Analysis II **Spring 2008**

Homework Problems

Do not print out this page. Keep checking for changes. Complete assignment will normally be available the day after the last lecture whose material is included in the test (Saturday, normally).

1 01/16 W

1. p13, q31
2. p13, q32
3. p14, q48 *using vectors only*
4. p32, q66 *using vectors and algebra only*
5. p32, q69
6. p32, q87
7. p54, q47 (30 points)

2 01/23 W

1. p78, q46
2. p78, q54
3. p78, q60
4. p79, q64
5. p79, q70
6. p80, q84
7. p80, q102
8. p102, q32
9. p103, q44

3 01/30 W

1. Derive $\vec{n}\,\mathrm{d}S$ in terms of $\mathrm{d}\theta$ and $\mathrm{d}\phi$, where (r, θ, ϕ) are spherical coordinates, assuming that the surface is given in the implicit form $F(r, \theta, \phi) = 0$ Use that:

   $$\vec{r} = r\,\hat{\imath}_r \qquad \frac{\partial\,\hat{\imath}_r}{\partial\theta} = \hat{\imath}_\theta \qquad \frac{\partial\,\hat{\imath}_r}{\partial\phi} = \sin\theta\,\hat{\imath}_\phi$$

   and that $\partial r/\partial\theta$ and $\partial r/\partial\phi$ can be found from the total differential

   $$\frac{\partial F}{\partial r}\mathrm{d}r + \frac{\partial F}{\partial\theta}\mathrm{d}\theta + \frac{\partial F}{\partial\phi}\mathrm{d}\phi = 0$$

   Express the vector part of the final expression in terms of vector calculus.

2. p104, q62 (Use the Cartesian expression for $\vec{n}\,\mathrm{d}S$ to formulate the integral, then switch to polar to do it.)

3. p132, q42 (use Stokes)
4. p132, q44 (use Stokes with $\vec{v} = (-y, x, 0)$, then $x = a\cos^3\alpha$, $y = a\sin^3\alpha$.)
5. p133, q56

4 02/06 W

1. p160, q38
2. Finish finding the derivatives of the unit vectors of the spherical coordinate system using the class formulae.
3. p160, q47 (finish)
4. Express the acceleration in terms of the spherical velocity components and their first time derivatives, instead of derivatives of position. Like $a_r = \dot{v}_r + \ldots$, etc. This is how you do it in fluid mechanics, where particle position coordinates are normally not used.
5. Derive the scale factors h_r, h_θ and h_z of cylindrical coordinates.
6. Use them to find the Laplacian in cylindrical coordinates.
7. Two-dimensional steady temperature distributions in a simple plate must satisfy a PDE called the Laplace equation $k\nabla^2 T = 0$. Which of the following possibilities are potential steady temperature distributions in a plate?
   (a) $T = \sqrt{2}y$
   (b) $T = 2\theta/\pi$
   (c) $T = \sin\theta$
8. Suppose additionally that the plate is triangular in shape and that the following boundary conditions (BC) must be satisfied.
   (a) the temperature is zero on the side $y = 0$ of the plate;
   (b) the heat flux coming out of the side $x = 1$ of the plate is zero;
   (c) the heat flux entering the third side $x = y$ is constant and equal to one, (per unit length in the z-direction).

   Write these BC as mathematical equations for the temperature distribution $T(x, y)$. According to Fourier's law, the heat flux per unit cross-sectional area is $-k\nabla T$, where the heat conduction constant k is here assumed to be one.
9. Which ones of the three solutions satisfy both the PDE and BC? Show why.

5 02/15 F

1. 2.19b, h.
2. 2.20.
3. 2.21, in 3D space.
4. 2.23. Reduce to canonical form by rotating the coordinate system. What is the angle the coordinate system must be rotated over?
5. 2.24.

6. Using the class formulae, completely convert the full problem 2.24 to the new coordinates.
7. 2.25, give the full equation
8. Get rid of the first order derivatives in the problem 2.25 by defining a new unknown $v = u/e^{\alpha\xi+\beta\eta+\gamma\theta}$ where ξ, η, θ are the rescaled coordinates and α, β, and γ are constants to be found from the condition that the first order derivatives disappear.

