Analysis II **Spring 2009**

Homework Problems

Do not print out this page. Keep checking for changes. Complete assignment will normally be available the day after the last lecture whose material is included in the assignment (Saturday, normally).

1 01/16 F

Use vector analysis wherever possible.

1. p13, q31 *if they can be vectors, count them as such.*
2. p13, q32
3. p32, q69
4. p32, q82
5. p33, q90
6. p54, q47 (30 points)

2 01/23 F

1. p78, q46
2. p78, q54
3. p78, q60
4. p79, q64
5. p79, q70

3 01/30 F

1. p80, q87
2. p80, q102
3. p103, q44. Do both with and without Stokes.
4. p104, q62. Do both directly and using the divergence theorem. Use the Cartesian expression for $\vec{n}\,\mathrm{d}S$ to formulate the surface integral, then switch to polar to do it.
5. p132, q50. Instead of $M_y = N_x$, show that the curl of the vector is zero and discuss Stokes.
6. Read through subsection 9.4 of QMFE[1] and write a half-page summary.

[1] http://www.eng.fsu.edu/~dommelen/quantum/style_a/node86.html

4 02/06 F

1. p133, q56.

2. Derive $\vec{n}\,\mathrm{d}S$ in terms of $\mathrm{d}\theta$ and $\mathrm{d}\phi$, where (r,θ,ϕ) are spherical coordinates, assuming that the surface is given as $r = f(\theta,\phi)$. Use that:

$$\vec{r} = r\,\hat{\imath}_r \qquad \frac{\partial\,\hat{\imath}_r}{\partial\theta} = \hat{\imath}_\theta \qquad \frac{\partial\,\hat{\imath}_r}{\partial\phi} = \sin\theta\,\hat{\imath}_\phi$$

as derived in class. Show that if instead the surface is given by the implicit expression $F(r,\theta,\phi) = 0$, then $\partial r/\partial\theta$ and $\partial r/\partial\phi$ can be found from the total differential

$$\frac{\partial F}{\partial r}\mathrm{d}r + \frac{\partial F}{\partial\theta}\mathrm{d}\theta + \frac{\partial F}{\partial\phi}\mathrm{d}\phi = 0$$

Express the vector part of the final expression in terms of vector calculus.

3. p160, q38.

4. Finish finding the derivatives of the unit vectors of the spherical coordinate system using the class formulae. Then finish p160, q47 as started in class, by finding the acceleration. Note that the metric indices h_i for spherical coordinates are in mathematical handbooks. Also,

$$\frac{\partial\hat{\imath}_i}{\partial u_i} = \frac{1}{h_i}\frac{\partial h_i}{\partial u_i}\hat{\imath}_i - \sum_{j=1}^{3}\frac{1}{h_j}\frac{\partial h_i}{\partial u_j}\hat{\imath}_j \qquad \frac{\partial\hat{\imath}_i}{\partial u_j} = \frac{1}{h_i}\frac{\partial h_j}{\partial u_i}\hat{\imath}_j$$

5. Express the acceleration in terms of the spherical velocity components v_r, v_θ, v_ϕ and their first time derivatives, instead of derivatives of position coordinates. Like $a_r = \dot{v}_r + \ldots$, etc. This is how you do it in fluid mechanics, where particle position coordinates are normally not used.

6. The Laplace equation

$$\text{PDE:} \quad u_{xx} + u_{yy} = 0$$

where subscripts indicate derivatives, is an elliptic equation. Such a steady-state equation needs boundary conditions at all points of the boundary. For example, one properly posed problem on the unit square is

$$\text{BC:} \quad u(x,0) = 1 \quad u_y(x,1) = 0 \quad u(0,y) = 0 \quad u_x(1,y) = 0$$

Identify Ω, $\delta\Omega$, and the type of each boundary condition.

