 \quad \Longrightarrow \quad k = \pm s/a$$

$$\hat{u} = Ae^{sy/a} + Be^{-sy/a}$$

Apply the BC at $y = \infty$:

$$A = 0$$

Apply the BC at $y = 0$:

$$\hat{u}_y - p\hat{u} = \hat{f} \quad \Longrightarrow \quad -\frac{s}{a}B - pB = \hat{f}$$

Solving for B and plugging it into the expression for $\hat{u}$ gives:

$$\hat{u} = -\frac{a\hat{f}}{s + ap}e^{-sy/a}$$

5 7.36, §5 Back

We need to find the original to

$$\hat{u} = -\frac{a}{s+ap}\hat{f}e^{-sy/a}$$

Looking in the tables:

$$\frac{1}{s+ap} \quad \xrightarrow[]{\text{Table 6.4, \# 3}} \quad e^{-apx}$$

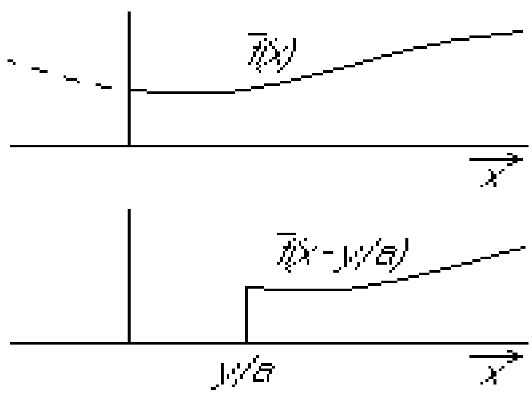
The other factor is a shifted function f, restricted to the interval that its argument is positive:

$$e^{-sy/a}\hat{f} \quad \xrightarrow[]{\text{Table 6.3, \# 6}} \quad \bar{f}\left(x-\frac{y}{a}\right)$$

With the bar, I indicate that I only want the part of the function for which the argument is positive. This could be written instead as

$$f\left(x-\frac{y}{a}\right)H\left(x-\frac{y}{a}\right)$$

where the Heaviside step function $H(x) = 0$ if x is negative and 1 if it is positive.



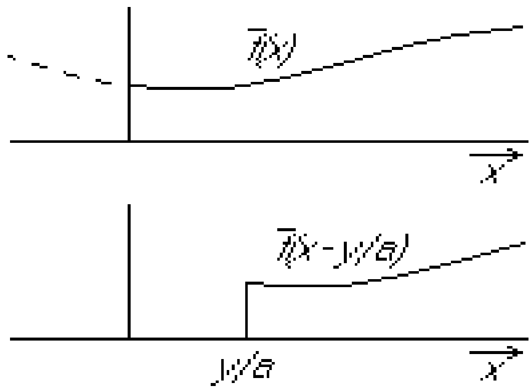
Use convolution, Table 6.3, # 7. again to get the product.

$$u(x,y) = -\int_0^x a\bar{f}\left(\xi-\frac{y}{a}\right)e^{-ap(x-\xi)}\,\mathrm{d}\xi$$

This *must* be cleaned up. I do not want bars or step functions in my answer.

I can do that by restricting the range of integration to only those values for which $\bar{f}$ is nonzero.

(Or H is nonzero, if you prefer)



Two cases now exist:

$$u(x,y) = -\int_{y/a}^{x} af\left(\xi - \frac{y}{a}\right) e^{-ap(x-\xi)}\,\mathrm{d}\xi \qquad (x > \frac{y}{a})$$

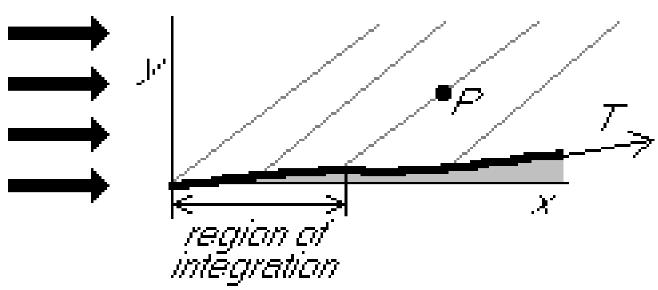
$$u(x,y) = 0 \qquad (x < \frac{y}{a})$$

It is neater if the integration variable is the argument of f. So, define $\phi = \xi - y/a$ and convert:

$$u(x,y) = -\int_{0}^{x-y/a} af\left(\phi\right) e^{-apx+py+ap\phi}\,\mathrm{d}\phi \qquad (x > \frac{y}{a})$$

$$u(x,y) = 0 \qquad (x < \frac{y}{a})$$

This allows me to see which physical f values I actually integrate over when finding the flow at an arbitrary point:



6 7.36, §6 Alternate

An alternate solution procedure is to define a new unknown:

$$v \equiv u_y - pu$$

You must derive the problem for v:

The boundary condition is simply:

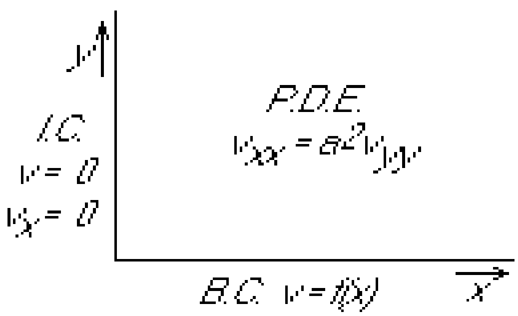
$$v(x,0) = f(x)$$

To get the PDE for v, use

$$\frac{\partial[PDE]}{\partial y} - p[PDE] \quad \Longrightarrow \quad v_{tt} = a^2 v_{xx}$$

Similarly, for the initial conditions:

$$\frac{\partial[ICs]}{\partial y} - p[ICs] \quad \Longrightarrow \quad v(0,y) = v_x(0,y) = 0$$



After finding v, I still need to find u from the definition of v:

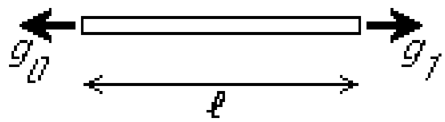
$$v \equiv u_y - pu$$

Where do you get the integration constant??

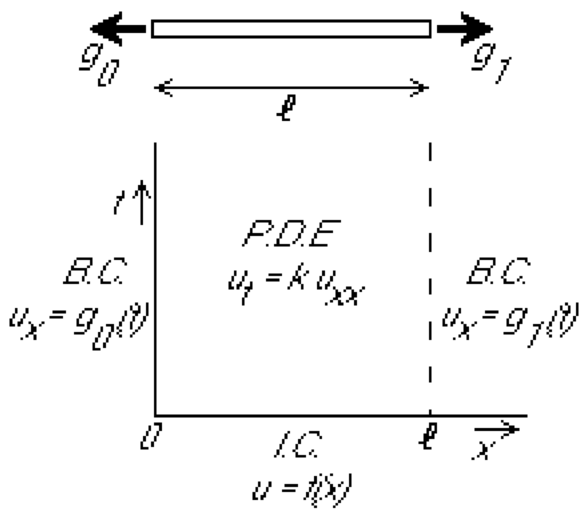
7.19

1 7.19, §1 Asked

Asked: Find the unsteady temperature distribution in the bar below for arbitrary position and time if the initial distribution at time zero and the heat fluxes out of the ends are known.



2 7.19, §2 P.D.E. Model



- Finite domain $\bar{\Omega}$: $0 \leq x \leq \ell$
- Unknown temperature $u = u(x, t)$
- Parabolic
- One initial condition
- Two Neumann boundary conditions
- Constant κ

As usual, we would like to use separation of variables to write the solution in a form that looks roughly like:

$$u(x,t) = \sum_n T_n(t) X_n(x)$$

The problems with that are:

- We will not find a Sturm-Liouville for $T_n(t)$ since the time coordinate is semi-infinite. (A Laplace transform in time is somewhat similar, but it would be a mess in this case.)
- We will not find a Sturm Liouville problem for $X_n(x)$ since the boundary conditions at the x-ends, $u_x(0,t) = g_0(t)$ and $u_x(\ell,t) = g_1(t)$, are inhomogeneous, assuming g_0 and g_1 are nonzero.

We will apply a trick to get around the second problem.

3 7.19, §3 Boundary Fix

To get rid of the inhomogeneous boundary conditions at $x = 0$ and $x = \ell$, use the following trick:

Trick: Find any u_0 that satisfies the inhomogeneous boundary conditions at $x = 0$ and $x = \ell$ and substract it from u. The remainder will have homogeneous boundary conditions.

So, we first try to find a $u_0(x,t)$ that satisfies the same boundary conditions as $u(x,t)$:

$$u_{0x}(0,t) = g_0(t) \qquad u_{0x}(\ell,t) = g_1(t)$$

This u_0 *does not* have to satisfy the PDE nor IC, which allows us to take something simple for it.

A linear function of x often works:

$$u_0(x,t) = A(t) + B(t)x$$

Unfortunately, if we put this in the two boundary conditions above, we get two equations for B alone, since A differentiates away.

Let's try quadratic:

$$u_0(x,t) = A(t) + B(t)x + C(t)x^2$$

If we put this in the two boundary conditions, we get two equations for B and C:

$$B(t) = g_0(t) \qquad B(t) + 2C(t)\ell = g_1(t)$$

We can solve this for B and C, (and take $A = 0$), which then gives u_0:

$$\boxed{u_0(x,t) = g_0(t)x + \frac{g_1(t) - g_0(t)}{2\ell}x^2}$$

This u_0 satisfies the BC but not the PDE or IC.

Please keep in mind what we know, and what we do not know. Since we (supposedly) have been given functions $f(x)$, $g_0(t)$, and $g_1(t)$, function u_0 is from now on a *known* quantity, given above.

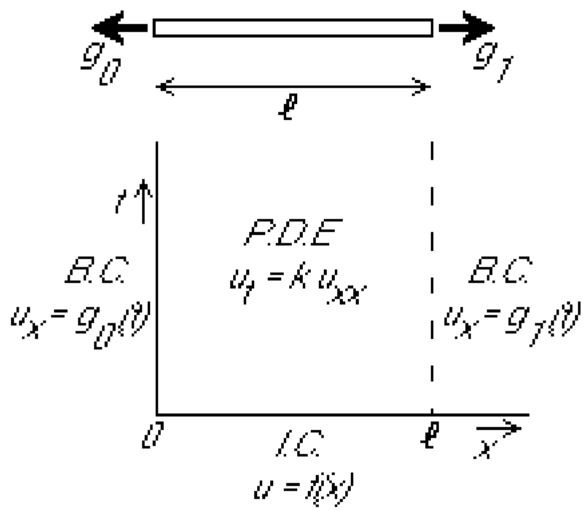
You might find a suitable u_0 in other ways. In problems with steady boundary conditions, the steady solution of the problem is often the best choice for u_0. Unfortunately, our boundary conditions are not steady. (Since g_0 and g_1 depend on time.) Also, a steady solution might be less easy to write down than a polynomial in x, for some problems.

