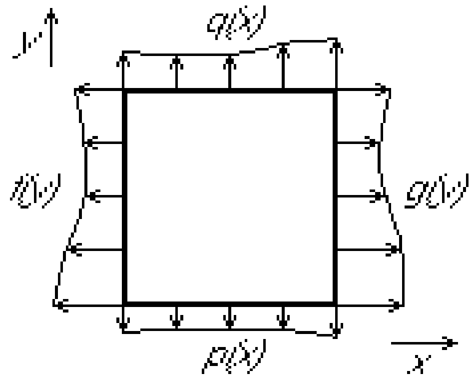
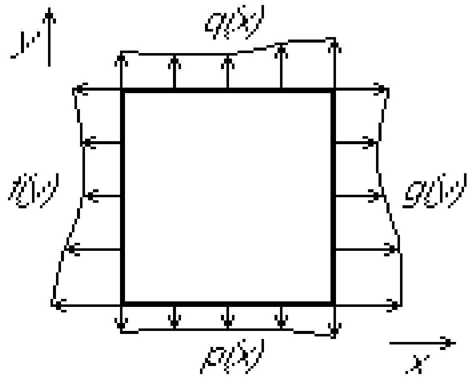
7.37

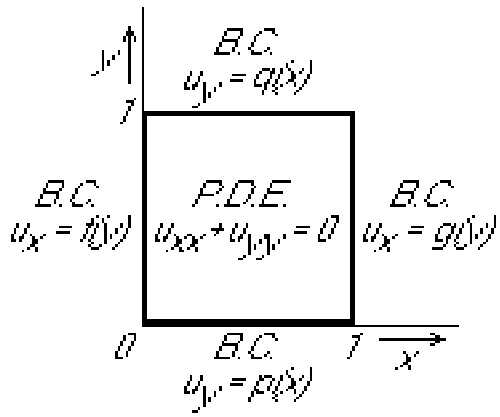
1 7.37, §1 Asked

Asked: Find the steady temperature distribution in the square plate/cross section below if the heat fluxes out of the sides are known.



2 7.37, §2 P.D.E. Model





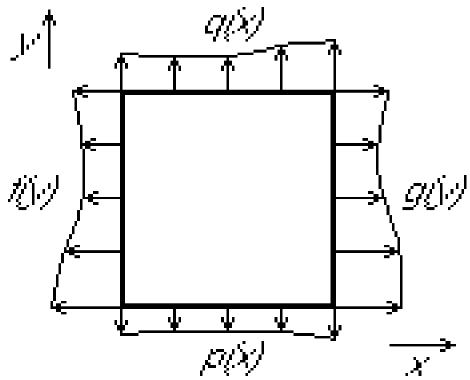
- Finite domain $\bar{\Omega}$: $0 \leq x \leq 1$, $0 \leq y \leq 1$.
- Unknown temperature $u = u(x, y)$
- Elliptic
- Four Neumann boundary conditions
- Integral constraint due to all Neumann B.C.s:

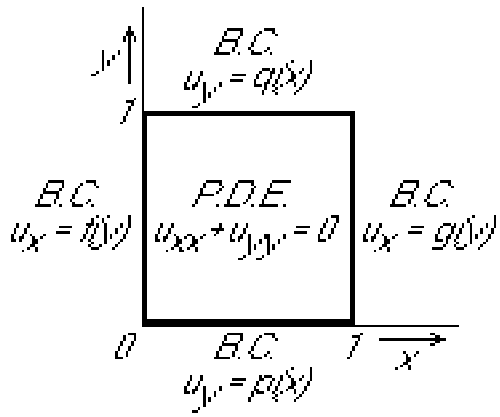
$$\int_0^1 p(x)\,\mathrm{d}x - \int_0^1 g(y)\,\mathrm{d}y - \int_0^1 q(x)\,\mathrm{d}x + \int_0^1 f(y)\,\mathrm{d}y = 0$$