The wave equation

$$\text{PDE:} \quad u_{xx} - u_{yy} = 0$$

is an hyperbolic equation. The above boundary conditions are *not* properly posed for the wave equation. (as you will see in a later homework.) For the wave equation, one of the coordinates must be time-like, and must have initial conditions instead of boundary conditions. The following initial and boundary conditions are properly posed for the wave equation,

$$\text{BC and IC:} \quad u(x,0) = 1 \quad u_y(x,0) = 0 \quad u(0,y) = 0 \quad u_x(1,y) = 0$$

For each of the four, determine whether it is an IC or BC, and if so, what kind of BC. The time-like coordinate is not normally included in the domain Ω. Under those conditions, identify Ω and $\delta\Omega$.

Check that the following proposed solution satisfies the PDE and all BC/IC of both the Laplace and wave equation problems:

$$u = \begin{cases} 1 \text{ for } y < x \\ 0 \text{ for } y > x \end{cases}$$

However, it is a valid solution to only the wave equation. Explain why it is not to the Laplace equation.

The Laplace equation problem as written does not have a simple solution. However, if you distort the domain into a quarter circle as in

$$\text{BC:} \quad u(x,0)=1 \qquad u(0,y)=0 \qquad u_n=0 \text{ on } x^2+y^2=1$$

then the correct solution is simply

$$u = 1 - \frac{2\theta}{\pi} \qquad \theta = \arctan(y/x)$$

Verify that solution.

5 02/13 F

1. 2.19b, h. Show a picture of the different regions.
2. 2.20.
3. 2.21, in three spatial dimensions, and time as appropriate.
4. 4.20. A number T is rational if it can be written as the ratio of a pair of integers, e.g. $1.5 = 3/2 = 6/4 = 9/6 = \ldots$. It is irrational if it cannot, like $\sqrt{2}$. Near any rational number, irrational numbers can be found infinitely closely nearby, and vice versa. For example, the value π to one billion digits, as found on the internet, is the rational number $31415927\ldots/10000000\ldots$; π itself is not rational. The wave equation problem when $T = \pi$ has no nonzero solutions, but when $T = \pi$ to 1 billion digits has infinitely many of them. Obviously, in physics it is impossible to determine the final time to infinitely many digits, so there is no physically meaningful solution to the stated problem.

   For nonzero solutions, try $u = \sin(n\pi x)\sin(n\pi t)$, which satisfies the wave equation and the boundary conditions at $x = 0$ and $x = 1$ and the initial condition at $t = 0$. See when it satisfies the end condition at $t = T$.

   The wave equation needs two initial conditions at $t = 0$, not one condition at $t = 0$ and one at $t = T$.
5. Show that the Laplace equation

   $$u_{xx} + u_{yy} = 0 \qquad 0 \le x \le 1 \quad 0 \le y \le T$$

   with the same boundary conditions, (replacing t by y), does not have the same problem. Hint: assume solutions of the form $u = \sin(n\pi x)f(y)$ and plug into the Laplace equation and $u(x,0) = 0$ to figure out what $f(y)$ is. (You may want to recall the graphs of the hyperbolic functions.)
6. Show that the Laplace equation

   $$u_{xx} + u_{yy} = 0 \qquad 0 \le x \le 1 \quad 0 \le y \le T$$

   is improperly posed for the initial/boundary value problem

   $$\text{BC:} \quad u(0,y) = u(1,y) = 0 \qquad \text{IC:} \quad u(x,0) = u_0(x), u_y(x,0) = 0$$

   because the solution at $y = T$ can be arbitrarily much larger than the given initial condition $u_0(x)$. To do so, assume that $u_0(x) = C\sin(n\pi x)$, in which case the solution is of the form $u = \sin(n\pi x)f(y)$ where $f(y)$ can be found from substitution into the Laplace equation and initial conditions.
7. Repeat the argument to show that the wave equation does not have a problem with the above initial value problem.

6 02/20 F

1. 2.23. Reduce to canonical form by rotating the coordinate system. (Not using characteristic coordinates as the book does.) What is the angle the coordinate system must be rotated over?

2. 2.24.