6 02/22 F

1. 2.22b,g. Sketch the characteristics.
2. 2.27a,b. Just give the characteristic coordinates, not the transformed equation.
3. 2.28d. First find a particular solution. Next convert the remaining homogeneous problem to characteristic coordinates.
4. Now solve the transformed PDE just like we did in class for the wave equation; by first solving for a first order derivative (which one?) and then integrating that derivative. Transform back to find u as a function of x and y.
5. 2.28f. In this case, leave the inhomogeneous term in there, don't try to find a particular solution for the original PDE. Transform the full problem to characteristic coordinates.
6. Now solve the transformed equation and transform back.

7 02/29 F

1. 2.28c, find the transformed equation.
2. Solve the transformed equation and convert the solution back to physical space.
3. 2.28k.
4. 2.28b. You could also diagonalized this equation by rotating the coordinate system and then stretching the axes. Is the two-dimensional canonical transformation equivalent to this? in particular, are the lines of constant ξ and η orthogonal?
5. 3.41. This is similar to the Laplace version discussed in class. Describe the reason that there is no solution physically, considering it as a heat conduction problem in a circular plate.
6. 3.44. This is essentially the uniqueness proof given in class, which can also be found in solved problems 3.14-3.16. However, you will want to write out the two parts of the surface integral separately since the boundary conditions are a mixture of the two cases 3.14 and 3.15 (with $c = 0$).
7. Show that the following Laplace equation problem has a unique solution, $u = 0$:

   $$\text{PDE: } \nabla^2 u = 0 \qquad \text{BC: } u(0,y) = u_y(x,0) = u_y(x,1) = u(1,y) + u_x(1,y) = 0$$

   This is essentially the uniqueness proof given in class, which can also be found in solved problems 3.14-3.16. However, you will want to write the four parts of the surface integral out separately since the boundary conditions are a mixture of the three cases 3.14-3.16.
8. Show that the following Laplace equation problem has infinitely many solutions beyond $u = 0$:

   $$\text{PDE: } \nabla^2 u = 0 \qquad \text{BC: } u(0,y) = u_y(x,0) = u_y(x,1) = u(1,y) - u_x(1,y) = 0$$

   Hint: Guess a very simple nonzero solution and check that it satisfies all boundary conditions and that its second order derivatives are zero. Since the equations are linear, any arbitrary multiple of this solution is also a solution.

9. Find the Green's function in three-dimensional unbounded space R^3. Use either the method of section 2.1 or 2.2 of the web page example[1] as you prefer.

8 03/07 F

1. Check whether the derivation for the expression (5) in the notes on elliptic equations[2] also applies in the three dimensional case. If it does, determine how (6) differs from the two-dimensional case.

2. 3.29a,b. Use the expressions for the Laplacian found in table books and the chain rule of differentiation. You may assume that $a = 1$ if you want.

3. Derive the Poisson integral formula for the Dirichlet problem in a sphere, as listed in the notes on elliptic equations[3] Remember that you want to get rid of the unknown derivative u_r on the circle and note that the source distribution does *not* disappear in this case.

4. 3.30. In (a), use the property mentioned in class that the minimum of a harmonic function must occur on the boundary. In (b), try $1 - y$ in the domain Ω given by $y \geq 0$. In (c), consider the functions $s = v - u$ and $t = w - v$.

5. 3.38. This does *not* require solution of the problem using the Poisson integral formula. You can just examine what symmetry properties the solution $u(x, y)$ should have to figure out the value at the origin.

6. 3.39. Again this does *not* require solution of the problem using the Poisson integral formula. You should be able to find the complete solution $u(x, y)$ by mere inspection.

7. 3.40. This is about the solution to the Dirichlet problem in a circle, which, as derived in class, is given by the Poisson integral formula (also listed in 3.37.) The question is really, suppose that the temperature f on the boundary is a narrow spike centered at a position $\frac{1}{2}(\phi_1 + \phi_2)$, will it cause a nonzero temperature at every point in the inside, or just in a limited range? (Note that the Poisson integral uses ϕ as the angular coordinate on the boundary and θ for the angle at which u is evaluated.)

