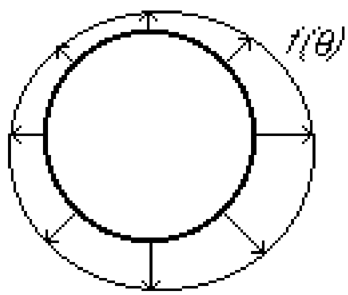
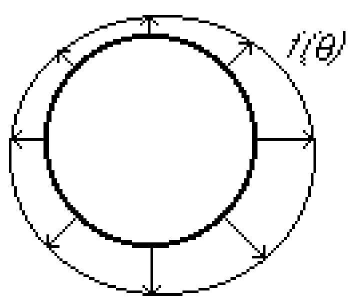
7.38

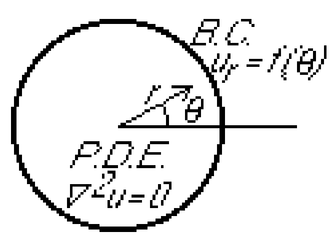
1 7.38, §1 Asked

Asked: Find the ideal flow in a cylinder if the normal (radial) velocity at the perimeter is known.



2 7.38, §2 P.D.E. Model





- Finite domain $\bar{\Omega}$: $0 \leq r \leq 1, 0 \leq \vartheta < 2\pi$
- Unknown velocity potential $u = u(r, \vartheta)$

- Elliptic equation
$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\vartheta\vartheta} = 0$$
- One Neumann boundary condition at $r = 1$.

We will try separation of variables.

3 7.38, §3 Eigenfunctions

If we substitute a trial solution $u = R(r)\Theta(\vartheta)$ into the homogeneous P.D.E. $u_{rr} + u_r/r + u_{\vartheta\vartheta}/r^2 = 0$, we get:
$$R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' = 0$$
which separates into
$$r^2 \frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \text{ constant } = \lambda$$
Make sure that *all* r terms are at the same side of the equation!

Now which ODE gives us the Sturm-Liouville problem, and thus the eigenvalues? Not the one for $R(r)$; u has an *inhomogeneous* boundary condition on the perimeter $r = 1$. Eigenvalue problems must be homogeneous; they simply don't work if anything is inhomogeneous.

We are in luck with $\Theta(\vartheta)$ however. The unknown $u(r, \vartheta)$ has "periodic" boundary conditions in the ϑ-direction. If ϑ increases by an amount 2π, $u(r, \vartheta)$ returns to exactly the same values as before: it is a "periodic function" of ϑ. Periodic boundary conditions are homogeneous: the zero solution satisfies them. After all, zero remains zero however many times you go around the circle.

The Sturm-Liouville problem for Θ is:
$$-\Theta'' = \lambda\Theta$$
$$\Theta(0) = \Theta(2\pi) \qquad \Theta'(0) = \Theta'(2\pi)$$
Note that for a second order ODE, we need two boundary conditions. So we wrote down that both Θ, as well as its derivative are exactly the same at $\vartheta = 0$ and 2π.

Pretend that we do not know the solution of this Sturm-Liouville problem! Write the characteristic equation of the ODE:
$$k^2 + \lambda = 0 \quad \Longrightarrow \quad k = \pm\mathrm{i}\sqrt{\lambda}$$

Lets look at all possibilities:

Case $\lambda = 0$:

Since $k_1 = k_2 = 0$:

$$\Theta = A + B\vartheta$$

Boundary conditions:

$$\Theta(0) = \Theta(2\pi) \quad \Longrightarrow \quad A = A + B2\pi$$

That can only be true if $B = 0$. Then the second boundary condition is

$$\Theta'(0) = \Theta'(2\pi) \quad \Longrightarrow \quad 0 = 0$$

hence $\Theta = A$. No undetermined constants in eigenfunctions! Simplest is to choose $A = 1$:

$$\Theta_0(\vartheta) = 1$$

Case $\lambda \neq 0$:

We will be lazy and try to do the cases of positive and negative λat the same time. For positive λ, the cleaned-up solution is

$$\Theta = A\cos\left(\sqrt{\lambda}\vartheta\right) + B\sin\left(\sqrt{\lambda}\vartheta\right)$$

This also applies for negative λ, except that the square roots are then imaginary.