Having found u_0, define a new unknown v as the remainder when u_0 is substracted from u:

$$v \equiv u - u_0$$

We now solve the problem by finding v. When we have found v, we simply add u_0, already known, back in to get u.

To do so, first, of course, we need the problem for v to solve. We get it from the problem for u by everywhere replacing u by $u_0 + v$. Let's take the picture of the problem for u in front of us and start converting.



First take the boundary conditions at $x = 0$ and $x = \ell$:

$$u_x(0,t) = g_0(t) \qquad u_x(\ell,t) = g_1(t)$$

Replacing u by $u_0 + v$:

$$u_{0x}(0,t) + v_x(0,t) = g_0(t) \qquad u_{0x}(\ell,t) + v_x(\ell,t) = g_1(t)$$

But since by construction $u_{0x}(0,t) = g_0$ and $u_{0x}(\ell,t) = g_1$,

$$v_x(0,t) = 0 \qquad v_x(\ell,t) = 0$$

Note the big thing: while the boundary conditions for v are similar to those for u, they are *homogeneous*. We will get a Sturm-Liouville problem in the x-direction for v where we did not for u. That is what u_0 does for us.

We continue finding the rest of the problem for v. We replace u by $u_0 + v$ into the PDE $u_t = \kappa u_{xx}$ and take all u_0 terms to the right hand side:

$$v_t = \kappa v_{xx} + q$$

where q is the collection of all the u_0 terms; $q(x,t) = -u_{0t} + \kappa u_{0xx}$, or, written out

$$\boxed{q(x,t) = -g_0'(t)x - \frac{g_1'(t) - g_0'(t)}{2\ell}x^2 + \kappa\frac{g_1(t) - g_0(t)}{\ell}}$$

Hence q is now a *known* function, just like u_0.

Note that the homogeneous PDE for u turned into an inhomogeneous PDE for v. But separation of variables can handle inhomogeneous PDEs just fine.

The final part of the problem for u that we have not converted yet is the initial condition. We replace u by $u_0 + v$ in $u(x,0) = f(x)$ and take u_0 to the other side:

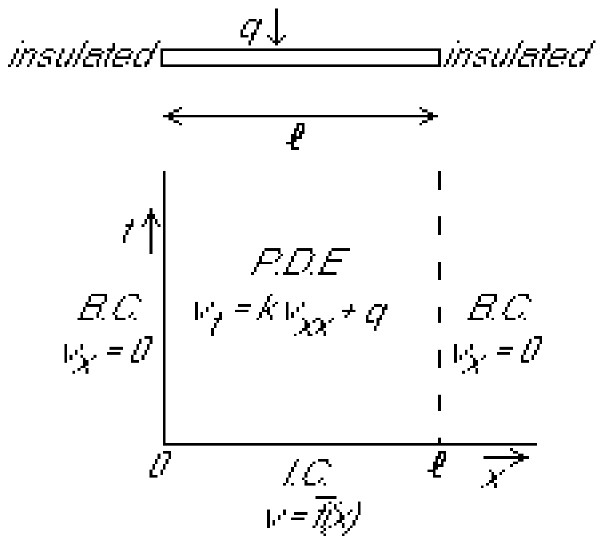
$$v(x,0) = \bar{f}(x)$$

where $\bar{f}(x)$ is $f(x) - u_0(x,0)$:

$$\boxed{\bar{f}(x) = f(x) - g_0(0)x - \frac{g_1(0) - g_0(0)}{2\ell}x^2}$$

Again, $\bar{f}$ is now a known function.

The problem for v can now be summarized as in the graph below, noting that $\bar{f}$ and q are known functions:



Using separation of variables, we can find the solution for v in the form:

$$v(x,t) = \sum_n v_n(t) X_n(x).$$

We already know how to do that! (Don't worry, we will go over the steps anyway.) Having found v, we will simply add u_0 to find the asked temperature u.

4 7.19, §4 Eigenfunctions

To find the eigenfunctions X_n, substitute a trial solution $v = T(t)X(x)$ into the *homogeneous part* of the PDE, $v_t = \kappa v_{xx} + q$. Remember: ignore the inhomogeneous part q when finding the eigenfunctions.

$$T'X = \kappa T X''$$

Separate variables:

$$\frac{T'}{\kappa T} = \frac{X''}{X} = \text{ constant } = -\lambda$$

As always, λ cannot depend on x since the left hand side does not. Also, λ cannot depend on t since the middle does not. So λ must be a constant.

We get the following Sturm-Liouville problem for the eigenfunctions $X(x)$:

$$-X'' = \lambda X \qquad X'(0) = 0 \qquad X'(\ell) = 0$$

The last two equations are the boundary conditions on v which we made homogeneous: the x-derivative is zero at both ends of the bar. It is a constant coefficient ODE, so we write the characteristic equation:

$$k^2 + \lambda = 0 \quad \Longrightarrow \quad k = \pm\sqrt{-\lambda} = \pm \mathrm{i}\sqrt{\lambda}$$

We again examine each possibility for λ in turn.

Case $\lambda < 0$*:*

Since $k = \pm\sqrt{-\lambda}$

$$X = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$$

Boundary conditions:

$$X'(0) = 0 = A\sqrt{-\lambda} - B\sqrt{-\lambda} \quad \Longrightarrow \quad B = A$$

$$X'(\ell) = 0 = A\left(\sqrt{-\lambda}e^{\sqrt{-\lambda}\ell} - \sqrt{-\lambda}e^{-\sqrt{-\lambda}\ell}\right) \quad \Longrightarrow \quad A = 0$$

No nontrivial (nonzero) solutions.

Case $\lambda = 0$*:*

Since $k_1 = k_2 = 0$:

$$X = A + Bx$$

Boundary conditions:

$$X'(0) = 0 = B \qquad X'(\ell) = 0 = B$$

hence $X = A$. No undetermined constants in eigenfunctions! Get a basis by choosing any nonzero value for A. Simplest is to choose 1:

$$X_0(x) = 1$$

Case $\lambda > 0$*:*

Since $k = \pm\sqrt{-\lambda} = \pm\mathrm{i}\sqrt{\lambda}$:

$$X = A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right)$$

Boundary conditions:

$$X'(0) = 0 = A\sqrt{\lambda}$$

We see that A must be zero, so X will be a cosine. The other boundary condition is:

$$X'(\ell) = 0 = -B\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)$$

Nontrivial solutions $B \neq 0$ exist only if

$$\sin\left(\sqrt{\lambda}\ell\right) = 0$$

A sine is zero when its argument, $\sqrt{\lambda}\ell$, equals π, 2π, 3π, so we get the eigenvalues:

$$\sqrt{\lambda_1} = \frac{\pi}{\ell}, \sqrt{\lambda_2} = \frac{2\pi}{\ell}, \sqrt{\lambda_3} = \frac{3\pi}{\ell}, \ldots$$

and corresponding eigenfunctions, taking each $B = 1$:

$$X_1 = \cos\left(\frac{\pi x}{\ell}\right), X_2 = \cos\left(\frac{2\pi x}{\ell}\right), X_3 = \cos\left(\frac{3\pi x}{\ell}\right), \ldots$$

Total:

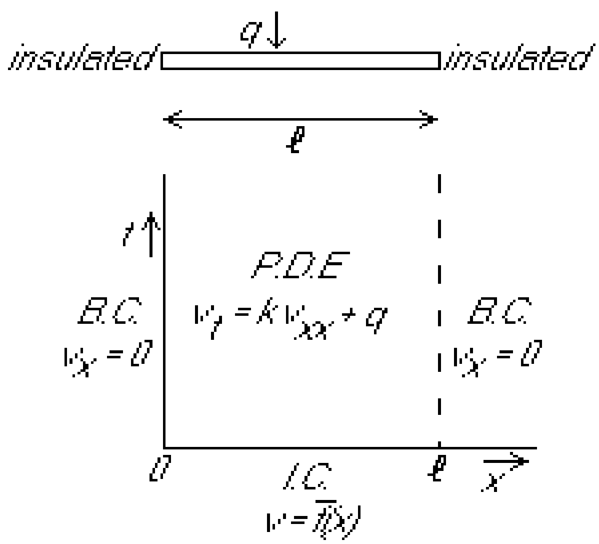
We found eigenfunctions both for $\lambda = 0$ and $\lambda > 0$, but we can write a single formula that works for all of them:

$$\boxed{\lambda_n = \frac{n^2\pi^2}{\ell^2} \qquad X_n = \cos\left(\frac{n\pi x}{\ell}\right) \qquad (n = 0, 1, 2, 3, \ldots)}$$

The case $n = 0$ gives the eigenvalue and eigenfunction for $\lambda = 0$, and the other values of n the ones for $\lambda > 0$.

5 7.19, §5 Solve

We again expand *everything* in the problem for v in a Fourier series:



$$v = \sum_{n=0}^{\infty} v_n(t) X_n(x) \quad \bar{f} = \sum_{n=0}^{\infty} \bar{f}_n X_n(x) \quad q = \sum_{n=0}^{\infty} q_n(t) X_n(x)$$

Since $q(x)$ and $\bar{f}(x)$ are known functions, we can find their Fourier coefficients from orthogonality:

$$\bar{f}_n = \frac{\int_0^\ell \bar{f}(x)\cos(n\pi x/\ell)\,\mathrm{d}x}{\int_0^\ell \cos^2(n\pi x/\ell)\,\mathrm{d}x}$$

$$q_n(t) = \frac{\int_0^\ell \bar{q}(x,t)\cos(n\pi x/\ell)\,\mathrm{d}x}{\int_0^\ell \cos^2(n\pi x/\ell)\,\mathrm{d}x}$$

Don't get caught! The integral in the bottom equals π *except* for $n = 0$, when it equals 2π. Do these two integral yourself and figure out why the answer for arbitrary n is not correct for $n = 0$.

(It is always a good idea to check that what you are doing for arbitrary n also works for $n = 0$.)

So the Fourier coefficients f_n are now known constants, and the $q_n(t)$ are now known functions of t. Though in actual application, numerical integration may be needed to find them. During finals, I usually make the functions f, g_0 and g_1 simple enough that you can do the integrals analytically.

