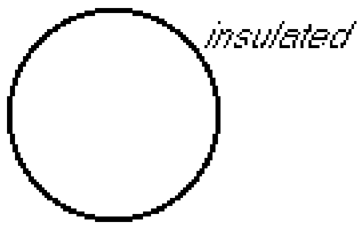
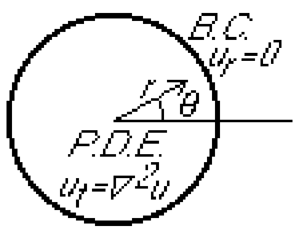
# 7.38 U

## 1 7.38 U, §1 Asked

**Asked:** Find the unsteady heat conduction in a disk if the perimeter is insulated. The initial temperature is given.




## 2 7.38 U, §2 PDE Model




- Finite domain $\bar{\Omega}$: $0 \le r \le a, 0 \le \vartheta < 2\pi$
- Unknown temperature $u = u(r, \vartheta, t)$
- Parabolic PDE:

$$u_t = \kappa \left( u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\vartheta\vartheta} \right)$$

- One homogeneous Neumann BC at $r = a$:

$$u_r(a, \vartheta, t) = 0$$

- One IC at $t = 0$:

$$u(r, \vartheta, 0) = f(r, \vartheta)$$

We will solve using separation of variables in the form

$$u(r,\vartheta,t) = \sum_n \left( \sum_m u_{nm}(t) R_{nm}(r) \right) \Theta_n(\vartheta)$$

The eigenfunctions $\Theta_n$ will get rid of the $\vartheta$ variable in the PDE, and the eigenfunctions $R_{nm}$ will get rid of the $r$ variable, leaving ODE for the Fourier coefficients $u_{nm}(t)$.

# 3 7.38 U, §3 Eigenfunctions

Let's start trying to get rid of one variable first. We might try a solution of the form

$$u(r,\vartheta,t) = \sum_n u_n(\vartheta,t) R_n(r)$$

where the $R_n$ would be the eigenfunctions and the $u_n(\vartheta,t)$ the corresponding Fourier coefficients. Unfortunately, if we try to substitute a single term of the form $C(\vartheta,t)R_n(r)$ into the homogeneous PDE, we are not able to take all $r$ terms to the same side of the equation and $\theta$ and $t$ terms to the other side. So we do not get a Sturm-Liouville problem for $R_n$.

Try again, this time

$$u(r,\vartheta,t) = \sum_n u_n(r,t)\Theta_n(\vartheta)$$

If we substitute $C(r,t)\Theta(\vartheta)$ into the homogeneous PDE $u_t/\kappa = u_{rr} + u_r/r + u_{\vartheta\vartheta}/r^2$ we get:

$$\frac{1}{\kappa}\dot{T}\Theta = C''\Theta + \frac{1}{r}C'\Theta + \frac{1}{r^2}C\Theta''$$

This, fortunately, *can* be separated:

$$r^2\frac{C''}{C} + r\frac{C'}{C} - r^2\frac{\dot{C}}{\kappa C} = -\frac{\Theta''}{\Theta} = \text{ constant } = \lambda$$

So we have a Sturm-Liouville problem for $\Theta$:

$$-\Theta'' = \lambda\Theta$$

with boundary conditions that are periodic of period $2\pi$. This problem was already fully solved in 7.38. It was the standard Fourier series for a function of period $2\pi$. In particular, the eigenfunctions were $\cos(n\vartheta)$, $n = 0, 1, 2, \ldots$, and $\sin(n\vartheta)$, $n = 1, 2, \ldots$.

Like we did in 7.38, in order to cut down on writing, we will indicate those eigenfunctions compactly as $\Theta_n^i$, where $\Theta_n^1 \equiv \cos(n\vartheta)$ and $\Theta_n^2 \equiv \sin(n\vartheta)$.

So we can concisely write

$$u = \sum_{n,i} u_n^i(r,t)\Theta_n^i(\vartheta)$$

Now, if you put this into the PDE, you will see that you get rid of the $\vartheta$ coordinate as usual, but that still leaves you with $r$ and $t$. So instead of ODE in $t$, you get PDE involving both $r$ and $t$ derivatives. That is not good enough.