8. 3.42. In both cases (a) and (b), you can assume that the solution u is given by the stated sum. You may also assume that, (using standard Fourier series results from a table book,) in both cases,

$$f = \sum_{m,n=1}^{\infty} b_{mn} \sin(mx) \sin(ny)$$

but in case (a), in which $f = 1$,

$$b_{mn} = \begin{cases} \dfrac{16}{\pi^2 mn} & \text{if } m \text{ and } n \text{ are both odd} \\ 0 & \text{otherwise} \end{cases}$$

while in case (b), where $f = \cos(x)\cos(y)$,

$$b_{mn} = \begin{cases} \dfrac{16mn}{\pi^2 (m-1)(m+1)(n-1)(n+1)} & \text{if } m \text{ and } n \text{ are both even} \\ 0 & \text{otherwise} \end{cases}$$

Plugging the sums into the PDE, in (a) there is a problem with equating left and right hand sides for some particular values of m and n, while there is no problem in (b); in (b) suitable a_{mn} values can be found, in (a) they cannot.

[1] http://www.eng.fsu.edu/~dommelen/courses/aim2/08/topics/pdes/elliptic/
[2] http://www.eng.fsu.edu/~dommelen/courses/aim2/08/topics/pdes/elliptic/
[3] http://www.eng.fsu.edu/~dommelen/courses/aim2/08/topics/pdes/elliptic/

9 03/21 F

1. 4.17a. (Question (b) was done in class, and the stated condition that F only needs to be continuous is not sufficient, but integrable and continuous would do it.)

2. 4.18. (This is the basic solution for the temperature u in a bar of length 1 where the ends of the bar are kept at zero temperature. Of course, the values of the constants C_n will normally follow from some given initial temperature, and the number of terms in the sum N will normally be infinite.)

3. 4.19. Plane wave solutions are solutions that take the form (2) in solved problem 4.12, with $\vec{\alpha}$ a constant vector and μ a constant. This sort of solutions are a multi-dimensional generalization of the $f(x - at)$ moving "wave" solution of the one-dimensional wave equation. In fact, if you take $\vec{\alpha}$ to be a unit vector, it gives the oblique direction of propagation of the wave and μ gives the wave propagation speed. However, in this case you will see that the function F cannot be an arbitrary function unless $b = 0$.

4. 4.20. A number T is rational if it can be written as the ratio of a pair of integers, e.g. $1.5 = 3/2 = 6/4 = 9/6 = \ldots$. It is irrational if it cannot, like $\sqrt{2}$. Near any rational number, irrational numbers can be found infinitely closely nearby, and vice versa. For example, the value π to one billion digits, as found on the internet, is the rational number $31415927\ldots/10000000\ldots$; π itself is not rational. The wave equation problem when $T = \pi$ has no nonzero solutions, but when $T = \pi$ to 1 billion digits has infinitely many of them. Obviously, in physics it is impossible to determine the final time to infinitely many digits, so there is no physically meaningful solution to the stated problem.

   For nonzero solutions, try $u = \sin(n\pi x)\sin(n\pi t)$, which satisfies the wave equation and the boundary conditions at $x = 0$ and $x = 1$ and the initial condition at $t = 0$. See when it satisfies the end condition at $t = T$.

   This is the boundary value problem for the wave equation, and would be perfectly OK for if it would have been the Laplace equation. (For the Laplace equation, the $\sin(n\pi t)$ becomes $\sinh(n\pi t)$ and only a unique, zero, solution is possible.) The wave equation needs two initial conditions at $t = 0$, not one condition at $t = 0$ and one at $t = T$.

5. 5.27. In (b), do not try to use an initial condition written in terms of two different, related, variables. Get rid of either x or y in the condition! In both problems, include a sketch of the characteristic lines.

6. 5.29. Making a picture of the characteristics and where the "initial" condition is given may help clarify the problem.

10 03/28 F

1. In 7.27, acoustics in a pipe with closed ends, graphically identify the extensions $F(x)$ and $G(x)$ of the given $f(x)$ and $g(x)$ to all x that allow the solution u to be written in terms of the infinite pipe D'Alembert solution.

