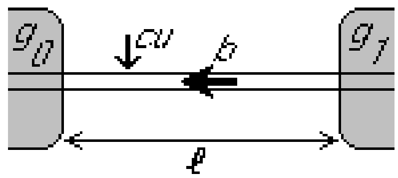
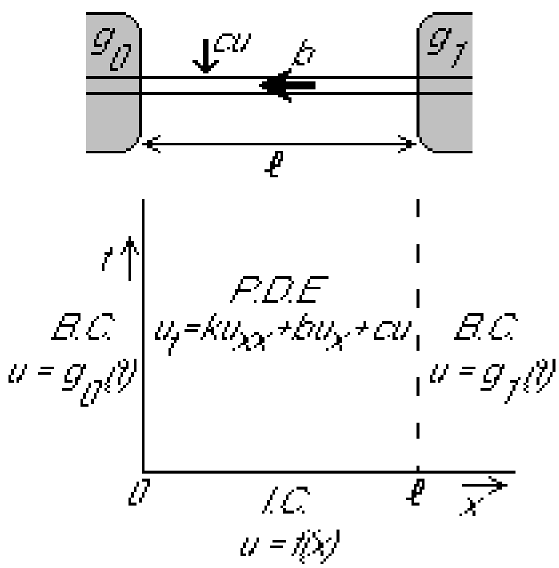
7.22

1 7.22, §1 Asked

Asked: Find the unsteady temperature distribution in the moving bar below for arbitrary position and time if the initial distribution at time zero and the temperatures of the ends are known.



2 7.22, §2 PDE Model



- Finite domain $\bar{\Omega}$: $0 \leq x \leq \ell$
- Unknown temperature $u = u(x, t)$
- Parabolic
- One initial condition

- Two Dirichlet boundary conditions
- Constant κ

Try separation of variables:

$$\sum_n C_n(t)X_n(x)$$

3 7.22, §3 Boundaries

Find u_0:

The x-boundary conditions are inhomogeneous:

$$u(0,t) = g_0(t) \qquad u(\ell,t) = g_1(t)$$

Try finding a u_0 satisfying these boundary conditions:

$$u_0(0,t) = g_0(t) \qquad u_0(\ell,t) = g_1(t)$$

Try a linear expression:

$$u_0 = A(t) + B(t)x$$

$$A(t) = g_0(t) \qquad A(t) + B(t)\ell = g_1(t)$$

This can be solved to find

$$\boxed{u_0(x,t) = g_0(t) + \frac{g_1(t) - g_0(t)}{\ell}x}$$

Identify the problem for the remainder:

Substitute $u = u_0 + v$ into the boundary conditions:

$$u_0(0,t) + v(0,t) = g_0(t) \qquad u_0(\ell,t) + v(\ell,t) = g_1(t)$$

gives

$$v(0,t) = 0 \qquad v(\ell,t) = 0$$

Substitute $u = u_0 + v$ into the PDE $u_t = \kappa u_{xx} + bu_x + cu$:

$$v_t = \kappa v_{xx} + bv_x + cv + q$$

where

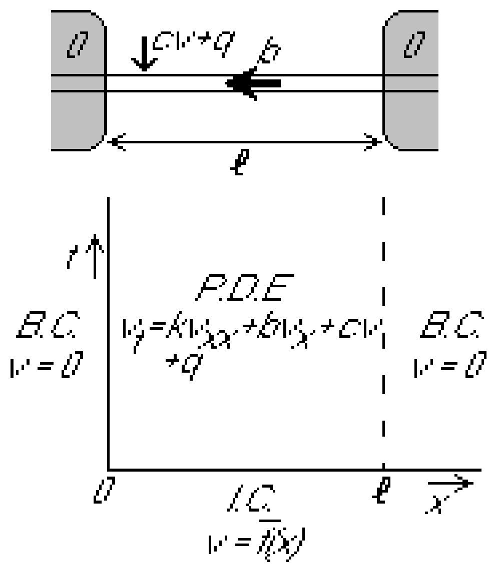
$$\boxed{q(x,t) = -g_0'(t) - \frac{g_1'(t) - g_0'(t)}{\ell}x + b\frac{g_1(t) - g_0(t)}{\ell} + c\left(g_0(t) + \frac{g_1(t) - g_0(t)}{\ell}x\right)}$$