We must go one step further: in addition we need to expand each Fourier coefficient $u_n^i(r,t)$ in a generalized Fourier series in $r$:

$$u(r,\vartheta,t) = \sum_{n,i}\left(\sum_m u_{nm}^i(t)R_{nm}^i(r)\right)\Theta_n^i(\vartheta)$$

Now, if you put a single term of the form $T_n(t)R_n(r)\Theta_n(\vartheta)$ into the homogeneous PDE, you get

$$\frac{1}{\kappa}\dot{T}_n^i R_n^i \Theta_n^i = T_n^i {R_n^i}'' \Theta_n^i + \frac{1}{r} T_n^i {R_n^i}' \Theta_n^i + \frac{1}{r^2} T_n^i R_n^i {\Theta_n^i}''$$

Since ${\Theta_n^i}'' = -\lambda\Theta_n^i = -n^2\Theta_n^i$, this is separable:

$$\frac{\dot{T}_n^i}{\kappa T_n^i} = \frac{{R_n^i}''}{R_n^i} + \frac{{R_n^i}'}{rR_n^i} - n^2\frac{1}{r^2} = \text{ constant } = -\mu_n$$

So we get a Sturm-Liouville problem for $R_n^i$ with eigenvalue $\mu_n$

$$r^2{R_n^i}'' + r{R_n^i}' + (\mu_n r^2 - n^2)R_n^i = 0$$

with again the same homogeneous boundary conditions as $u$:

$$R_n^i \text{ regular at } r=0 \qquad {R_n^i}'(a) = 0$$

We need to find all solutions to this problem.

Unfortunately, the ODE above is not a constant coefficient one, so we cannot write a characteristic equation. However, we have seen the special case that $\mu_n = 0$ before, 7.38. It was a Euler equation. We found in 7.38 that the only solutions that are regular at $r=0$ were found to be $A_n r^n$. But over here, the only one of that form that also satisfies the boundary condition ${R_n^i}' = 0$ at $r=a$ is the case $n=0$. So, for $\mu = 0$, we only get a single eigenfunction

$$R_{00} = 1$$

For the case $\mu_n \neq 0$, the trick is to define a stretched $r$ coordinate $\rho$ as

$$\rho = \sqrt{\mu_n}r \quad\Longrightarrow\quad \rho^2\frac{\mathrm{d}^2R_n^i}{\mathrm{d}\rho^2} + \rho\frac{\mathrm{d}R_n^i}{\mathrm{d}\rho} + (\rho^2 - n^2)R_n^i = 0$$

This equation can be found in any mathematical handbook in the section on Bessel functions. It says there that solutions are the Bessel functions of the first kind $J_n$ and of the second kind $Y_n$:

$$R_n^i = A_nJ_n(\sqrt{\mu_n}r) + B_nY_n(\sqrt{\mu_n}r)$$

Now we need to apply the boundary conditions. Now if you look up the graphs for the functions $Y_n$, or their power series around the origin, you will see that they are all singular at $r = 0$. So, regularity at $r = 0$ requires $B_n = 0$.

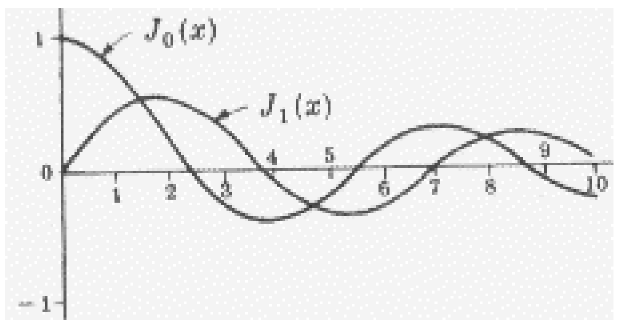
The boundary condition at the perimeter is

$$R^{i\,\prime}_n(a) = 0 = A_n\sqrt{\mu_n}J'_n(\sqrt{\mu_n}a)$$

Since $\mu_n$ is nonzero, nontrivial solutions only occur if

$$J'_n(\sqrt{\mu_n}a) = 0$$

Now if you look up the graphs of the various functions $J_0$, $J_1$, ..., you will see that they are all oscillatory functions, like decaying sines, and have an infinity of maxima and minima where the derivative is zero.




Each of the extremal points gives you a value of $\mu_n$, so you will get an infinite of values $\mu_{n1}$, $\mu_{n2}$, $\mu_{n3}$, ..., $\mu_{nm}$, .... There is no simple formula for these values, but you can read them off from the graph. Better still, you can find them in tables for low values of $n$ and $m$. (Schaum's gives a table containing both the zeros of the Bessel functions and the zeros of their derivatives.)