Lets write down the boundary conditions first:

$$\Theta(0) = \Theta(2\pi) \quad \Longrightarrow \quad A = A\cos\left(\sqrt{\lambda}2\pi\right) + B\sin\left(\sqrt{\lambda}2\pi\right)$$

$$\Theta'(0) = \Theta'(2\pi) \quad \Longrightarrow \quad B\sqrt{\lambda} = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}2\pi\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}2\pi\right)$$

These two equations are a bit less simple than the ones we saw so far. Rather than directly trying to solve them and make mistakes, this time let us write out the augmented matrix of the system of equations for A and B:

$$\left(\begin{array}{cc|c} 1-\cos\left(\sqrt{\lambda}2\pi\right) & -\sin\left(\sqrt{\lambda}2\pi\right) & 0 \\ \sin\left(\sqrt{\lambda}2\pi\right) & 1-\cos\left(\sqrt{\lambda}2\pi\right) & 0 \end{array}\right)$$

Any nontrivial solution must be nonunique (since zero is also a solution). So the determinant of the matrix must be zero, which is:

$$1 - 2\cos\left(\sqrt{\lambda}2\pi\right) + \cos^2\left(\sqrt{\lambda}2\pi\right) + \sin^2\left(\sqrt{\lambda}2\pi\right) = 0$$

or

$$\cos\left(\sqrt{\lambda}2\pi\right) = 1$$

A cosine is only equal to 1 when its argument is an integer multiple of 2π. Hence the only possible eigenvalues are

$$\sqrt{\lambda_1} = 1 \quad \sqrt{\lambda_2} = 2 \quad \sqrt{\lambda_3} = 3 \quad \ldots$$

If λ is negative, $\cos\left(i\sqrt{-\lambda}2\pi\right) = \cosh\left(\sqrt{-\lambda}2\pi\right)$ which is always greater than one for nonzero λ.

For the found eigenvalues, the system of equations for A and B becomes:

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Hence we can find *neither* A or B; there are *two* undetermined constants in the solution:

$$\Theta_n = A\cos(n\vartheta) + B\sin(n\vartheta)$$

We had this situation before with eigenvector in the case of double eigenvalues, where an eigenvalue gave rise two linearly independent eigenvectors. Basically we have the same situation here: each eigenvalue is double. Similar to the case of eigenvectors of symmetric matrices, here we want two linearly independent, and more specifically, orthogonal eigenfunctions. A suitable pair is

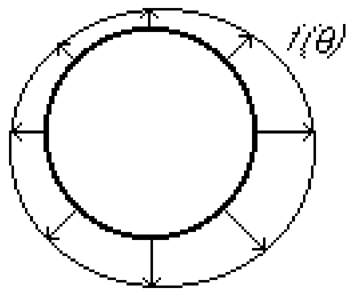
$$\Theta_n^1(\vartheta) = \cos(n\vartheta)$$
$$\Theta_n^2(\vartheta) = \sin(n\vartheta)$$

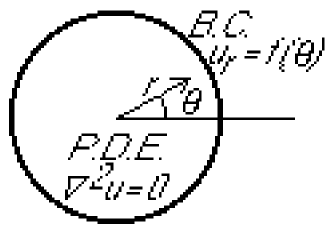
Total:

We can tabulate the complete set of eigenvalues and eigenfunctions now as:

$$\begin{array}{rcc}
\lambda_0 = 0 & \Theta_0 = 1 & \\
\lambda_1 = 1 & \Theta_1^1 = \cos(\vartheta) & \Theta_1^2 = \sin(\vartheta) \\
\lambda_2 = 4 & \Theta_2^1 = \cos(2\vartheta) & \Theta_2^2 = \sin(2\vartheta) \\
\lambda_3 = 9 & \Theta_3^1 = \cos(3\vartheta) & \Theta_3^2 = \sin(3\vartheta) \\
\lambda_4 = 16 & \Theta_4^1 = \cos(4\vartheta) & \Theta_4^2 = \sin(4\vartheta) \\
\vdots & \vdots & \vdots
\end{array}$$

4 7.38, §4 Solve





We will again expand all variables in the problem in a Fourier series. Let's start with the function $f(\vartheta)$ giving the outflow through the perimeter.

$$f(\vartheta) = f_0 + \sum_{n=1}^{\infty} f_n^1 \cos(n\vartheta) + \sum_{n=1}^{\infty} f_n^2 \sin(n\vartheta)$$

This is the way a Fourier series of a periodic function with period 2π always looks.