Try separation of variables:

$$\sum_n u_n(y) X_n(x) \text{ or } \sum_n u_n(x) Y_n(y)$$

3 7.37, §3 Boundaries





Standard approach:

All boundary conditions are inhomogeneous. Our standard approach would be to set $u = u_0 + v$ where

$$u_{0x}(0, y) = f(y) \qquad u_{0x}(1, y) = g(y)$$

and then set

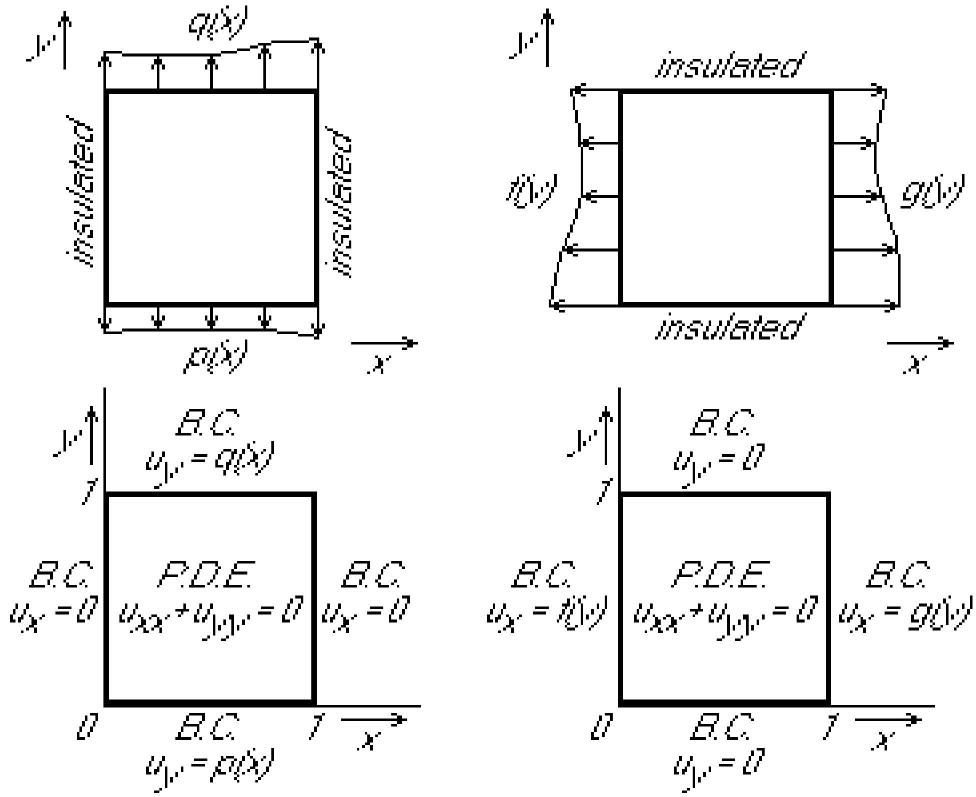
$$v = \sum_n v_n(y) X_n(x)$$

This would work without any problems. A u_0 quadratic in x would be fine. Of course, this choice for u_0 is quite arbitrary.

Alternative approach:

Instead, *we* will follow a more elegant procedure that does not require us to arbitrarily choose a u_0. Unfortunately, this alternative procedure will get us into some trouble.

The idea is that the given problem can be seen as the sum of two problems, each with homogeneous boundary conditions in one direction.



If we add the solutions u to the two problems together, we should get the solution to the original problem.

The *instructor* will solve the left hand problem. The *students* will solve the right hand problem, identify the difficulty, and fix it. Note that the four-problem procedure in the book has the problem even worse.