3. By identifying d' show that the PDE of the previous question may be reduced to

$$u_{\xi_1\xi_1} + 3u_{\xi_2\xi_2} + 4u_{\xi_3\xi_3} \pm \frac{16}{\sqrt{6}}u_{\xi_1} \pm \frac{8}{\sqrt{2}}u_{\xi_2} \pm \frac{20}{\sqrt{3}}u_{\xi_3} = 0$$

   where the $\pm$ depend on how you choose the sign of your eigenvectors. Multiply the above equation by 6 and then rescale the independent variables to get

$$u_{\xi\xi} + u_{\eta\eta} + u_{\theta\theta} \pm 16u_\xi \pm 8u_\eta \pm 10\sqrt{2}u_\theta = 0$$

4. Get rid of the first derivatives in the previous equation by defining a new unknown $v = u/e^{\alpha\xi+\beta\eta+\gamma\theta}$ where α, β, and γ are constants to be found from the condition that the first order derivatives disappear. Write and name the final equation.

5. 2.22b,g. Draw the characteristics in the xy-plane,

6. 2.28d. First find a particular solution. Next convert the remaining homogeneous problem to characteristic coordinates. Show that the homogeneous solution satisfies

$$2u_{h,\xi\eta} = u_{h,\eta}$$

   Solve this ODE for $u_{h,\eta}$, then integrate with respect to η to find u_h and u. Write the solution in terms of x and y.

7. 2.28f. In this case, leave the inhomogeneous term in there, don't try to find a particular solution for the original PDE. Transform the full problem to characteristic coordinates. Show that the solution satisfies

$$4u_{\xi\eta} - 2u_\xi \pm e^\eta = 0$$

   where $\pm$ indicates the sign of xy, or

$$4\xi\eta u_{\xi\eta} - 2\xi u_\xi + \eta = 0$$

   or

$$4\xi\eta u_{\xi\eta} + 2\xi u_\xi - \frac{1}{\eta} = 0$$

   or equivalent, depending on exactly how you define the characteristic coordinates. Solve this ODE for u_ξ, then integrate with respect to ξ to find u. Write the solution in terms of x and y.

7 02/27 F

1. 2.28c. Show that the equation may be simplified to

$$u_{\xi\xi} = 0$$

   Solve this equation and write the solution in terms of x and y.

2. 2.28b. Reduce to canonical form II. In 2.23 you diagonalized essentially the same equation by rotating the coordinate system; and you could then have stretched the coordinates to reduce it to the Laplace equation. Are the coordinates that you find now equivalent to those? In particular, are the lines of constant ξ and η orthogonal like in 2.23? If not, how come that more than one linear coordinate transformation can turn the equation into the Laplace equation?

3. 3.41. This is similar to the Laplace version discussed earlier in class. Describe the reason that there is no solution physically, considering it as a heat conduction problem in a circular plate.

8 03/06 F

1. 3.44. This is mostly the uniqueness proof given in class, which can also be found in solved problems 3.14-3.16. However, here you will want to write out the two parts of the surface integral separately since the boundary conditions are a mixture of the two cases 3.14 and 3.15 (with $c = 0$).

2. Show that the following Laplace equation problem has a unique solution, $u = 0$:

$$\text{PDE: } \nabla^2 u = 0 \qquad \text{BC: } u(0, y) = u_y(x, 0) = u_y(x, 1) = u(1, y) + u_x(1, y) = 0$$

   This is essentially the uniqueness proof given in class, which can also be found in solved problems 3.14-3.16. However, you will want to write the four parts of the surface integral out separately since the boundary conditions are a mixture of the three cases 3.14-3.16.

3. Show that the following Laplace equation problem has infinitely many solutions beyond $u = 0$:

$$\text{PDE: } \nabla^2 u = 0 \qquad \text{BC: } u(0, y) = u_y(x, 0) = u_y(x, 1) = u(1, y) - u_x(1, y) = 0$$

   Hint: Guess a very simple nonzero solution and check that it satisfies all boundary conditions and that its second order derivatives are zero. Since the equations are linear, any arbitrary multiple of this solution is also a solution. Verify whether or not the uniqueness proof of the previous section conflicts with the nonunique solution of this problem. Why would a slight difference in one boundary condition make a difference?