Since $f(\vartheta)$ is supposedly known, we should again be able to find its Fourier coeficients using orthogonality. The formulae are as before.

$$f_0 = \frac{\int_{\vartheta=0}^{2\pi} f(\vartheta) 1 \,\mathrm{d}\vartheta}{\int_{\vartheta=0}^{2\pi} 1^2 \,\mathrm{d}\vartheta}$$

(the bottom is of course equal to 2π,)

$$f_n^1 = \frac{\int_{\vartheta=0}^{2\pi} f(\vartheta) \cos(n\vartheta) \,\mathrm{d}\vartheta}{\int_{\vartheta=0}^{2\pi} \cos^2(n\vartheta) \,\mathrm{d}\vartheta} \quad (n = 1, 2, \ldots)$$

$$f_n^2 = \frac{\int_{\vartheta=0}^{2\pi} f(\vartheta) \sin(n\vartheta) \,\mathrm{d}\vartheta}{\int_{\vartheta=0}^{2\pi} \sin^2(n\vartheta) \,\mathrm{d}\vartheta} \quad (n = 1, 2, \ldots)$$

(the bottoms are equal to π.)

Since I hate typing big formulae, allow me to write the Fourier series for $f(\vartheta)$ much more compactly as

$$f(\vartheta) = \sum_{n,i}^{\infty} f_n^i \Theta_n^i(\vartheta)$$

where $\Theta_n^1 = \cos(n\vartheta)$ and $\Theta_n^2 = \sin(n\vartheta)$. Also, all three formulae for the Fourier coefficients can be summarized as

$$f_n^i = \frac{\int_{\vartheta=0}^{2\pi} f(\vartheta)\Theta_n^i(\vartheta)\,\mathrm{d}\vartheta}{\int_{\vartheta=0}^{2\pi} \Theta_n^{i2}(\vartheta)\,\mathrm{d}\vartheta} \qquad (n = 0, 1, 2, \ldots; i = 1, 2)$$

For $n = 0$, only the value $i = 1$ is relevant, of course; $\Theta_0^1 = \cos(0\vartheta) = 1 = \Theta_0$. There is no $\Theta_0^2 = \sin(0\vartheta) = 0$.

Next, let's write the unknown $u(r, \vartheta)$ as a compact Fourier series:

$$u(r, \vartheta) = \sum_{n,i} u_n^i(r)\Theta_n^i(\vartheta)$$

We put this into P.D.E. $u_{rr} + u_r/r + u_{\vartheta\vartheta}/r^2 = 0$:

$$\sum_{n,i} u_n^i(r)''\Theta_n^i(\vartheta) + \frac{1}{r}\sum_{n,i} u_n^i(r)'\Theta_n^i(\vartheta) + \frac{1}{r^2}\sum_{n,i} u_n^i(r)\Theta_n^i(\vartheta)'' = 0$$

Using the Sturm-Liouville equation $\Theta_n^i(\vartheta)'' = -\lambda\Theta_n^i(\vartheta)$, where λ was found to be n^2, this simplifies to

$$\sum_{n,i} u_n^i(r)''\Theta_n^i(\vartheta) + \frac{1}{r}\sum_{n,i} u_n^i(r)'\Theta_n^i(\vartheta) - \frac{1}{r^2}\sum_{n,i} n^2 u_n^i(r)\Theta_n^i(\vartheta) = 0$$

We get the following ODE for $u_n^i(r)$:

$$u_n^i(r)'' + \frac{1}{r}u_n^i(r)' - \frac{n^2}{r^2}u_n^i(r) = 0$$

or multiplying by r^2:

$$r^2 u_n^i(r)'' + r u_n^i(r)' - n^2 u_n^i(r) = 0$$

This is *not* a constant coefficient equation. Writing down a characteristic equation is no good.