4 7.37, §4 Eigenfunctions

Substitute $u = T(y)X(x)$ into the homogeneous P.D.E. $u_{xx} + u_{yy} = 0$:

$$TX'' + T''X = 0$$
$$\frac{T''}{T} = -\frac{X''}{X} = \text{ constant } = \lambda$$

Since the instructor's x-boundary conditions are homogeneous, he has a Sturm-Liouville problem for X:

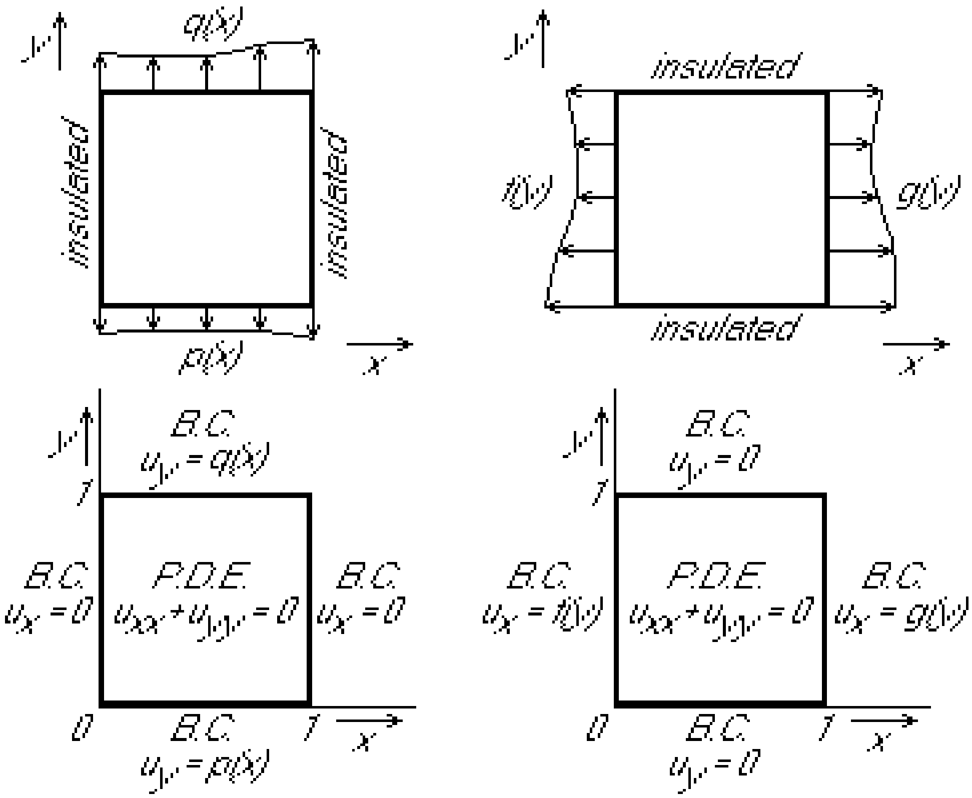
$$-X'' = \lambda X \qquad X'(0) = 0 \qquad X'(1) = 0$$

This was already solved in problem 7.19. Looking back there, substituting $\ell = 1$,

$$\lambda_n = n^2\pi^2 \qquad X_n = \cos(n\pi x) \qquad (n = 0, 1, 2, 3, \ldots)$$

5 7.37, §5 Solve

Expand all variables in the problem for u in a Fourier series:



$$u = \sum_{n=0}^{\infty} u_n(y) X_n(x) \quad p(x) = \sum_{n=0}^{\infty} p_n X_n(x) \quad q(x) = \sum_{n=0}^{\infty} q_n X_n(x)$$

$$p_n = \frac{\int_0^1 p(x) X_n(x)\,\mathrm{d}x}{\int_0^1 X_n^2(x)\,\mathrm{d}x}$$

$$q_n = \frac{\int_0^1 q(x) X_n(x)\,\mathrm{d}x}{\int_0^1 X_n^2(x)\,\mathrm{d}x}$$

Remember that the expression you find for the integrals in the bottom, $\frac{1}{2}$, does not work for $n = 0$, in which case it turns out to be 1.