Now write the PDE using the Fourier series: $v_t = \kappa v_{xx} + q$:

$$\sum_{n=0}^{\infty} \dot{v}_n(t)X_n(x) = \kappa \sum_{n=0}^{\infty} v_n(t)X_n''(x) + \sum_{n=0}^{\infty} q_n(t)X_n(x)$$

Looking in the previous section, the Sturm-Liouville ODE was $-X'' = \lambda X$, so the PDE simplifies to:

$$\sum_{n=0}^{\infty} \dot{v}_n(t)X_n(x) = -\kappa \sum_{n=0}^{\infty} \lambda_n v_n(t)X_n(x) + \sum_{n=0}^{\infty} q_n(t)X_n(x)$$

It will always simplify or you made a mistake.

Don't start dividing by the sum now. A sum is a symbol, not a number, and you cannot divide by symbols. Instead, bring all terms together in one sum:

$$\sum_{n=0}^{\infty} \left(\dot{v}_n(t) + \kappa\lambda_n v_n(t) - q_n(t)\right) X_n(x) = 0$$

Since the eigenfunctions are independent, they can only produce zero if every single coefficient is zero:

$$\dot{v}_n(t) + \kappa\lambda_n v_n(t) = q_n(t)$$

The result is still like we "divided" the previous equations by $\sum_n X_n$, but you would of course never want to be heard saying that!

In any case, we have obtained an ODE for each v_n which is again constant coefficient, but is inhomogeneous. Solve the homogeneous equation first. The characteristic polynomial is

$$k + \kappa\lambda = 0$$

so the homogeneous solution is

$$v_{nh} = A_n e^{-\kappa\lambda_n t}$$

For the inhomogeneous equation, since we do not know the actual form of the functions q, undetermined constants is not a possibility. So we use variation of parameter:

$$v_n = A_n(t)e^{-\kappa\lambda_n t}$$

Plugging into the ODE produces

$$A_n' e^{-\kappa\lambda_n t} + 0 = q_n(t)$$

We integrate this equation to find A_n. I could write the solution using an indefinite integral:

$$A_n(t) = \int q_n(t)e^{\kappa\lambda_n t}\,\mathrm{d}t$$

But that has the problem that the integration constant is not explicitly shown, which makes it impossible to apply the initial condition. We can however write the anti-derivative using an integral with limits and an explicit integration constant as:

$$A_n(t) = \int_{\tau=0}^{t} q_n(\tau)e^{\kappa\lambda_n \tau}\,\mathrm{d}\tau + A_{n0}$$

You can check using Leibnitz that the derivative is exactly what it should be. (Also, the lower limit does not really have to be zero; you could start the integration from 1, if it would be simpler. The important thing is that the upper limit is the independent variable t.)

Putting this in

$$v_n = A_n(t)e^{-\kappa\lambda_n t}$$

we get, cleaned up:

$$v_n(t) = \int_{\tau=0}^{t} q_n(\tau)e^{-\kappa\lambda_n (t-\tau)}\,\mathrm{d}\tau + A_{n0}e^{-\kappa\lambda_n t}$$

Next, write the IC $v(x,0) = \bar{f}(x)$ using Fourier series:

$$\sum_{n=0}^{\infty} v_n(0)X_n(x) = \sum_{n=0}^{\infty} \bar{f}_n X_n(x)$$

This gives us initial conditions for the v_n:

$$v_n(0) = \bar{f}_n = A_{n0}$$

the latter from above, and hence

$$\boxed{v_n(t) = \int_{\tau=0}^{t} q_n(\tau)e^{-\kappa\lambda_n (t-\tau)}\,\mathrm{d}\tau + \bar{f}_n e^{-\kappa\lambda_n t}}$$

6 7.19, §6 Total

Collecting it all together, the solution is to compute the $\bar{f}_n$:

$$\bar{f}_0 = \frac{1}{2\pi}\int_0^\ell \bar{f}(x)\cos(n\pi x/\ell)\,\mathrm{d}x$$

$$\bar{f}_n = \frac{1}{\pi}\int_0^\ell \bar{f}(x)\cos(n\pi x/\ell)\,\mathrm{d}x \qquad (n > 0)$$

where

$$\bar{f}(x) = f(x) - g_0(0)x - \frac{g_1(0) - g_0(0)}{2\ell}x^2$$

Also compute:

$$q_0(t) = \frac{1}{2\pi}\int_0^\ell q(x,t)\cos(n\pi x/\ell)\,\mathrm{d}x$$

$$q_n(t) = \frac{1}{\pi}\int_0^\ell q(x,t)\cos(n\pi x/\ell)\,\mathrm{d}x \qquad (n > 0)$$

where

$$q(x,t) = -g_0'(t)x - \frac{g_1'(t) - g_0'(t)}{2\ell}x^2 + \kappa\frac{g_1(t) - g_0(t)}{\ell}$$

Then the temperature is:

$$\begin{aligned} u(x,t) &= g_0(t)x + \frac{g_1(t) - g_0(t)}{2\ell}x^2 \\ &+ \sum_{n=0}^{\infty}\left[\int_{\tau=0}^{t} q_n(\tau)e^{-\kappa n^2\pi^2(t-\tau)/\ell^2}\,\mathrm{d}\tau + \bar{f}_n e^{-\kappa n^2\pi^2 t/\ell^2}\right]\cos(n\pi x/\ell) \end{aligned}$$

7 7.19, §7 More Fun

We can, if we want, write the solution for v in other ways that may be more efficient numerically. The solution was:

$$v(x,t) = \sum_{n=0}^{\infty}\left[\int_{\tau=0}^{t} q_n(\tau)e^{-\kappa n^2\pi^2(t-\tau)/\ell^2}\,\mathrm{d}\tau + \bar{f}_n e^{-\kappa n^2\pi^2 t/\ell^2}\right]\cos(n\pi x/\ell)$$

The first part is due to the inhomogeneous term q in the PDE, the second due to the initial condition $v(x,0) = \bar{f}(x)$

Looking at the second term first,

$$v_f \equiv \sum_{n=0}^{\infty}\bar{f}_n e^{-\kappa n^2\pi^2 t/\ell^2}\cos(n\pi x/\ell)$$

we can substitute in the orthogonality relationship for $\bar f(x)$:

$$v_f = \sum_{n=0}^{\infty} \frac{1}{[2]\pi} \int_0^\ell \bar f(\xi) \cos(n\pi\xi/\ell)\,\mathrm{d}\xi\, e^{-\kappa n^2\pi^2 t/\ell^2} \cos(n\pi x/\ell)$$

and change the order of the terms to get:

$$v_f = \int_0^\ell \left[\sum_{n=0}^{\infty} \frac{1}{[2]\pi} \cos(n\pi\xi/\ell) e^{-\kappa n^2\pi^2 t/\ell^2} \cos(n\pi x/\ell) \right] \bar f(\xi) d\xi$$

We define a shorthand symbol for the term within the square brackets:

$$G(x,t,\xi) \equiv \sum_{n=0}^{\infty} \frac{1}{[2]\pi} \cos(n\pi\xi/\ell) \cos(n\pi x/\ell) e^{-\kappa n^2\pi^2 t/\ell^2}$$

Since this does not depend on what function $\bar f(x)$ is, we can evaluate G once and for. For any $\bar f(x)$, the corresponding temperature is then simply found as:

$$v_f = \int_0^\ell G(x,t,\xi) \bar f(\xi) d\xi$$

Function $G(x,t,\xi)$ by itself is the temperature $v(x,t)$ if $\bar f$ is a single spike of heat initially located at $x = \xi$. Mathematically, G is the solution for v if $\bar f(x)$ is the "delta function" $\delta(x-\xi)$.

Now look at the first term,

$$v_q \equiv \sum_{n=0}^{\infty} \int_{\tau=0}^{t} q_n(\tau) e^{-\kappa n^2\pi^2 (t-\tau)/\ell^2}\,\mathrm{d}\tau \cos(n\pi x/\ell)$$

We again plug in the orthogonality expression for q_n:

$$v_q = \sum_{n=0}^{\infty} \int_{\tau=0}^{t} \frac{1}{[2]\pi} \int_0^\ell q(\xi,\tau) \cos(n\pi\xi/\ell)\,\mathrm{d}\xi\, e^{-\kappa n^2\pi^2 (t-\tau)/\ell^2}\,\mathrm{d}\tau \cos(n\pi x/\ell)$$

and rewrite

$$v_q = \int_{\tau=0}^{t} \int_0^\ell \left[\sum_{n=0}^{\infty} \alpha_n \cos(n\pi\xi/\ell) \cos(n\pi x/\ell) e^{-\kappa n^2\pi^2 (t-\tau)/\ell^2} \right] q(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

We see that

$$v_q = \int_{\tau=0}^{t} \int_0^\ell G(x,t-\tau,\xi) q(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

where the function G is exactly the same as it was before. However, $G(x,t-\tau,\xi)$ describes the temperature due to a spike of heat added to the bar at a time τ and position ξ; it is called the Green's function.

The fact that solving the initial value problem ($\bar f$), also solves the inhomogeneous PDE problem (q) is known as Duhamel principle. The idea behind this principle is that fuction $q(x,t)$ can be "sliced up" as a cake. The contribution of each slice $\tau \le t \le \tau + \mathrm{d}\tau$ of the cake to the solution v can be found as an initial value problem with $q\,\mathrm{d}\tau$ as the initial condition at time τ.

Procedure

1 Description

This document describes the separation of variables method to solve some partial (not ordinary) differential equations. You might want to look at one or two examples first and then read this.

2 Form of the solution

Before starting the process, you should have some idea of the form of the solution you are looking for. Some experience helps here.

For example, for unsteady heat conduction in a bar of length ℓ, with homogeneous end conditions, the temperature u would be written

$$u(x,t) = \sum_n u_n(t) X_n(x)$$

where the X_n are chosen eigenfunctions and the u_n are computed Fourier coefficients of u. The separation of variables procedure allows you to choose the eigenfunctions cleverly.

For a uniform bar, you will find sines and/or cosines for the functions X_n. In that case the above expansion for u is called a Fourier series. In general it is called a generalized Fourier series.

After the functions X_n have been found, the Fourier coefficients u_n can simply be found from substituting the expression above for u in the given PDE and initial conditions. (The boundary conditions are satisfied when you choose the eigenfunctions X_n.) If there are other functions in the PDE or IC, they too need to be expanded in a Fourier series.

If the problem was axially symmetric heat conduction through the wall of a pipe, the temperature would still be written

$$u(r,t) = \sum_n u_n(t) R_n(r)$$

but the expansion functions R_n would now be found to be Bessel functions, not sines or cosines.