Fortunately, we have seen this one before: it is the Euler equation. You solved that one by changing to the logaritm of the independent variable, in other words, by rewriting the equation in terms of

$$\rho \equiv \ln r$$

instead of r. The r-derivatives can be converted as in:

$$\frac{\mathrm{d}u_n^i}{\mathrm{d}r} = \frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{1}{r}$$

$$\frac{\mathrm{d}^2u_n^i}{\mathrm{d}r^2} = \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{1}{r}\right] = \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\right]\frac{1}{r} - \frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{1}{r^2}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\right]\frac{\mathrm{d}\rho}{\mathrm{d}r}\frac{1}{r} - \frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{1}{r^2} = \frac{\mathrm{d}^2u_n^i}{\mathrm{d}\rho^2}\frac{1}{r^2} - \frac{\mathrm{d}u_n^i}{\mathrm{d}\rho}\frac{1}{r^2}$$

The ODE becomes in terms of ρ:

$$\frac{\mathrm{d}^2u_n^i}{\mathrm{d}\rho^2} - n^2u_n^i = 0$$

This is now a constant coefficient equation, so we can write the characteristic polynomial, $k^2 - n^2 = 0$, or $k = \pm n$, which has a double root when $n = 0$. So we get for $n = 0$:

$$u_0^1 = A_0^1 + B_0^1\rho = A_0^1 + B_0^1 \ln r$$

while for $n \neq 0$:

$$u_n^i = A_n^i e^{n\rho} + B_n^i e^{-n\rho} = A_n^i r^n + B_n^i r^{-n}$$

Now both $\ln r$ as well as r^{-n} are infinite when $r = 0$. But that is in the middle of our flow region, and the flow is obviously not infinite there. So from the 'boundary condition' at $r = 0$ that the flow is not singular, we conclude that all the B-coefficients must be zero. Since $r^0 = 1$, all coefficients are of the form $A_n^i r^n$, including the one for $n = 0$.

Hence our solution can be more precisely written

$$u(r,\vartheta) = \sum_{n,i} A_n^i r^n \Theta_n^i(\vartheta)$$

Next we expand the boundary condition $u_r(1,\vartheta) = f(\vartheta)$ at $r = 1$ in a Fourier series:

$$\sum_{n,i} nA_n^i \Theta_n^i(\vartheta) = \sum_{n,i} f_n^i \Theta_n^i(\vartheta)$$

producing

$$nA_n^i = f_n^i$$

For $n = 0$, we see immediately that A_0 can be anything, but we need $f_0 = 0$ for a solution to exist! According to the orthogonality relationship for f_0, this requires:

$$\int_0^{2\pi} f(\vartheta)\,\mathrm{d}\vartheta = 0$$

Are you surprised that the net outflow through the perimeter must be zero for this steady flow?

For nonzero n:

$$A_n^i = \frac{f_n^i}{n}$$

and our solution becomes

$$u = A_0 + \sum_{n,i} f_n^i \frac{r^n}{n} \Theta_n^i(\vartheta)$$

where A_0 can be anything.

5 7.38, §5 Total

Let's summarize our results, and write the eigenfunctions out in terms of the individual sines and cosines.

Required for a solution is that:

$$\int_0^{2\pi} f(\vartheta)\,\mathrm{d}\vartheta = 0$$

Then:

$$f_n^1 = \frac{1}{\pi}\int_{\vartheta=0}^{2\pi} f(\vartheta)\cos(n\vartheta)\,\mathrm{d}\vartheta \quad (n = 1, 2, \ldots)$$
$$f_n^2 = \frac{1}{\pi}\int_{\vartheta=0}^{2\pi} f(\vartheta)\sin(n\vartheta)\,\mathrm{d}\vartheta \quad (n = 1, 2, \ldots)$$