4. Find the Green's function in three-dimensional unbounded space R^3. Use either the method of section 2.1 or 2.2 of the web page example[2] as you prefer.

9 03/20 F

1. See whether any terms must be changed in expression (5) in the notes on elliptic equations[3] in the three dimensional case. Then determine how (6) differs from the two-dimensional case.

2. Assume that you choose

$$u_{\text{out}}(r, \vartheta, \varphi) = A\frac{1}{r}\bar{u}(\bar{r}, \vartheta, \varphi) \qquad \text{with} \quad \bar{r} \equiv \frac{1}{r}$$

   where A is a constant still to be chosen and function

$$\bar{u}(\bar{r}, \vartheta, \varphi) \quad \text{and} \quad u(r, \vartheta, \varphi)$$

   are the same function using different names. Show that then on the surface of the unit sphere,

$$u_{\text{out}}(1, \vartheta, \varphi) = A\bar{u}(1, \vartheta, \varphi) = Au(1, \vartheta, \varphi)$$

   Show next that

$$u_{\text{out},r} = -A\left(\frac{\bar{u}}{r^2} + \frac{1}{r^3}\bar{u}_{\bar{r}}\right)$$

   where subscripts indicate partial derivatives with respect to that variable. Conclude that on the surface on the unit sphere

$$u_{\text{out},r}(1, \vartheta, \varphi) = -A\bar{u}(1, \vartheta, \varphi) - A\bar{u}_r(1, \vartheta, \varphi) = -Au(1, \vartheta, \varphi) - Au_r(1, \vartheta, \varphi)$$

   Now look up the Laplacian in spherical coordinates and evaluate $\nabla^2 u_{\text{out}}$ in terms of $\bar{u}(\bar{r}, \vartheta, \varphi)$ using the chain rule of differentiation. Show that

$$\nabla^2 u_{\text{out}} = \frac{A}{r^5}\left(\frac{1}{\bar{r}^2}(\bar{r}^2\bar{u}_{\bar{r}})_{\bar{r}} + \frac{1}{\bar{r}^2 \sin\vartheta}(\sin\vartheta\, \bar{u}_\vartheta)_\vartheta + \frac{1}{\bar{r}^2 \sin^2\vartheta}\bar{u}_{\varphi\varphi}\right)$$

   Hint: in the right hand side, differentiate out the first product before comparing with what you got from differentiating out $\nabla^2 u_{\text{out}}$. Conclude that the Laplacian of u_{out} is zero.

[2] http://www.eng.fsu.edu/~dommelen/courses/aim2/09/topics/pdes/elliptic/
[3] http://www.eng.fsu.edu/~dommelen/courses/aim2/09/topics/pdes/elliptic/

3. In view of the formulae derived in the previous question, show that A must be $A = -1$ in order for the surface integrals in (6) only to involve the function

$$u(1, \vartheta, \varphi) = g(\vartheta, \varphi)$$

given in a Dirichlet problem, and not the then unknown function $u_r(1, \vartheta, \varphi)$. Also show that it would not be possible to choose A so that u drops out from both integrals, so that the Neumann problem cannot be solved this way. For the Dirichlet problem, use spherical coordinates for the point $\vec{x}$ at which u is to be evaluated and for the generic integration point $\vec{\xi}$ as in

$$\vec{x} = r\hat{\imath}_r \quad \hat{\imath}_r = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} \qquad \vec{\xi} = \rho\hat{\imath}_\rho \quad \hat{\imath}_\rho = \begin{pmatrix} \sin\bar{\vartheta}\cos\bar{\varphi} \\ \sin\bar{\vartheta}\sin\bar{\varphi} \\ \cos\bar{\vartheta} \end{pmatrix}$$

In those terms, show that the integrals to be evaluated reduce to

$$\int_S g(\bar{\vartheta}, \bar{\varphi}) \left(2\frac{\partial G}{\partial \rho} + G \right) \mathrm{d}S \qquad \mathrm{d}S = \rho^2 \sin\bar{\vartheta}\, \mathrm{d}\bar{\vartheta}\mathrm{d}\bar{\varphi}$$

Show that the parenthetical expression may be simplified to

$$2\frac{\partial G}{\partial \rho} + G = \frac{1}{4\pi d}\left(\frac{\rho^2 - r^2 + d^2}{d^2 \rho} - 1 \right) \qquad d \equiv |\vec{x} - \vec{\xi}|$$

Use this plus the made assumption that $\rho = R = 1$ on the boundary to derive the Poisson integral (9).