For heat conduction through a pipe wall without axial symmetry, still with homogeneous boundary conditions, the temperature would be written

$$u(r,\theta,t) = \sum_{n,i} u_n^i(r,t)\Theta_n^i(\theta) = \sum_{n,i}\sum_m u_{nm}^i(t) R_{nm}(r)\Theta_n^i(\theta)$$

where the eigenfunctions Θ_n^i turn out to be sines and cosines and the eigenfunctions R_{nm} Bessel functions. Note that in the first sum, the temperature is written as a simple Fourier series in θ, with coefficients u_n that of course depend on r and t. Then in the second sum, these coefficients themselves are written as a (generalized) Fourier series in r with coefficients u_{nm} that depend on t.

(For steady heat conduction, the coordinate "t" might actually be a second spatial coordinate. For convenience, we will refer to conditions at given values of t as "initial conditions", even though they might physically really be boundary conditions.)

3 Limitations

First, the equation must be linear. (After all, the solution is found as an sum of simple solutions.)

The PDE does not necessarily have to be a constant coefficient equation, but the coefficients cannot be too complicated. You should be able to separate variables. Something like $\sin(xt)$ is not separable.

Further, the boundaries must be at constant values of the coordinates. For example, for the heat conduction in a bar, the ends of the bar must be at fixed locations $x = 0$ and $x = \ell$. The bar cannot expand, since then the end points would depend on time.

You may be able to find fixes for problems such as the ones above, though. For example, the nonlinear Burger's equation can be converted into the linear heat equation.

4 Procedure

The general lines of the procedure are to choose the eigenfunctions and then to find the (generalized) Fourier coefficients of the desired solution u. In more detail, the steps are:

1. **Make the boundary conditions for the eigenfunctions X_n homogeneous**

   For heat conduction in a bar, this means that if nonzero end temperatures or heat fluxes through the ends are given, you will need to eliminate these.

   Typically, you eliminate nonzero boundary conditions for the eigenfunctions by substracting a function u_0 from u that satisfies these boundary conditions. Since u_0 only needs to satisfy the boundary conditions, not the partial differential equation or the initial conditions, such a function is easy to find.

   If the boundary conditions are steady, you can try substracting the steady solution, if it exists. More generally, a low degree polynomial can be tried, say $u_0 = A + Bx + Cx^2$, where the coefficients are chosen to satisfy the boundary conditions.

   Afterwards, carefully identify the partial differential equation and initial conditions satisfied by the new unknown $v = u - u_0$. (They are typically different from the ones for u.)

2. **Identify the eigenfunctions X_n**

   To do this substitute a single term TX into the *homogeneous* partial differential equation. Then take all terms involving X and the corresponding independent variable to one side of the equation, and T and the other independent variables to the other side. (If that turns out to be impossible, the PDE cannot be solved using separation of variables.)

Now, since the two sides of the equation depends on different coordinates, they must both be equal to some constant. The constant is called the eigenvalue.

Setting the X-side equal to the eigenvalue gives an ordinary differential equation. Solve it to get the eigenfunctions X_n. In particular, you get the complete set of eigenfunctions X_n by finding *all* possible solutions to this ordinary differential equation. (If the ordinary differential equation problem for the X_n turns out to be a regular Sturm-Liouville problem of the type described in the next section, the method is guaranteed to work.)

The equation for T is usually safest ignored. The book tells you to also solve for the T_n, to get the Fourier coefficients v_n, but if you have an inhomogeneous PDE, you have to mess around to get it right. Also, it is confusing, since the eigenfunctions X_n *do not* have undetermined constants, but the coefficients v_n *do*. It are the undetermined constants in v_n that allow you to satisfy the initial conditions. They probably did not make this fundamental difference between the functions X_n and the coefficients u_n clear in your undergraduate classes.

There is one case in which you do need to use the equation for the T_n: in problems with more than two independent variables, where you want to expand the T_n themselves in a generalized Fourier series. That would be the case for the pipe wall without axial symmetry. Simply repeat the above separation of variables process for the PDE satisfied by the T_n.

3. **Find the coefficients**

   Now find the Fourier coefficients v_n (or v_{nm} for three independent variables) by putting the Fourier series expansion into the PDE and IC.

   While doing this, you will also need to expand the inhomogeneous terms in the PDE and IC into a Fourier series of the same form. You can find the coefficients of these Fourier series using the orthogonality property described in the next section.

   You will find that the PDE produces ordinary differential equations for the individual coefficients. And the integration constants in solving those equations follow from the initial conditions.

Afterwards you can play around with the solution to get other equivalent forms. For example, you can interchange the order of summation and integration (which results from the orthogonality property) to put the result in a Green's function form, etcetera.

5 Sturm-Liouville Problems

The eigenvalue problems you get for the eigenfunctions may vary, and sometimes that produces a different orthogonality expression. You can figure out what is the correct expression by writing your ODE in the standard form of a Sturm-Liouville problem:

$$\boxed{-pX'' - p'X' + qX = \lambda \bar{\mathrm{r}} X,}$$

where $X(x)$ is the eigenfunction to be found and $p(x) > 0$, $q(x)$, and $\bar{\mathrm{r}}(x) > 0$ are given functions. The distinguishing feature is that the coefficient of the second, X', term is the derivative of the coefficient of the first, X'' term.

Starting with an arbitrary second order linear O.D.E., you can achieve such a form by multiplying the entire O.D.E. with a suitable factor.

The boundary conditions may either be periodic ones,

$$X(b) = X(a) \qquad X'(b) = X'(a),$$

or they can be homogeneous of the form

$$AX(a) + BX'(a) = 0 \qquad CX(b) + DX'(b) = 0,$$

where A, B, C, and D are given constants. Note the important fact that a Sturm-Liouville problem must be completely homogeneous: $X = 0$ must be a solution.

If you have a Sturm-Liouville problem, simply (well, simply ...) solve it. The solutions only exists for certain values of λ. Make sure you find *all* solutions, or you are in trouble. They will form an infinite sequence of 'eigenfunctions', say $X_1(x)$, $X_2(x)$, $X_3(x)$, ... with corresponding 'eigenvalues' λ_1, λ_2, λ_3, ... that go off to positive infinity.

You can represent *arbitrary* functions, say $f(x)$, on the interval $[a, b]$ as a generalized Fourier series:

$$f(x) = \sum_n f_n X_n(x).$$

If you know $f(x)$, the orthogonality relation that gives the generalized Fourier coefficients f_n is

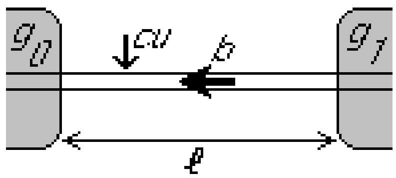
$$f_n = \frac{\int_a^b f(x) X_n(x) \bar{\mathrm{r}}(x) dx}{\int_a^b X_n^2(x) \bar{\mathrm{r}}(x) dx}$$

Now you know why you need to write your Sturm-Liouville problem in standard form: it allows you to pick out the weight factor $\bar{\mathrm{r}}$ that you need to put in the orthogonality relation!

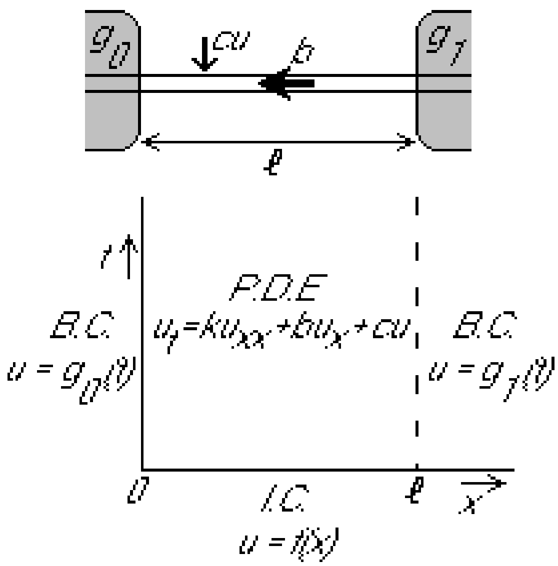
7.22

1 7.22, §1 Asked

Asked: Find the unsteady temperature distribution in the moving bar below for arbitrary position and time if the initial distribution at time zero and the temperatures of the ends are known.



2 7.22, §2 PDE Model



- Finite domain $\bar{\Omega}$: $0 \leq x \leq \ell$
- Unknown temperature $u = u(x,t)$
- Parabolic
- One initial condition

- Two Dirichlet boundary conditions
- Constant κ

Try separation of variables:

$$\sum_n C_n(t)X_n(x)$$

3 7.22, §3 Boundaries

Find u_0:

The x-boundary conditions are inhomogeneous:

$$u(0,t) = g_0(t) \qquad u(\ell,t) = g_1(t)$$

Try finding a u_0 satisfying these boundary conditions:

$$u_0(0,t) = g_0(t) \qquad u_0(\ell,t) = g_1(t)$$

Try a linear expression:

$$u_0 = A(t) + B(t)x$$

$$A(t) = g_0(t) \qquad A(t) + B(t)\ell = g_1(t)$$

This can be solved to find

$$\boxed{u_0(x,t) = g_0(t) + \frac{g_1(t) - g_0(t)}{\ell}x}$$

Identify the problem for the remainder:

Substitute $u = u_0 + v$ into the boundary conditions:

$$u_0(0,t) + v(0,t) = g_0(t) \qquad u_0(\ell,t) + v(\ell,t) = g_1(t)$$

gives

$$v(0,t) = 0 \qquad v(\ell,t) = 0$$

Substitute $u = u_0 + v$ into the PDE $u_t = \kappa u_{xx} + bu_x + cu$:

$$v_t = \kappa v_{xx} + bv_x + cv + q$$

where

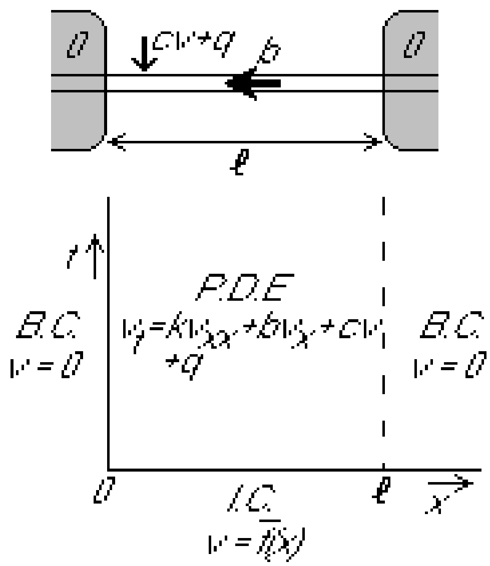
$$\boxed{q(x,t) = -g_0'(t) - \frac{g_1'(t) - g_0'(t)}{\ell}x + b\frac{g_1(t) - g_0(t)}{\ell} + c\left(g_0(t) + \frac{g_1(t) - g_0(t)}{\ell}x\right)}$$

Substitute $u = u_0 + v$ into the IC $u(x,0) = f(x)$:

$$v(x,0) = \bar{f}(x)$$

$$\boxed{\bar{f}(x) = f(x) - g_0(0) - \frac{g_1(0) - g_0(0)}{\ell} x}$$

The problem for v is therefor:



4 7.22, §4 Eigenfunctions

Substitute $v = T(t)X(x)$ into the *homogeneous* PDE $v_t = \kappa v_{xx} + b v_x + cv$:

$$T'X = \kappa T X'' + bTX' + cTX$$

Separate:

$$\frac{T'}{T} = \kappa \frac{X''}{X} + b\frac{X'}{X} + c = \text{ constant } = -\lambda$$

The Sturm-Liouville problem for X is now:

$$-\kappa X'' - bX' - cX = \lambda X \qquad X(0) = 0 \qquad X(\ell) = 0$$

This is a constant coefficient ODE, with a characteristic polynomial:

$$\kappa k^2 + bk + (c + \lambda) = 0$$

The fundamentally different cases are now two real roots (discriminant positive), a double root (discriminant zero), and two complex conjugate roots (discriminant negative.) We do each in turn.