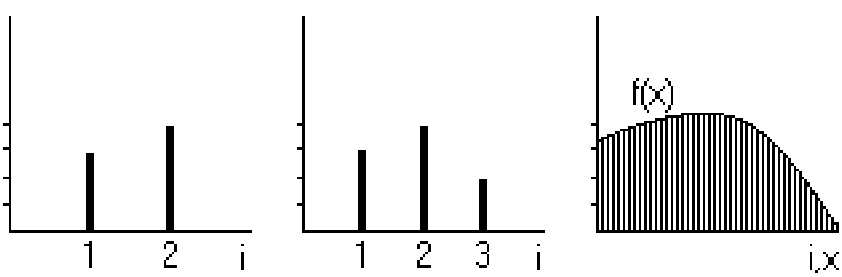
$$u = A_0 + \sum_{n=1}^{\infty}\left\{f_n^1\frac{r^n}{n}\cos(n\vartheta) + f_n^2\frac{r^n}{n}\sin(n\vartheta)\right\}$$

where A_0 can be anything.

6 7.38, §6 Notes

(The material in this section is elective).

It may be interesting to see exactly how functions are similar to vectors. Let's start with a vector in two dimensions, like the vector $\vec{v} = (3, 4)$. I can represent this vector graphically as a point in a plane, but I can also represent it as the 'spike function' in the first figure below:



The first coefficient, v_1, is 3, giving a spike of height of 3 when the subscript, call it i, is 1. The second coefficient $v_2 = 4$, so we have a spike of height 4 at $i = 2$. Similarly, the three-dimensional vector $\vec{v} = (3, 4, 2)$ can be graphed as the three-spike function in the second figure. If I keep adding more dimensions, going to the limit of infinite-dimensional space, my spike graph v_i becomes a function graph f of a continuous coordinate x instead of i. You can think of function $f(x)$ as a column vector of numbers, with the numbers being the successive values of $f(x)$. In this way, vectors become functions. And vector analysis turns into functional analysis.

To take the dot product of two vectors $\vec{v}$ and $\vec{w}$, we multiply corresponding coefficients and sum:

$$\vec{v} \cdot \vec{w} \equiv \sum_{i=0}^{n} v_i w_i$$

For functions $f(x)$ and $g(x)$, the sum over i becomes an integral over x:

$$(f, g) \equiv \int_{x=0}^{\ell} f(x) g(x) \, \mathrm{d}x$$

Since we now have a dot product, or inner product, for functions we can define the "norm" of a function, $||f|| \equiv \sqrt{(f, f)}$, corresponding to length for vectors. More importantly, we can define orthogonality for functions. Functions f and g are orthogonal if the integral above is zero.

For vectors we have matrices that turn vectors into other vectors: a matrix A turns a vector $\vec{v}$ into another vector $A\vec{v}$. For functions we have "operators" that turn functions into other functions. For example, the operator $\partial^2/\partial x^2$ turns a function $f(x)$ into another function $f''(x)$. Among the functions of period 2π, a function such as $\cos(nx)$ is an eigenfunction of this operator:

$$\frac{\partial^2}{\partial x^2} \cos(nx) = -n^2 \cos(nx)$$

The eigenvalue is n^2.

Symmetry for matrices can be expressed as $\vec{v} \cdot (A\vec{w}) = (A\vec{v}) \cdot \vec{w}$ because this can be written using matrix multiplication as $\vec{v}^T A\vec{w} = \vec{v}^T A^T \vec{w}$, which can only be true for all vectors $\vec{v}$ and $\vec{w}$ if $A = A^T$. And since $(f, \partial^2 g/\partial x^2) = (\partial^2 f/\partial x^2, g)$, as can be seen from integration by parts, $\partial^2/\partial x^2$ is a symmetric, or "self-adjoint", operator, with orthogonal eigenfunctions.

How about orthogonality relations? Given eigenfunctions X_n, we have seen that you get the Fourier coefficients of an arbitrary function $f(x)$ by the following formula:

$$f_n = \frac{\int f X_n \, \mathrm{d}x}{\int X_n^2 \, \mathrm{d}x} \equiv \frac{(X_n, f)}{(X_n, X_n)}$$

But where does this come from? Remember that we get the coordinates in the new coordinate system, (here, the Fourier coefficients f_n), by multiplying the original vector, (here $f(x)$), by the inverse transformation matrix P^1.