2. Using the solution of the previous problem, and taking $\ell = 1$, $a = 1$, $f(x) = x$, and $g(x) = 1$, draw $u(x, 0)$ as a function of x between say $x = -6$ and 6. Use raster paper or equivalent and a ruler, with 4 raster cells per unit length. Then draw $u(x, 0.5)$, by first drawing $\frac{1}{2}F(x - at)$, $\frac{1}{2}F(x + at)$, and $\int_{x-at}^{x+at} G(\xi)\mathrm{d}\xi$, and then u from that. If you are unable to draw a neat and clear graph, use a plotting package to do it. Note that the boundary conditions are now satisfied, though the initial condition did not.

3. Repeat, but now draw $u(x, 0.25)$. Are the boundary conditions still satisfied? Does that mean they will be satisfied for all nonzero times?

4. Using the solution of the previous problems, find $u(0.25, 3)$.

5. Write the *complete* (Sturm-Liouville) eigenvalue problem for the eigenfunctions of 7.27.

6. Find the eigenfunctions of that problem. Make very sure you do not miss one. Write a single symbolic expression for the eigenfunctions in terms of an index, and identify all the values that index takes.

7. Write $f = x$ and $g = 1$ in terms of these eigenfunctions for the case $\ell = 1$. Be very careful with one particular eigenfunction.

8. Substitute $u(x,t) = \sum_n u_n(t)X_n(x)$ into the PDE to convert it into an ordinary differential for each separate coefficient $u_n(t)$.

11 04/04 F

1. Solve the ODE of the previous question. Apply the initial conditions to find the integration constants. Write the complete solution to the PDE, BC, and IC.

2. Plot the solution using a computer and compare with the D'Alembert solution you got earlier. You should get the same results.

   To help you get started, a Matlab program that plots the solution to problem 7.28 is provided as an example. You need both p7_28.m[4] and p7_28u.m[5]. This program is valid for the PDE and BC solved in class, with the additional data

   $$a = \tfrac{1}{2}, \quad \ell = \tfrac{1}{2}\pi, \qquad f(x) = \tfrac{1}{2}\pi - x \Rightarrow f_n = \frac{1}{(2n-1)^2}, \qquad g(x) = 0 \Rightarrow g_n = 0.$$

   These may of course not apply for your problem.

   To run the program, enter matlab and type in `p7_28`. If you do not have matlab, a free replacement is octave. Or you can use some other programming and plotting facilities.

3. Refer to problem 7.19. Find a function $u_0(x,t)$ that satisfies the inhomogeneous boundary conditions.

4. Continuing the previous problem, define $v = u - u_0$. Find the PDE, BC and IC satisfied by v.

5. Find eigenfunctions in terms of which v may be written, and that satisfy the homogeneous boundary conditions. Work out the Fourier coefficients of the relevant functions in the problem for v as far as possible.

6. Solve for v using separation of variables in terms of expressions in terms of the known functions $f(x)$, $g_0(t)$, and $g_1(t)$. Write the solution for u completely.

7. Assume that $f = 0$, $k = \ell = 1$, and that $u_x = t$ at both $x = 0$ and $x = \ell$. Work out the solution completely.

8. Plot the solution numerically at some relevant times. I suspect for large times the solution is given by

   $$u = (x - \tfrac{1}{2})t + \tfrac{1}{6}(x - \tfrac{1}{2})^3 - \tfrac{1}{8}(x - \tfrac{1}{2})$$

   Do your results agree?

12 04/11 F

1. In problem 7.26, show that if we define a new unknown v by setting

   $$u = e^{\alpha x + \beta t} v$$

   the PDE for v simplifies into the normal heat equation $v_t = \kappa v_{xx}$ for suitable values of the constants α and β. Find those values. Show that the initial condition for v is the same as the one for u, but that the boundary condition becomes $v(0,t) = \bar{f}(t)$ where $\bar{f}$ is a different function than f. Identify what $\bar{f}$ is.

[4] `p7_28.m`
[5] `p7_28u.m`

2. Solve the problem for v using Laplace transformation in time. Write the solution in terms of u and the given function f, of course. Note that the answer in the book is wrong.