Case $b^2 - 4\kappa(c + \lambda) > 0$*:*

Roots k_1 and k_2 real and distinct:

$$X = Ae^{k_1 x} + Be^{k_2 x}$$

Boundary conditions:

$$X(0) = 0 = A + B \quad \Longrightarrow \quad B = -A$$
$$X(\ell) = 0 = A\left(e^{k_1\ell} - e^{k_2\ell}\right) = 0$$

No nontrivial solutions since the roots are different.

Case $b^2 - 4\kappa(c + \lambda) = 0$*:*

Since $k_1 = k_2 = k$:

$$X = Ae^{kx} + Bxe^{kx}$$

Boundary conditions:

$$X(0) = 0 = A \qquad X(\ell) = 0 = B\ell e^{k\ell}$$

No nontrivial solutions.

Case $b^2 - 4\kappa(c + \lambda) < 0$*:*

For convenience, we will write the roots of the characteristic polynomial more concisely as:

$$k_1 = -\mu + \mathrm{i}\omega \quad k_2 = -\mu - \mathrm{i}\omega$$

where according to the solution of the quadratic

$$\mu = \frac{b}{2\kappa} \qquad \omega = \frac{\sqrt{4\kappa(c + \lambda) - b^2}}{2\kappa}$$

Since it can be confusing to have too many variables representing the same thing, let's agree that μ is our "representative" for b, and ω our "representative" for λ. In terms of these representatives, the solution is, after clean-up,

$$X = e^{-\mu x}\left(A\cos(\omega x) + B\sin(\omega x)\right)$$

Boundary conditions:

$$X(0) = 0 = A \qquad X(\ell) = 0 = e^{-\mu\ell}B\sin(\omega\ell)$$

Nontrivial solutions $B \neq 0$ can only occur if

$$\sin(\omega\ell) = 0 \quad \Longrightarrow \quad \omega_n = n\pi/\ell \quad (n = 1, 2, \ldots)$$

which gives us our eigenvalues, by substituting in for ω:

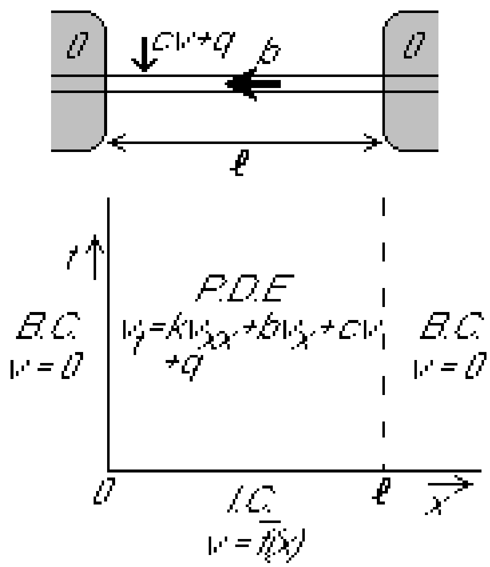
$$\boxed{\lambda_n = \frac{\kappa n^2\pi^2}{\ell^2} + \frac{b^2}{4\kappa} - c \quad (n = 1, 2, 3, ...)}$$

Also, choosing each $B = 1$:

$$\boxed{X_n = e^{-\mu x} \sin\left(n\pi x/\ell\right) \quad (n = 1, 2, 3, ...)}$$

5 7.22, §5 Solve

Expand all variables in the problem for v in a Fourier series:



$$v = \sum_{n=1}^{\infty} v_n(t) X_n(x) \quad \bar{f} = \sum_{n=1}^{\infty} \bar{f}_n X_n(x) \quad q = \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

We want to first find the Fourier coefficients of the known functions $\bar{f}$ and q. Unfortunately, the ODE found in the previous section,

$$-\kappa X'' - bX' - cX = \lambda X$$

is *not* in standard Sturm-Liouville form: the derivative of the first, X'', coefficient, $-\kappa$, is zero, not $-b$. Let's try to make it OK by multiplying the entire equation by a factor, which will then be our $\bar{r}$.

$$-\bar{\mathrm{r}}\kappa X'' - \bar{\mathrm{r}}bX' - \bar{\mathrm{r}}cX = \lambda\bar{\mathrm{r}}X$$

We want that the second coefficient is the derivative of the first:

$$\bar{\mathrm{r}}b = \frac{\mathrm{d}}{\mathrm{d}x}\left(\bar{\mathrm{r}}\kappa\right)$$

This is a simple ODE for the $\bar{\mathrm{r}}$ we are trying to find, and a valid solution is:

$$\bar{\mathrm{r}} = e^{bx/\kappa} = e^{2\mu x}$$

Having found $\bar{\mathrm{r}}$, we can write the orthogonality relationships for the generalized Fourier coefficients of $\bar{f}$ and q (remember that $X_n = e^{-\mu x}\sin(n\pi x/\ell)$):

$$\boxed{\bar{f}_n = \frac{\int_{x=0}^{\ell} e^{\mu x}\bar{f}(x)\sin(n\pi x/\ell)\,\mathrm{d}x}{\int_{x=0}^{\ell}\sin^2(n\pi x/\ell)\,\mathrm{d}x}}$$

$$\boxed{q_n(t) = \frac{\int_{x=0}^{\ell} e^{\mu x}q(x,t)\sin(n\pi x/\ell)\,\mathrm{d}x}{\int_{x=0}^{\ell}\sin^2(n\pi x/\ell)\,\mathrm{d}x}}$$

The integrals in the bottoms equal $\ell/2$.

Expand the PDE $v_t = \kappa v_{xx} + bv_x + cv + q$ in a generalized Fourier series:

$$\sum_{n=1}^{\infty}\dot{v}_n(t)X_n(x) = \kappa\sum_{n=1}^{\infty}v_n(t)X_n''(x) + b\sum_{n=1}^{\infty}v_n(t)X_n'(x) + c\sum_{n=1}^{\infty}v_n(t)X_n(x) + \sum_{n=1}^{\infty}q_n(t)X_n(x)$$

Because of the choice of the X_n, $\kappa X'' + bX' + cX = -\lambda X$:

$$\sum_{n=1}^{\infty}\dot{v}_n(t)X_n(x) = -\sum_{n=1}^{\infty}\lambda_n v_n(t)X_n(x) + \sum_{n=1}^{\infty}q_n(t)X_n(x)$$

So, the ODE for the generalized Fourier coefficients of v becomes:

$$\dot{v}_n(t) + \lambda_n v_n(t) = q_n(t)$$

Expand the IC $v(x,0) = \bar{f}(x)$ in a generalized Fourier series:

$$\sum_{n=1}^{\infty}v_n(0)X_n(x) = \sum_{n=1}^{\infty}\bar{f}_n X_n(x)$$

so

$$v_n(0) = \bar{f}_n$$

Solve this O.D.E. and initial condition for v_n:

Homogeneous equation:

$$v_{nh} = A_n e^{-\lambda_n t}$$

Inhomogeneous equation:

$$A_n' e^{-\lambda_n t} + 0 = q_n(t)$$

$$A_n = \int_{\tau=0}^{t} q_n(\tau) e^{\lambda_n \tau} \, \mathrm{d}\tau + A_{n0}$$

$$v_n = A_n e^{-\lambda_n t}$$

$$v_n = \int_{\tau=0}^{t} q_n(\tau) e^{-\lambda_n (t-\tau)} \, \mathrm{d}\tau + A_{n0} e^{-\lambda_n t}$$

Initial condition: $A_{n0} = \bar{f}_n$.

$$v_n = \int_{\tau=0}^{t} q_n(\tau) e^{-\lambda_n (t-\tau)} \, \mathrm{d}\tau + \bar{f}_n e^{-\lambda_n t}$$

6 7.22, §6 Total

Total solution:

$$\mu = \frac{b}{2\kappa} \qquad \lambda_n = \frac{\kappa n^2 \pi^2}{\ell^2} + \lambda_0 \qquad \lambda_0 = \frac{b^2}{4\kappa} - c$$

$$\bar{f}(x) = f(x) - g_0(0) - \frac{g_1(0) - g_0(0)}{\ell} x$$

$$\bar{f}_n = \frac{2}{\ell} \int_{x=0}^{\ell} \bar{f}(x) e^{\mu x} \sin(n\pi x/\ell) \, \mathrm{d}x$$

$$q(x,t) = -g_0'(t) - \frac{g_1'(t) - g_0'(t)}{\ell} x + b \frac{g_1(t) - g_0(t)}{\ell} + c \left(g_0(t) + \frac{g_1(t) - g_0(t)}{\ell} x \right)$$

$$q_n(t) = \frac{2}{\ell} \int_{x=0}^{\ell} q(x,t) e^{\mu x} \sin(n\pi x/\ell) \, \mathrm{d}x$$

$$u = g_0(t) + \frac{g_1(t) - g_0(t)}{\ell} x + \sum_{n=1}^{\infty} \left[\int_{\tau=0}^{t} q_n(\tau) e^{-\lambda_n (t-\tau)} \, \mathrm{d}\tau + \bar{f}_n e^{-\lambda_n t} \right] e^{-\mu x} \sin(n\pi x/\ell)$$

Solution in the book is no good (check the boundary conditions.)