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{pmatrix} = P^{-1} f(x)$$

Now the transformation matrix P has the eigenfunctions as columns:

$$P = (X_1 \; X_2 \; X_3 \; \ldots)$$

Since the eigenfunctions are orthogonal, you get P^{-1} by simply taking the transpose, so:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} X_1^T \\ X_2^T \\ X_3^T \\ \vdots \end{pmatrix} f(x) = \begin{pmatrix} (X_1, f) \\ (X_2, f) \\ (X_3, f) \\ \vdots \end{pmatrix}$$

giving $f_n = (X_n, f)$. The difference from $f_n = (X_n, f)/(X_n, X_n)$ above is simply due to the fact that we usually do not normalize eigenfunctions to norm 1.

Note: I simply told you that the proper orthogonal eigenfunctions for the double eigenvalues in 7.38 are $\cos(nx)$ and $\sin(nx)$, but I could actually have derived it from Gram-Schmidt! There is really nothing new for PDE, if you think of it this way.

7 7.38, §7 More Fun

Our final result was

$$u = A_0 + \sum_{n=1}^{\infty} \left\{ f_n^1 \frac{r^n}{n} \cos(n\vartheta) + f_n^2 \frac{r^n}{n} \sin(n\vartheta) \right\}$$

We can write it directly in terms of the given $f(x)$ if we substitute in the expressions for the Fourier coefficients:

$$u = A_0 + \sum_{n=1}^{\infty} \int_0^{2\pi} f(\phi) \cos(n\phi) \, \mathrm{d}\phi \frac{r^n}{n\pi} \cos(n\vartheta) + \int_0^{2\pi} f(\phi) \sin(n\phi) \, \mathrm{d}\phi \frac{r^n}{n\pi} \sin(n\vartheta)$$

We can clean it up by combining terms and interchanging integration and summation:

$$u = A_0 + \int_0^{2\pi} \sum_{n=1}^{\infty} \left\{ \frac{r^n}{n\pi} [\cos(n\phi)\cos(n\vartheta) + \sin(n\phi)\sin(n\vartheta)] \right\} f(\phi) \, \mathrm{d}\phi$$

$$u = A_0 + \int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} \frac{r^n}{n\pi} \cos(n[\vartheta - \phi]) \right\} f(\phi) \, \mathrm{d}\phi$$

This we can clean up even more by giving a name to the function within the curly brackets:

$$u = A_0 + \int_0^{2\pi} G(r, \vartheta - \phi) f(\phi) \, \mathrm{d}\phi$$

Nice, not? We can even simplify G by converting to complex exponentials and differentiating:

$$G(r,\vartheta) = \sum_{n=1}^{\infty} \frac{r^n}{n\pi}\cos(n\vartheta) = \sum_{n=1}^{\infty}\left\{\frac{r^n}{2n\pi}e^{in\vartheta} + \frac{r^n}{2n\pi}e^{-in\vartheta}\right\}$$

$$2\pi\frac{\partial G}{\partial r} = \sum_{n=1}^{\infty}\left\{r^{-1}\left(re^{i\vartheta}\right)^n + r^{-1}\left(re^{-i\vartheta}\right)^n\right\} = \frac{e^{i\vartheta}}{1-re^{i\vartheta}} + \frac{e^{-i\vartheta}}{1-re^{-i\vartheta}}$$

The last because the sums are geometric series.

Integrating and cleaning up produces

$$G(r,\vartheta) = -\frac{1}{2\pi}\ln\left(1 - 2r\cos(\vartheta) + r^2\right)$$

So, we finally have the following Poisson-type integral expression giving u directly in terms of the given $f(\vartheta)$, with no sums:

$$u(r,\vartheta) = A_0 - \frac{1}{2\pi}\int_0^{2\pi}\ln\left(1 - 2r\cos(\vartheta-\phi) + r^2\right) f(\phi)\,\mathrm{d}\phi$$

Neat!