So the $r$-eigenvalues and eigenfunctions are:

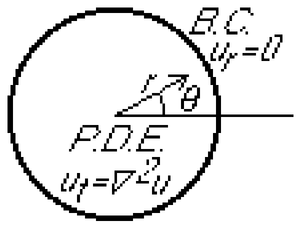
$$\begin{array}{cccccc} \mu_{n1} & \mu_{n2} & \ldots & \mu_{nm} & \ldots \\ R^i_{n1} = J_n\left(\sqrt{\mu_{n1}}r\right) & R^i_{n2} = J_n\left(\sqrt{\mu_{n2}}r\right) & \ldots & R^i_{nm} = J_n\left(\sqrt{\mu_{n3}}r\right) & \ldots \end{array}$$

where $m$ is the counter over the nonzero stationary points of $J_n$. To include the special case $\mu_n = 0$, we can simply add $\mu_{00} = 0$, $R^i_{00} = J_0(0) = 1$ to the list above.

In case of negative $\mu_n$, the Bessel function $J_n$ of imaginary argument becomes a modified Bessel function $I_n$ of real argument, and looking at the graph of those, you see that there are no solutions.

# 4 7.38 U, §4 Solve

We again expand all variables in the problem in generaized Fourier series:




Let's start with the initial condition:

$$f(r,\vartheta) = \sum_{n,i}\sum_m f^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)$$

To find the Fourier coefficients $f^i_{nm}$, we need orthogonality for both the $r$ and $\vartheta$ eigenfunctions. Now the ODE for the $\Theta$ eigenfunctions was in standard form,

$$-\Theta'' = \lambda\Theta$$

but the one for $R_n$ was not:

$$r^2 R^{i\,\prime\prime}_n + rR^{i\,\prime}_n - n^2 R^i_n = -\mu_n r^2 R^i_n$$

The derivative of the first coefficient is $2r$, not $r$. To fix it up, we must divide the equation by $r$. And that makes the weight factor $\bar{r}$ that we need to put in the orthogonality relationship equal to $r$.

As a result, our orthogonality relation for the Fourier coefficients of initial condition $f(r,\vartheta)$ becomes

$$f^i_{nm} = \frac{\int_0^a J_n(\sqrt{\mu_{nm}}r)\left[\int_0^{2\pi}\Theta^i_n(\vartheta)f(r,\vartheta)\,\mathrm{d}\vartheta\right]r\,\mathrm{d}r}{\int_0^a J_n^2(\sqrt{\mu_{nm}}r)\,r\,\mathrm{d}r\,\int_0^{2\pi}\Theta^{i2}_n(\vartheta)\,\mathrm{d}\vartheta}$$

The integral within the square brackets turns $f(r,\vartheta)$ into its $\theta$-Fourier coefficient $f^i_n(r)$ and the outer integral turns that coefficient in its generalized $r$-Fourier coefficient $f^i_{nm}$. Note that the total numerator is an integral of $f$ over the area of the disk against a mode shape $J_n(\sqrt{\mu_{nm}}r)\Theta^i_n(\vartheta)$.

The $r$-integral in the denominator can be worked out using Schaum's Mathematical Handbook 24.88/27.88:

$$\int_0^a J_n^2(\sqrt{\mu_{nm}}r)\,r\,\mathrm{d}r = \left(\frac{a^2}{2} - \frac{n^2}{2\mu_{nm}}\right)J_n^2\left(\sqrt{\mu_{nm}}a\right)$$

(setting the second term to zero for $\mu_{00}$.)

Hence, while akward, there is no fundamental problem in evaluating as many $f^i_{nm}$ as you want numerically. We will therefor consider them now "known".