7 7.22, §7 Poor Method

Define a new unknown w by $u = we^{-\alpha x - \beta t}$. Put this in the PDE for u and choose α and β so that the w_x and w terms drop out. This requires:

$$u = we^{-\mu x - \lambda_0 t}$$

Then:

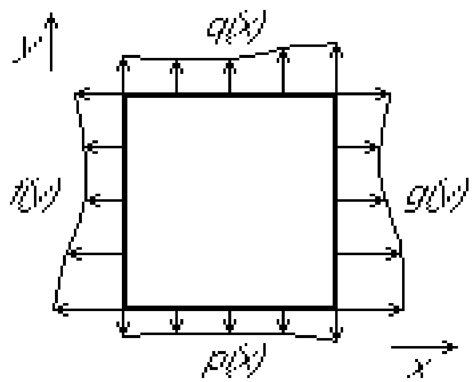
$$w_t = \kappa w_{xx} \quad w(x, 0) = e^{\mu x} f(x) \quad w(0, t) = e^{\lambda_0 t} g_0(t) \quad w(\ell, t) = e^{\mu \ell + \lambda_0 t} g_1(t)$$

No fun! Note that the generalized Fourier series coefficients for u become normal Fourier coefficients for w.

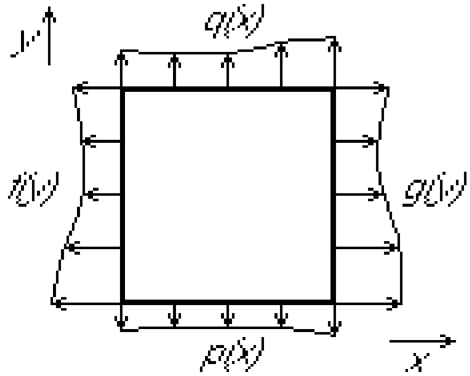
7.37

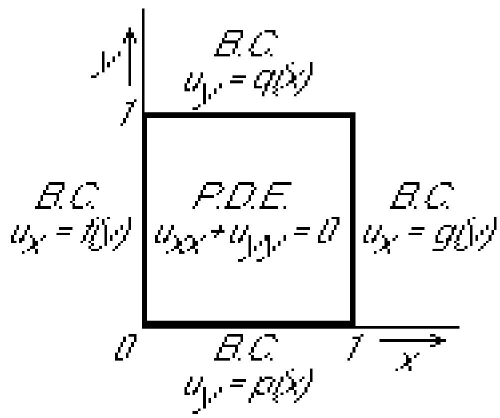
1 7.37, §1 Asked

Asked: Find the steady temperature distribution in the square plate/cross section below if the heat fluxes out of the sides are known.



2 7.37, §2 P.D.E. Model





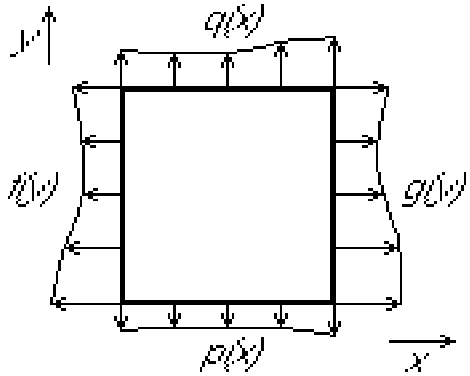
- Finite domain $\bar{\Omega}$: $0 \le x \le 1$, $0 \le y \le 1$.
- Unknown temperature $u = u(x, y)$
- Elliptic
- Four Neumann boundary conditions
- Integral constraint due to all Neumann B.C.s:

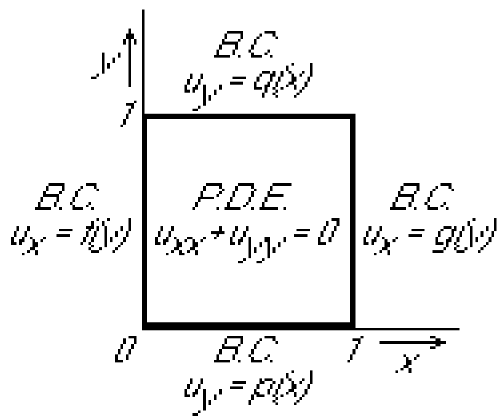
$$\int_0^1 p(x)\,\mathrm{d}x - \int_0^1 g(y)\,\mathrm{d}y - \int_0^1 q(x)\,\mathrm{d}x + \int_0^1 f(y)\,\mathrm{d}y = 0$$

Try separation of variables:

$$\sum_n u_n(y) X_n(x) \text{ or } \sum_n u_n(x) Y_n(y)$$

3 7.37, §3 Boundaries





Standard approach:

All boundary conditions are inhomogeneous. Our standard approach would be to set $u = u_0 + v$ where

$$u_{0x}(0,y) = f(y) \qquad u_{0x}(1,y) = g(y)$$

and then set

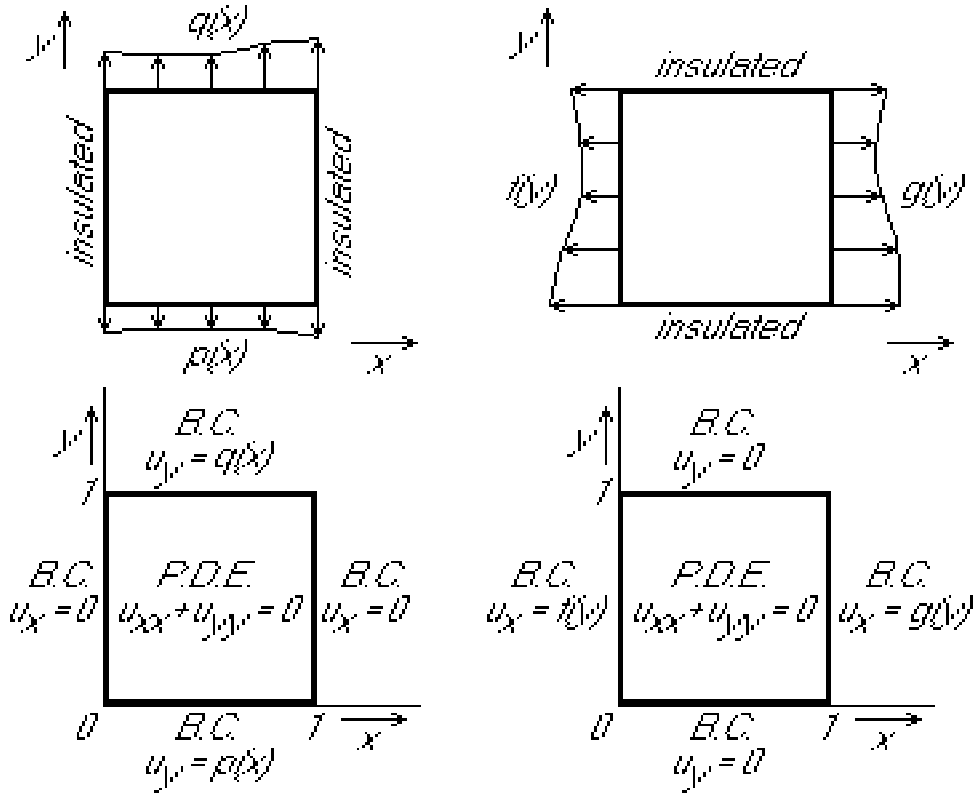
$$v = \sum_n v_n(y) X_n(x)$$

This would work without any problems. A u_0 quadratic in x would be fine. Of course, this choice for u_0 is quite arbitrary.

Alternative approach:

Instead, *we* will follow a more elegant procedure that does not require us to arbitrarily choose a u_0. Unfortunately, this alternative procedure will get us into some trouble.

The idea is that the given problem can be seen as the sum of two problems, each with homogeneous boundary conditions in one direction.



If we add the solutions u to the two problems together, we should get the solution to the original problem.

The *instructor* will solve the left hand problem. The *students* will solve the right hand problem, identify the difficulty, and fix it. Note that the four-problem procedure in the book has the problem even worse.

4 7.37, §4 Eigenfunctions

Substitute $u = T(y)X(x)$ into the homogeneous P.D.E. $u_{xx} + u_{yy} = 0$:

$$TX'' + T''X = 0$$

$$\frac{T''}{T} = -\frac{X''}{X} = \text{ constant } = \lambda$$

Since the instructor's x-boundary conditions are homogeneous, he has a Sturm-Liouville problem for X:

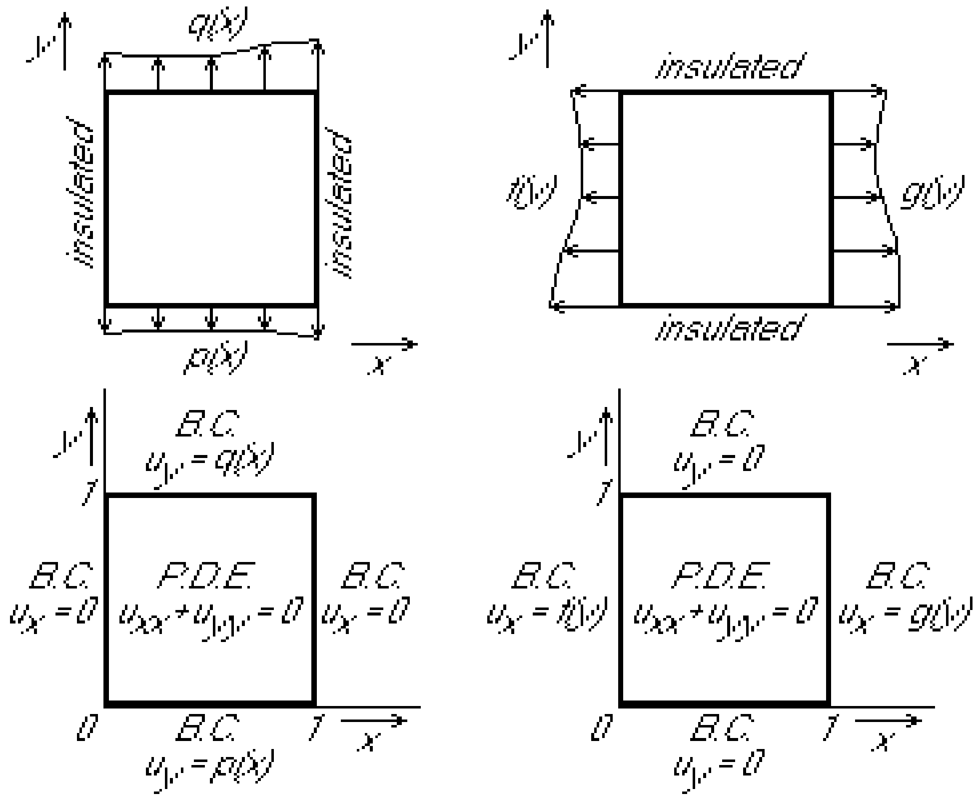
$$-X'' = \lambda X \qquad X'(0) = 0 \qquad X'(1) = 0$$

This was already solved in problem 7.19. Looking back there, substituting $\ell = 1$,

$$\lambda_n = n^2\pi^2 \qquad X_n = \cos(n\pi x) \qquad (n = 0, 1, 2, 3, \ldots)$$

5 7.37, §5 Solve

Expand all variables in the problem for u in a Fourier series:



$$u = \sum_{n=0}^{\infty} u_n(y) X_n(x) \quad p(x) = \sum_{n=0}^{\infty} p_n X_n(x) \quad q(x) = \sum_{n=0}^{\infty} q_n X_n(x)$$

$$p_n = \frac{\int_0^1 p(x) X_n(x)\,\mathrm{d}x}{\int_0^1 X_n^2(x)\,\mathrm{d}x}$$

$$q_n = \frac{\int_0^1 q(x) X_n(x)\,\mathrm{d}x}{\int_0^1 X_n^2(x)\,\mathrm{d}x}$$

Remember that the expression you find for the integrals in the bottom, $\frac{1}{2}$, does not work for $n = 0$, in which case it turns out to be 1.