Substitute $u = u_0 + v$ into the IC $u(x,0) = f(x)$:

$$v(x,0) = \bar{f}(x)$$

$$\boxed{\bar{f}(x) = f(x) - g_0(0) - \frac{g_1(0) - g_0(0)}{\ell} x}$$

The problem for v is therefor:



4 7.22, §4 Eigenfunctions

Substitute $v = T(t)X(x)$ into the *homogeneous* PDE $v_t = \kappa v_{xx} + b v_x + cv$:

$$T'X = \kappa T X'' + bTX' + cTX$$

Separate:

$$\frac{T'}{T} = \kappa \frac{X''}{X} + b\frac{X'}{X} + c = \text{ constant } = -\lambda$$

The Sturm-Liouville problem for X is now:

$$-\kappa X'' - bX' - cX = \lambda X \qquad X(0) = 0 \qquad X(\ell) = 0$$

This is a constant coefficient ODE, with a characteristic polynomial:

$$\kappa k^2 + bk + (c + \lambda) = 0$$

The fundamentally different cases are now two real roots (discriminant positive), a double root (discriminant zero), and two complex conjugate roots (discriminant negative.) We do each in turn.

Case $b^2 - 4\kappa(c+\lambda) > 0$*:*

Roots k_1 and k_2 real and distinct:

$$X = Ae^{k_1 x} + Be^{k_2 x}$$

Boundary conditions:

$$X(0) = 0 = A + B \quad \Longrightarrow \quad B = -A$$
$$X(\ell) = 0 = A\left(e^{k_1\ell} - e^{k_2\ell}\right) = 0$$

No nontrivial solutions since the roots are different.

Case $b^2 - 4\kappa(c+\lambda) = 0$*:*

Since $k_1 = k_2 = k$:

$$X = Ae^{kx} + Bxe^{kx}$$

Boundary conditions:

$$X(0) = 0 = A \qquad X(\ell) = 0 = B\ell e^{k\ell}$$

No nontrivial solutions.

Case $b^2 - 4\kappa(c+\lambda) < 0$*:*

For convenience, we will write the roots of the characteristic polynomial more concisely as:

$$k_1 = -\mu + \mathrm{i}\omega \quad k_2 = -\mu - \mathrm{i}\omega$$

where according to the solution of the quadratic

$$\mu = \frac{b}{2\kappa} \qquad \omega = \frac{\sqrt{4\kappa(c+\lambda) - b^2}}{2\kappa}$$

Since it can be confusing to have too many variables representing the same thing, let's agree that μ is our "representative" for b, and ω our "representative" for λ. In terms of these representatives, the solution is, after clean-up,

$$X = e^{-\mu x}\left(A\cos(\omega x) + B\sin(\omega x)\right)$$

Boundary conditions:

$$X(0) = 0 = A \qquad X(\ell) = 0 = e^{-\mu\ell}B\sin(\omega\ell)$$

Nontrivial solutions $B \neq 0$ can only occur if

$$\sin(\omega\ell) = 0 \quad \Longrightarrow \quad \omega_n = n\pi/\ell \quad (n = 1, 2, \ldots)$$

which gives us our eigenvalues, by substituting in for ω:

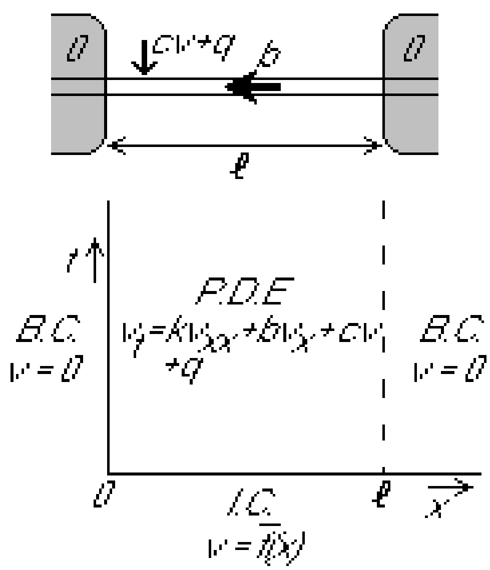
$$\boxed{\lambda_n = \frac{\kappa n^2\pi^2}{\ell^2} + \frac{b^2}{4\kappa} - c \quad (n = 1, 2, 3, ...)}$$

Also, choosing each $B = 1$:

$$\boxed{X_n = e^{-\mu x}\sin\left(n\pi x/\ell\right) \quad (n = 1, 2, 3, ...)}$$

5 7.22, §5 Solve

Expand all variables in the problem for v in a Fourier series:



$$v = \sum_{n=1}^{\infty} v_n(t)X_n(x) \quad \bar{f} = \sum_{n=1}^{\infty} \bar{f}_n X_n(x) \quad q = \sum_{n=1}^{\infty} q_n(t)X_n(x)$$

We want to first find the Fourier coefficients of the known functions $\bar{f}$ and q. Unfortunately, the ODE found in the previous section,

$$-\kappa X'' - bX' - cX = \lambda X$$

is *not* in standard Sturm-Liouville form: the derivative of the first, X'', coefficient, $-\kappa$, is zero, not $-b$. Let's try to make it OK by multiplying the entire equation by a factor, which will then be our $\bar{r}$.

$$-\bar{\mathrm{r}}\kappa X'' - \bar{\mathrm{r}}bX' - \bar{\mathrm{r}}cX = \lambda\bar{\mathrm{r}}X$$

We want that the second coefficient is the derivative of the first:

$$\bar{\mathrm{r}}b = \frac{\mathrm{d}}{\mathrm{d}x}\left(\bar{\mathrm{r}}\kappa\right)$$

This is a simple ODE for the $\bar{\mathrm{r}}$ we are trying to find, and a valid solution is:

$$\bar{\mathrm{r}} = e^{bx/\kappa} = e^{2\mu x}$$

Having found $\bar{\mathrm{r}}$, we can write the orthogonality relationships for the generalized Fourier coefficients of $\bar{f}$ and q (remember that $X_n = e^{-\mu x}\sin(n\pi x/\ell)$):

$$\boxed{\bar{f}_n = \frac{\int_{x=0}^{\ell} e^{\mu x}\bar{f}(x)\sin(n\pi x/\ell)\,\mathrm{d}x}{\int_{x=0}^{\ell}\sin^2(n\pi x/\ell)\,\mathrm{d}x}}$$

$$\boxed{q_n(t) = \frac{\int_{x=0}^{\ell} e^{\mu x}q(x,t)\sin(n\pi x/\ell)\,\mathrm{d}x}{\int_{x=0}^{\ell}\sin^2(n\pi x/\ell)\,\mathrm{d}x}}$$

The integrals in the bottoms equal $\ell/2$.

Expand the PDE $v_t = \kappa v_{xx} + bv_x + cv + q$ in a generalized Fourier series:

$$\begin{aligned}&\sum_{n=1}^{\infty}\dot{v}_n(t)X_n(x) = \\ &\quad\kappa\sum_{n=1}^{\infty}v_n(t)X_n''(x) + b\sum_{n=1}^{\infty}v_n(t)X_n'(x) + c\sum_{n=1}^{\infty}v_n(t)X_n(x) \\ &\quad+\sum_{n=1}^{\infty}q_n(t)X_n(x)\end{aligned}$$