Fourier-expand the PDE $u_{xx} + u_{yy} = 0$:

$$\sum_{n=0}^{\infty} u_n(y) X_n(x)'' + \sum_{n=0}^{\infty} u_n(y)'' X_n(x) = 0$$

Because of the Sturm-Liouville equation in the previous section

$$-\sum_{n=0}^{\infty} \lambda_n u_n(y) X_n(x) + \sum_{n=0}^{\infty} u_n(y)'' X_n(x) = 0$$

giving the ODE

$$u_n(y)'' - \lambda_n u_n(y) = 0$$

or substituting in the eigenvalue

$$u_n(y)'' - n^2\pi^2 u_n(y) = 0$$

Fourier-expand the BC $u_y(x, 0) = p(x)$:

$$\sum_{n=0}^{\infty} u_n(0)' X_n(x) = \sum_{n=0}^{\infty} p_n X_n(x) \quad \Longrightarrow \quad u_n'(0) = p_n$$

Fourier-expand the BC $u_y(x, 1) = q(x)$:

$$\sum_{n=0}^{\infty} u_n(1)' X_n(x) = \sum_{n=0}^{\infty} q_n X_n(x) \quad \Longrightarrow \quad u_n'(1) = q_n$$

Solve the above ODE and boundary conditions for u_n. It is a constant coefficient one, with a characteristic equation

$$k^2 - n^2\pi^2 = 0$$

Caution! Note that both roots are the same when $n = 0$. So we need to do the $n = 0$ case separately.

For $n \neq 0$ the solution is

$$u_n = A_n e^{n\pi y} + B_n e^{-n\pi y}$$

The boundary conditions above give two linear equations for A_n and B_n:

$$\left(\begin{array}{cc|c} n\pi & -n\pi & p_n \\ n\pi e^{n\pi} & -n\pi e^{-n\pi} & q_n \end{array} \right)$$

whic are best solved using Gaussian elimination. Rewriting the various exponentials in terms of sinh and cosh, the solution for the Fourier coefficients of u except $n = 0$ is:

$$u_n = -\frac{\cosh(n\pi[y-1])}{n\pi \sinh(n\pi)} p_n + \frac{\cosh(n\pi y)}{n\pi \sinh(n\pi)} q_n \quad (n = 1, 2, 3, \ldots)$$

For $n = 0$ the solution of the ODE is

$$u_0 = A_0 + B_0 y$$

Put in the boundary conditions to get equations for the integration constants A_0 and B_0:

$$u_0'(0) = B_0 = p_0 \qquad u_0'(1) = B_0 = q_0$$

Oops! We can only solve this if

$$p_0 = q_0$$

Looking above for the definition odf those Fourier coefficients, we see we only have a solution if

$$\int_0^1 p(x)\,\mathrm{d}x = \int_0^1 q(x)\,\mathrm{d}x$$

Unfortunately, these two integrals will normally *not* be equal! Also, A_0 remains unknown. No problem! Students will explain and fix the problem.

6 7.37, §6 Total

First compute the Fourier coefficients of the given boundary conditions:

$$p_0 = \int_0^1 p(x)\,\mathrm{d}x \qquad p_n = 2\int_0^1 p(x)\cos(n\pi x)\,\mathrm{d}x \quad (n = 1, 2, \ldots)$$

$$q_0 = \int_0^1 q(x)\,\mathrm{d}x \qquad q_n = 2\int_0^1 q(x)\cos(n\pi x)\,\mathrm{d}x \quad (n = 1, 2, \ldots)$$

Then the solution is equal to:

$$\begin{aligned} u &= A_0 + p_0 x \\ &+ \sum_{n=1}^{\infty}\left[-\frac{\cosh(n\pi[y-1])}{n\pi\sinh(n\pi)}p_n + \frac{\cosh(n\pi y)}{n\pi\sinh(n\pi)}q_n\right]\cos(n\pi x) \end{aligned}$$

But this only satisfies the BC on the top of the plate if

$$\int_0^1 q(x)\,\mathrm{d}x = \int_0^1 p(x)\,\mathrm{d}x$$