Fourier-expand the PDE $u_{xx} + u_{yy} = 0$:

$$\sum_{n=0}^{\infty} u_n(y) X_n(x)'' + \sum_{n=0}^{\infty} u_n(y)'' X_n(x) = 0$$

Because of the Sturm-Liouville equation in the previous section

$$-\sum_{n=0}^{\infty} \lambda_n u_n(y) X_n(x) + \sum_{n=0}^{\infty} u_n(y)'' X_n(x) = 0$$

giving the ODE

$$u_n(y)'' - \lambda_n u_n(y) = 0$$

or substituting in the eigenvalue

$$u_n(y)'' - n^2\pi^2 u_n(y) = 0$$

Fourier-expand the BC $u_y(x,0) = p(x)$:

$$\sum_{n=0}^{\infty} u_n(0)' X_n(x) = \sum_{n=0}^{\infty} p_n X_n(x) \quad \Longrightarrow \quad u_n'(0) = p_n$$

Fourier-expand the BC $u_y(x,1) = q(x)$:

$$\sum_{n=0}^{\infty} u_n(1)' X_n(x) = \sum_{n=0}^{\infty} q_n X_n(x) \quad \Longrightarrow \quad u_n'(1) = q_n$$

Solve the above ODE and boundary conditions for u_n. It is a constant coefficient one, with a characteristic equation

$$k^2 - n^2\pi^2 = 0$$

Caution! Note that both roots are the same when $n = 0$. So we need to do the $n = 0$ case separately.

For $n \neq 0$ the solution is

$$u_n = A_n e^{n\pi y} + B_n e^{-n\pi y}$$

The boundary conditions above give two linear equations for A_n and B_n:

$$\left(\begin{array}{cc|c} n\pi & -n\pi & p_n \\ n\pi e^{n\pi} & -n\pi e^{-n\pi} & q_n \end{array} \right)$$

whic are best solved using Gaussian elimination. Rewriting the various exponentials in terms of sinh and cosh, the solution for the Fourier coefficients of u except $n = 0$ is:

$$u_n = -\frac{\cosh(n\pi[y-1])}{n\pi \sinh(n\pi)} p_n + \frac{\cosh(n\pi y)}{n\pi \sinh(n\pi)} q_n \quad (n = 1, 2, 3, \ldots)$$

For $n = 0$ the solution of the ODE is

$$u_0 = A_0 + B_0 y$$

Put in the boundary conditions to get equations for the integration constants A_0 and B_0:

$$u_0'(0) = B_0 = p_0 \qquad u_0'(1) = B_0 = q_0$$

Oops! We can only solve this if

$$p_0 = q_0$$

Looking above for the definition of those Fourier coefficients, we see we only have a solution if

$$\int_0^1 p(x)\,\mathrm{d}x = \int_0^1 q(x)\,\mathrm{d}x$$

Unfortunately, these two integrals will normally *not* be equal! Also, A_0 remains unknown. No problem! Students will explain and fix the problem.

6 7.37, §6 Total

First compute the Fourier coefficients of the given boundary conditions:

$$p_0 = \int_0^1 p(x)\,\mathrm{d}x \qquad p_n = 2\int_0^1 p(x)\cos(n\pi x)\,\mathrm{d}x \quad (n = 1, 2, \ldots)$$

$$q_0 = \int_0^1 q(x)\,\mathrm{d}x \qquad q_n = 2\int_0^1 q(x)\cos(n\pi x)\,\mathrm{d}x \quad (n = 1, 2, \ldots)$$

Then the solution is equal to:

$$\begin{aligned} u &= A_0 + p_0 x \\ &+ \sum_{n=1}^{\infty}\left[-\frac{\cosh(n\pi[y-1])}{n\pi\sinh(n\pi)}p_n + \frac{\cosh(n\pi y)}{n\pi\sinh(n\pi)}q_n\right]\cos(n\pi x) \end{aligned}$$

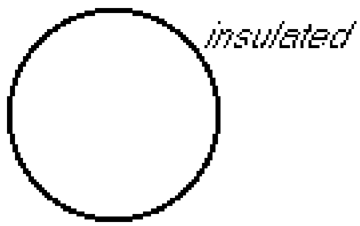
But this only satisfies the BC on the top of the plate if

$$\int_0^1 q(x)\,\mathrm{d}x = \int_0^1 p(x)\,\mathrm{d}x$$

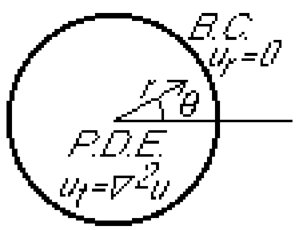
7.38 U

1 7.38 U, §1 Asked

Asked: Find the unsteady heat conduction in a disk if the perimeter is insulated. The initial temperature is given.



2 7.38 U, §2 PDE Model



- Finite domain $\bar{\Omega}$: $0 \le r \le a, 0 \le \vartheta < 2\pi$
- Unknown temperature $u = u(r, \vartheta, t)$
- Parabolic PDE:
$$u_t = \kappa \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\vartheta\vartheta} \right)$$
- One homogeneous Neumann BC at $r = a$:
$$u_r(a, \vartheta, t) = 0$$
- One IC at $t = 0$:
$$u(r, \vartheta, 0) = f(r, \vartheta)$$

We will solve using separation of variables in the form

$$u(r,\vartheta,t)=\sum_n\left(\sum_m u_{nm}(t)R_{nm}(r)\right)\Theta_n(\vartheta)$$

The eigenfunctions Θ_n will get rid of the ϑ variable in the PDE, and the eigenfunctions R_{nm} will get rid of the r variable, leaving ODE for the Fourier coefficients $u_{nm}(t)$.

3 7.38 U, §3 Eigenfunctions

Let's start trying to get rid of one variable first. We might try a solution of the form

$$u(r,\vartheta,t)=\sum_n u_n(\vartheta,t)R_n(r)$$

where the R_n would be the eigenfunctions and the $u_n(\vartheta,t)$ the corresponding Fourier coefficients. Unfortunately, if we try to substitute a single term of the form $C(\vartheta,t)R_n(r)$ into the homogeneous PDE, we are not able to take all r terms to the same side of the equation and θ and t terms to the other side. So we do not get a Sturm-Liouville problem for R_n.

Try again, this time

$$u(r,\vartheta,t)=\sum_n u_n(r,t)\Theta_n(\vartheta)$$

If we substitute $C(r,t)\Theta(\vartheta)$ into the homogeneous PDE $u_t/\kappa=u_{rr}+u_r/r+u_{\vartheta\vartheta}/r^2$ we get:

$$\frac{1}{\kappa}\dot{T}\Theta=T''\Theta+\frac{1}{r}T'\Theta+\frac{1}{r^2}T\Theta''$$

This, fortunately, *can* be separated:

$$r^2\frac{T''}{T}+r\frac{T'}{T}-r^2\frac{\dot{T}}{\kappa T}=-\frac{\Theta''}{\Theta}=\text{ constant }=\lambda$$

So we have a Sturm-Liouville problem for Θ:

$$-\Theta''=\lambda\Theta$$

with boundary conditions that are periodic of period 2π. This problem was already fully solved in 7.38. It was the standard Fourier series for a function of period 2π. In particular, the eigenfunctions were $\cos(n\vartheta)$, $n=0,1,2,\ldots$, and $\sin(n\vartheta)$, $n=1,2,\ldots$.

Like we did in 7.38, in order to cut down on writing, we will indicate those eigenfunctions compactly as Θ_n^i, where $\Theta_n^1\equiv\cos(n\vartheta)$ and $\Theta_n^2\equiv\sin(n\vartheta)$.

So we can concisely write

$$u=\sum_{n,i}u_n^i(r,t)\Theta_n^i(\vartheta)$$

Now, if you put this into the PDE, you will see that you get rid of the ϑ coordinate as usual, but that still leaves you with r and t. So instead of ODE in t, you get PDE involving both r and t derivatives. That is not good enough.

We must go one step further: in addition we need to expand each Fourier coefficient $u_n^i(r,t)$ in a generalized Fourier series in r:

$$u(r,\vartheta,t) = \sum_{n,i}\left(\sum_m u_{nm}^i(t)R_{nm}^i(r)\right)\Theta_n^i(\vartheta)$$

Now, if you put a single term of the form $T_n(t)R_n(r)\Theta_n(\vartheta)$ into the homogeneous PDE, you get

$$\frac{1}{\kappa}\dot{T}_n^i R_n^i \Theta_n^i = T_n^i {R_n^i}'' \Theta_n^i + \frac{1}{r} T_n^i {R_n^i}' \Theta_n^i + \frac{1}{r^2} T_n^i R_n^i {\Theta_n^i}''$$

Since ${\Theta_n^i}'' = -\lambda\Theta_n^i = -n^2\Theta_n^i$, this is separable:

$$\frac{\dot{T}_n^i}{\kappa T_n^i} = \frac{{R_n^i}''}{R_n^i} + \frac{{R_n^i}'}{rR_n^i} - n^2\frac{1}{r^2} = \text{ constant } = -\mu_n$$

So we get a Sturm-Liouville problem for R_n^i with eigenvalue μ_n

$$r^2{R_n^i}'' + r{R_n^i}' + (\mu_n r^2 - n^2)R_n^i = 0$$

with again the same homogeneous boundary conditions as u:

$$R_n^i \text{ regular at } r=0 \qquad {R_n^i}'(a) = 0$$

We need to find all solutions to this problem.