3. Now solve 7.26 directly, by Laplace transforming the problem as given in time. This is a good way to practice back transform methods. Note that one factor in $\widehat{u}$ is a simpler function at a shifted value of coordinate s.

4. Solve 7.35 by Laplace transform in time. Clean up completely; only the given function may be in your answer, no Heaviside functions or other weird stuff. There is a minor error in the book's answer.

5. Replot the solution to 7.27, summing only one, two, and three eigenfunctions and examine how the solution approaches the exact one if you sum more and more terms. Comment in particularly on the slope of the profiles at time $t = 0$ and $t = 1$ at the boundaries. What should the slope be at those times? What is it?

6. Derive the (real) Fourier series for the 2π-periodic function $f(\theta)$ with values

$$f = 0 \quad |\theta| < 0.5\pi \qquad f = 1 \quad 0.5\pi < |\theta| < \pi$$

If possible, check your answer versus 24.17 in the math handbook.

7. Plot the Fourier series for increasing number of terms, $n_{\max} = 1, 3, 7, 15$ using 360 equally spaced plot points in the range from 0 to 2π.

8. Based on these plots, discuss whether the Fourier series really converges to the given function for high enough number of terms:

   (a) When does an actual jump first show up?

   (b) The original function is undefined when $\theta = \frac{1}{2}\pi$ and $\frac{3}{2}\pi$. What happens with the Fourier series at those locations?

   (c) Are there points where the Fourier series does not converge? If so, give the θ values for which it does not converge.

9. Supposing that the Fourier series converges *everywhere,* and it does, the difference between the converged function and the nonconverged function should become vanishingly small, not? In particular the maximum difference between the Fourier series and the converged function should become zero. Or does it? If not, explain why not. In particular, plot the function for $n_{\max} = 100$ using 1800 equally spaced plot points to explain your reasoning.

13 04/18 F

1. A relatively basic problem for which the Sturm-Liouville theorem is helpful is for unsteady heat conduction in a bar, or acoustics in a pipe, or steady heat conduction in a plate, when there is a mixed boundary condition, like 7.21. Such problems will lead during separation of variables to a Sturm-Liouville problem maybe like

$$X'' + \lambda X = 0 \qquad X(0) = 0 \qquad X(1)' + pX(1) = 0$$

where $p > 0$ is a positive constant, assuming the problem is stable. Show that there are no eigenfunctions for λ negative nor for λ zero. Also show that there are eigenfunctions $\sin\sqrt{\lambda}x$ for λ positive, assuming that some nonlinear function of $\sqrt{\lambda}$ is zero.

2. Examine the zeros of the nonlinear function to establish the qualitative behavior of the eigenvalues. To examine the zeros of the function, you can plot it on a computer; there are also analytical ways to examine them, by formulating it into two functions that must be equal, then sketching both functions and examining their intersection points. Do the predictions of Sturm Liouville theory stand up, as found in chapter 6 or on the web page?[6]

[6] http://www.eng.fsu.edu/~dommelen/courses/aim2/08/topics/pdes/sepvar/proc/node5.html

3. What is the orthogonality property? In particular, how can any arbitrary function defined from $x = 0$ to $x = 1$ be written in terms of the eigenfunctions? What is the expression for the Fourier coefficients? Would any of this have been self-evident without Sturm-Liouville theory? (These are sines, but the integration is over an irrational fraction of their period.)

4. Write out the first few terms of the eigenfunction expansion for u for the unsteady heat conduction in a disk with insulated boundary explicitly (not using summation symbols). In particular, include $n = 0$ and 1, and the first two eigenvalues $\mu_{nm} > 0$ for each n. Also include the constant term.

5. Identify the actual values of the four μ_{nm} using the tables in the Mathematical Handbook.

6. Consider the three modes $J_0(\sqrt{\mu_{01}}r)$, $J_1(\sqrt{\mu_{11}}r)\cos\theta$, and $J_1(\sqrt{\mu_{11}}r)\sin\theta$, where $\sqrt{\mu_{01}}$ and $\sqrt{\mu_{11}}$ are the first nonzero stationary points of J_0 respectivel J_1. In three separate circles, sketch the positions where each of these modes are zero.