Next we expand the desired temperature in a generalized Fourier series:

$$u(r,\vartheta,t) = \sum_{n,i}\sum_{m} u^i_{nm}(t)\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)$$

Put into PDE $u_t/\kappa = u_{rr} + u_r/r + u_{\vartheta\vartheta}/r^2$:

$$\begin{aligned}
&\frac{1}{\kappa}\sum_{n,i}\sum_{m} \dot{u}^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)\\
&= \sum_{n,i}\sum_{m} u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)''\\
&+ \frac{1}{r}\sum_{n,i}\sum_{m} u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)'\\
&+ \frac{1}{r^2}\sum_{n,i}\sum_{m} u^i_{nm}\Theta^i_n(\vartheta)''J_n(\sqrt{\mu_{nm}}r)
\end{aligned}$$

Because of the SL equation satisfied by the $\Theta^i_n$:

$$\begin{aligned}
&\frac{1}{\kappa}\sum_{n,i}\sum_{m} \dot{u}^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)\\
&= \sum_{n,i}\sum_{m} u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)''\\
&+ \frac{1}{r}\sum_{n,i}\sum_{m} u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)'\\
&- \frac{1}{r^2}\sum_{n,i}\sum_{m} n^2u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)
\end{aligned}$$

Because of the SL equation satisfied by the $J_n$:

$$\begin{aligned}
&\frac{1}{\kappa}\sum_{n,i}\sum_{m} \dot{u}^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)\\
&= -\sum_{n,i}\sum_{m} \mu_{nm}u^i_{nm}\Theta^i_n(\vartheta)J_n(\sqrt{\mu_{nm}}r)
\end{aligned}$$

Hence the ODE for the Fourier coefficients is:

$$\dot{u}^i_{nm} + \kappa\mu_{nm}u^i_{nm} = 0$$

with solution:

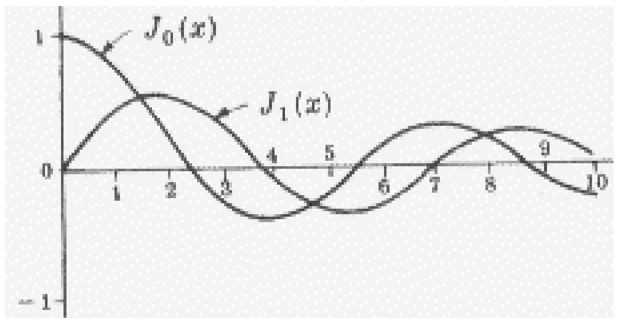
$$u^i_{nm}(t) = u^i_{nm}(0)e^{-\kappa\mu_{nm}t}$$

At time zero, the series expansion for $u$ must be the same as the one for the given initial condition $f$:

$$u_{nm}^i(0) = f_{nm}^i$$

Hence we have found the Fourier coefficients of $u$ and solved the problem.

# 5 7.38 U, §5 Total




Find the set $\sqrt{\mu_{nm}}a$ of positive stationary points of the Bessel functions $J_n$, $n = 0, 1, 2, ...$ and add $\mu_{00} = 0$.

Find the generalized Fourier coefficients of the initial condition:

$$f_{0m}^1 = \frac{\int_0^{2\pi}\int_0^a f(r,\vartheta) J_0(\sqrt{\mu_{0m}}r)\, r\, \mathrm{d}\vartheta\, \mathrm{d}r}{\pi a^2 J_0^2\left(\sqrt{\mu_{0m}}a\right)}$$

$$f_{nm}^1 = \frac{2\mu_{nm}\int_0^{2\pi}\int_0^a f(r,\vartheta)\cos(n\vartheta) J_n(\sqrt{\mu_{nm}}r)\, r\, \mathrm{d}\vartheta\, \mathrm{d}r}{\pi\left(\mu_{nm}a^2 - n^2\right) J_n^2\left(\sqrt{\mu_{nm}}a\right)}$$

$$f_{nm}^2 = \frac{2\mu_{nm}\int_0^{2\pi}\int_0^a f(r,\vartheta)\sin(n\vartheta) J_n(\sqrt{\mu_{nm}}r)\, r\, \mathrm{d}\vartheta\, \mathrm{d}r}{\pi\left(\mu_{nm}a^2 - n^2\right) J_n^2\left(\sqrt{\mu_{nm}}a\right)}$$

Then:

$$\begin{aligned} u(r,\vartheta,t) &= \sum_{m=0}^{\infty} f_{0m} e^{-\kappa\mu_{0m}t} J_0\left(\sqrt{\mu_{0m}}r\right) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f_{nm}^1 e^{-\kappa\mu_{nm}t}\cos(n\vartheta) J_n\left(\sqrt{\mu_{nm}}r\right) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f_{nm}^2 e^{-\kappa\mu_{nm}t}\sin(n\vartheta) J_n\left(\sqrt{\mu_{nm}}r\right) \end{aligned}$$

That was easy!