Unfortunately, the ODE above is not a constant coefficient one, so we cannot write a characteristic equation. However, we have seen the special case that $\mu_n = 0$ before, 7.38. It was a Euler equation. We found in 7.38 that the only solutions that are regular at $r=0$ were found to be $A_n r^n$. But over here, the only one of that form that also satisfies the boundary condition ${R_n^i}' = 0$ at $r=a$ is the case $n=0$. So, for $\mu = 0$, we only get a single eigenfunction

$$R_{00} = 1$$

For the case $\mu_n \neq 0$, the trick is to define a stretched r coordinate ρ as

$$\rho = \sqrt{\mu_n}r \quad \Longrightarrow \quad \rho^2\frac{\mathrm{d}^2R_n^i}{\mathrm{d}\rho^2} + \rho\frac{\mathrm{d}R_n^i}{\mathrm{d}\rho} + (\rho^2 - n^2)R_n^i = 0$$

This equation can be found in any mathematical handbook in the section on Bessel functions. It says there that solutions are the Bessel functions of the first kind J_n and of the second kind Y_n:

$$R_n^i = A_n J_n(\sqrt{\mu_n}r) + B_n Y_n(\sqrt{\mu_n}r)$$

Now we need to apply the boundary conditions. Now if you look up the graphs for the functions Y_n, or their power series around the origin, you will see that they are all singular at $r = 0$. So, regularity at $r = 0$ requires $B_n = 0$.

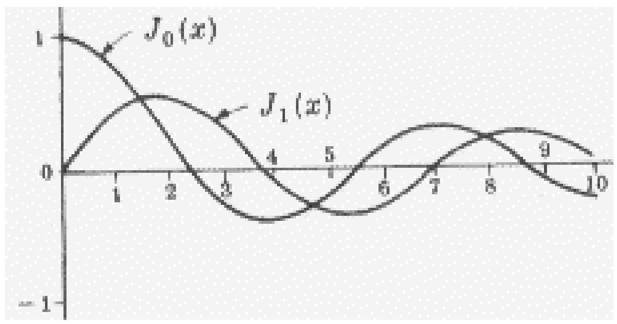
The boundary condition at the perimeter is

$$R_n^{i\,\prime}(a) = 0 = A_n \sqrt{\mu_n} J_n'(\sqrt{\mu_n} a)$$

Since μ_n is nonzero, nontrivial solutions only occur if

$$J_n'(\sqrt{\mu_n} a) = 0$$

Now if you look up the graphs of the various functions J_0, J_1, ..., you will see that they are all oscillatory functions, like decaying sines, and have an infinity of maxima and minima where the derivative is zero.



Each of the extremal points gives you a value of μ_n, so you will get an infinite of values μ_{n1}, μ_{n2}, μ_{n3}, ..., μ_{nm}, ... None of those values will be simple, but you can read them off from the graph. Better still, you can find tables of values for low values of n and m, which is often all you need.

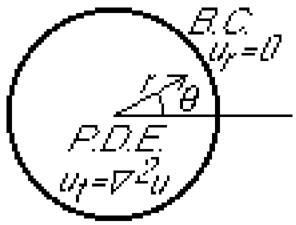
So the r-eigenvalues and eigenfunctions are:

$$\begin{array}{ccccc} \mu_{n1} & \mu_{n2} & \ldots & \mu_{nm} & \ldots \\ R_{n1}^i = J_n\left(\sqrt{\mu_{n1}} r\right) & R_{n2}^i = J_n\left(\sqrt{\mu_{n2}} r\right) & \ldots & R_{nm}^i = J_n\left(\sqrt{\mu_{n3}} r\right) & \ldots \end{array}$$

where m is the counter over the nonzero stationary points of J_n. To include the special case $\mu_n = 0$, we can simply add $\mu_{00} = 0$, $R_{00}^i = J_0(0) = 1$ to the list above.

4 7.38 U, §4 Solve

We again expand all variables in the problem in generaized Fourier series:



Let's start with the initial condition:

$$f(r,\vartheta)=\sum_{n,i}\sum_{m} f^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)$$

To find the Fourier coefficients f^i_{nm}, we need orthogonality for both the r and ϑ eigenfunctions. Now the ODE for the Θ eigenfunctions was in standard form,

$$-\Theta''=\lambda\Theta$$

but the one for R_n was not:

$$r^2R^{i\,\prime\prime}_n+rR^{i\,\prime}_n-n^2R^i_n=-\mu_n r^2R^i_n$$

The derivative of the first coefficient is $2r$, not r. To fix it up, we must divide the equation by r. And that makes the weight factor $\bar r$ that we need to put in the orthogonality relationship equal to r.

As a result, our orthogonality relation for the Fourier coefficients of initial condition $f(r,\vartheta)$ becomes

$$f^i_{nm}=\frac{\int_0^a J_n(\sqrt{\mu_{nm}}r)\left[\int_0^{2\pi}\Theta^i_n(\vartheta)f(r,\vartheta)\,\mathrm{d}\vartheta\right]r\,\mathrm{d}r}{\int_0^a J_n^2(\sqrt{\mu_{nm}}r)\,r\,\mathrm{d}r\,\int_0^{2\pi}\Theta^{i2}_n(\vartheta)\,\mathrm{d}\vartheta}$$

The integral within the square brackets turns $f(r,\vartheta)$ into its θ-Fourier coefficient $f^i_n(r)$ and the outer integral turns that coefficient in its generalized r-Fourier coefficient f^i_{nm}. Note that the total numerator is an integral of f over the area of the disk against a mode shape $J_n(\sqrt{\mu_{nm}}r)\Theta^i_n(\vartheta)$.

The r-integral in the denominator can be worked out using Schaum's Mathematical Handbook 24.88/27.88:

$$\int_0^a J_n^2(\sqrt{\mu_{nm}}r)\,r\,\mathrm{d}r=\left(\frac{a^2}{2}-\frac{n^2}{2\mu_{nm}}\right)J_n^2\left(\sqrt{\mu_{nm}}a\right)$$

(setting the second term to zero for μ_{00}.)

Hence, while akward, there is no fundamental problem in evaluating as many f^i_{nm} as you want numerically. We will therefor consider them now "known".

Next we expand the desired temperature in a generalized Fourier series:

$$u(r,\vartheta,t) = \sum_{n,i}\sum_{m} u^i_{nm}(t)\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)$$

Put into PDE $u_t/\kappa = u_{rr} + u_r/r + u_{\vartheta\vartheta}/r^2$:

$$\begin{aligned}
&\frac{1}{\kappa}\sum_{n,i}\sum_{m} \dot{u}^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r) \\
&= \sum_{n,i}\sum_{m} u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)'' \\
&+ \frac{1}{r}\sum_{n,i}\sum_{m} u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)' \\
&+ \frac{1}{r^2}\sum_{n,i}\sum_{m} u^i_{nm}\Theta^i_n(\vartheta)''J_n(\sqrt{\mu_{nm}}r)
\end{aligned}$$

Because of the SL equation satisfied by the Θ^i_n:

$$\begin{aligned}
&\frac{1}{\kappa}\sum_{n,i}\sum_{m} \dot{u}^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r) \\
&= \sum_{n,i}\sum_{m} u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)'' \\
&+ \frac{1}{r}\sum_{n,i}\sum_{m} u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)' \\
&- \frac{1}{r^2}\sum_{n,i}\sum_{m} n^2 u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)
\end{aligned}$$

Because of the SL equation satisfied by the J_n:

$$\begin{aligned}
&\frac{1}{\kappa}\sum_{n,i}\sum_{m} \dot{u}^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r) \\
&= -\sum_{n,i}\sum_{m} \mu_{nm}u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)
\end{aligned}$$

Hence the ODE for the Fourier coefficients is:

$$\dot{u}^i_{nm} + \kappa\mu_{nm}u^i_{nm} = 0$$

with solution:

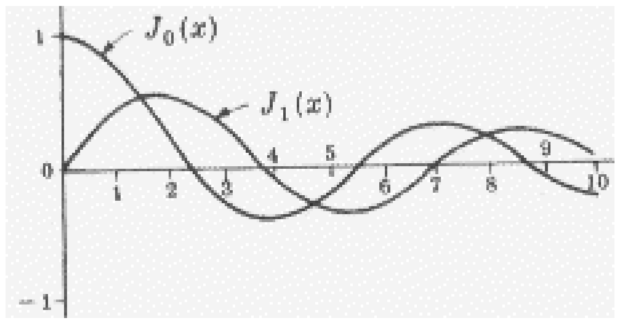
$$u^i_{nm}(t) = u^i_{nm}(0)e^{-\kappa\mu_{nm}t}$$

At time zero, the series expansion for u must be the same as the one for the given initial condition f:

$$u^i_{nm}(0) = f^i_{nm}$$

Hence we have found the Fourier coefficients of u and solved the problem.

5 7.38 U, §5 Total



Find the set $\sqrt{\mu_{nm}}a$ of positive stationary points of the Bessel functions J_n, $n = 0, 1, 2, ...$ and add $\mu_{00} = 0$.

Find the generalized Fourier coefficients of the initial condition:

$$f^1_{0m} = \frac{\int_0^{2\pi} \int_0^a f(r,\vartheta) J_0(\sqrt{\mu_{0m}}r)\, r \,\mathrm{d}\vartheta\, \mathrm{d}r}{\pi a^2 J_0^2\left(\sqrt{\mu_{0m}}a\right)}$$

$$f^1_{nm} = \frac{2\mu_{nm} \int_0^{2\pi} \int_0^a f(r,\vartheta) \cos(n\vartheta) J_n(\sqrt{\mu_{nm}}r)\, r \,\mathrm{d}\vartheta\, \mathrm{d}r}{\pi \left(\mu_{nm}a^2 - n^2\right) J_n^2\left(\sqrt{\mu_{nm}}a\right)}$$

$$f^2_{nm} = \frac{2\mu_{nm} \int_0^{2\pi} \int_0^a f(r,\vartheta) \sin(n\vartheta) J_n(\sqrt{\mu_{nm}}r)\, r \,\mathrm{d}\vartheta\, \mathrm{d}r}{\pi \left(\mu_{nm}a^2 - n^2\right) J_n^2\left(\sqrt{\mu_{nm}}a\right)}$$

Then:

$$\begin{aligned} u(r,\vartheta,t) &= \sum_{m=0}^{\infty} f_{0m} e^{-\kappa\mu_{0m}t} J_0\left(\sqrt{\mu_{0m}}r\right) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f^1_{nm} e^{-\kappa\mu_{nm}t} \cos(n\vartheta) J_n\left(\sqrt{\mu_{nm}}r\right) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f^2_{nm} e^{-\kappa\mu_{nm}t} \sin(n\vartheta) J_n\left(\sqrt{\mu_{nm}}r\right) \end{aligned}$$

That was easy!