Because of the choice of the X_n, $\kappa X'' + bX' + cX = -\lambda X$:

$$\sum_{n=1}^{\infty}\dot{v}_n(t)X_n(x) = -\sum_{n=1}^{\infty}\lambda_n v_n(t)X_n(x) + \sum_{n=1}^{\infty}q_n(t)X_n(x)$$

So, the ODE for the generalized Fourier coefficients of v becomes:

$$\dot{v}_n(t) + \lambda_n v_n(t) = q_n(t)$$

Expand the IC $v(x,0) = \bar{f}(x)$ in a generalized Fourier series:

$$\sum_{n=1}^{\infty}v_n(0)X_n(x) = \sum_{n=1}^{\infty}\bar{f}_n X_n(x)$$

so

$$v_n(0) = \bar{f}_n$$

Solve this O.D.E. and initial condition for v_n:

Homogeneous equation:

$$v_{nh} = A_n e^{-\lambda_n t}$$

Inhomogeneous equation:

$$A_n' e^{-\lambda_n t} + 0 = q_n(t)$$

$$A_n = \int_{\tau=0}^{t} q_n(\tau) e^{\lambda_n \tau} \, \mathrm{d}\tau + A_{n0}$$

$$v_n = A_n e^{-\lambda_n t}$$

$$v_n = \int_{\tau=0}^{t} q_n(\tau) e^{-\lambda_n (t-\tau)} \, \mathrm{d}\tau + A_{n0} e^{-\lambda_n t}$$

Initial condition: $A_{n0} = \bar{f}_n$.

$$v_n = \int_{\tau=0}^{t} q_n(\tau) e^{-\lambda_n (t-\tau)} \, \mathrm{d}\tau + \bar{f}_n e^{-\lambda_n t}$$

6 7.22, §6 Total

Total solution:

$$\mu = \frac{b}{2\kappa} \qquad \lambda_n = \frac{\kappa n^2 \pi^2}{\ell^2} + \lambda_0 \qquad \lambda_0 = \frac{b^2}{4\kappa} - c$$

$$\bar{f}(x) = f(x) - g_0(0) - \frac{g_1(0) - g_0(0)}{\ell} x$$

$$\bar{f}_n = \frac{2}{\ell} \int_{x=0}^{\ell} \bar{f}(x) e^{\mu x} \sin(n\pi x/\ell) \, \mathrm{d}x$$

$$q(x,t) = -g_0'(t) - \frac{g_1'(t) - g_0'(t)}{\ell} x + b \frac{g_1(t) - g_0(t)}{\ell} + c \left(g_0(t) + \frac{g_1(t) - g_0(t)}{\ell} x \right)$$

$$q_n(t) = \frac{2}{\ell} \int_{x=0}^{\ell} q(x,t) e^{\mu x} \sin(n\pi x/\ell) \, \mathrm{d}x$$

$$\begin{aligned} u = {} & g_0(t) + \frac{g_1(t) - g_0(t)}{\ell} x \\ & + \sum_{n=1}^{\infty} \left[\int_{\tau=0}^{t} q_n(\tau) e^{-\lambda_n (t-\tau)} \, \mathrm{d}\tau + \bar{f}_n e^{-\lambda_n t} \right] e^{-\mu x} \sin(n\pi x/\ell) \end{aligned}$$

Solution in the book is no good (check the boundary conditions.)

7 7.22, §7 Poor Method

Define a new unknown w by $u = we^{-\alpha x - \beta t}$. Put this in the PDE for u and choose α and β so that the w_x and w terms drop out. This requires:

$$u = we^{-\mu x - \lambda_0 t}$$

Then:

$$w_t = \kappa w_{xx} \quad w(x,0) = e^{\mu x} f(x) \quad w(0,t) = e^{\lambda_0 t} g_0(t) \quad w(\ell,t) = e^{\mu \ell + \lambda_0 t} g_1(t)$$

No fun! Note that the generalized Fourier series coefficients for u become normal Fourier coefficients for w.