4. 3.38. This does *not* require solution of the problem using the Poisson integral formula. You can just examine what symmetry properties the solution $u(x, y)$ should have to figure out the value at the origin. However, feel free to check your result against the Poisson integral.

5. 3.39. Again this does *not* require solution of the problem using the Poisson integral formula. You should be able to find the complete solution $u(x, y)$ by mere inspection. However, feel free to check your result against the Poisson integral; in that case, first write the boundary values in the form

$$f(\phi) = A + B\cos(\phi - \theta) + C\sin(\phi - \theta)$$

The integrals for A and B can be found in a table of definite integrals.

6. 3.40. This is about the solution to the Dirichlet problem in a circle, which, as derived in class, is given by the Poisson integral formula (also listed in 3.37.) Because the problem for u is linear, the *change* in u due to the change in boundary condition is the Poisson integral solution when f is zero everywhere except in the interval (θ_1, θ_2), where it equals the change in boundary condition.

10 03/27 F

1. 3.30. In (a), use the property mentioned in class that the minimum of a harmonic function must occur on the boundary. In (b), try $1 - y$ in the domain Ω given by $y \geq 0$. In (c), consider the functions $s = v - u$ and $t = w - v$.

2. 4.16. This is the heat equation equivalent of the uniqueness proof of the Poisson equation. You need to use method 2 of solved problem 4.2. Ignore the hint, which is wrong. Instead, you can assume that $\beta \neq 0$, since $\beta = 0$ has already been covered in 4.2. Use that to eliminate $\partial v/\partial n$.

3. 4.17a. (Question (b) was done in class, and the stated condition that F only needs to be continuous is not sufficient, but integrable and continuous would do it.)

4. 4.18. (This is the basic solution for the temperature u in a bar of length 1 where the ends of the bar are kept at zero temperature. Of course, the values of the constants C_n will normally follow from some given initial temperature, and the number of terms in the sum N will normally be infinite.)

5. 4.19. Plane wave solutions are solutions that take the form (2) in solved problem 4.12, with $\vec{\alpha}$ a constant vector and μ a constant. This sort of solutions are a multi-dimensional generalization of the $f(x - at)$ moving "wave" solution of the one-dimensional wave equation. In fact, if you take $\vec{\alpha}$ to be a unit vector, it gives the oblique direction of propagation of the wave and μ gives the wave propagation speed. However, in this case you will see that the function F cannot be an arbitrary function unless $b = 0$. You may want to do the case $b = 0$ separately. And also split up the cases for μ.

6. Solve the wave equation $u_{tt} = a^2 u_{xx}$ using a Fourier transform in x. From it, show that the solution takes the form
$$u(x,t) = f_1(x + at) + f_2(x - at)$$
Hint: stick with complex exponentials in the solution of the ODE.

7. 5.27(a). Include a sketch of the characteristic lines. Is the solution you get valid everywhere?

11 04/03 F

1. 5.27(b). Do not try to use an initial condition written in terms of two different, related, variables. Get rid of either x or y in the condition! Include a sketch of the characteristic lines.

2. Solve the Burgers' equation, as given in 5.34, using the method of characteristics. Give two different forms of the solution, one in which you set $C_1 = C_1(C_2)$ and another in which you set $C_2 = C_2(C_1)$ and then re-express that function argument in terms of the physical variables. Check that the solutions v and w are in each of the seven regions given by at least one of these two forms, and identify the corresponding function C_1 or C_2. If a function C_1 or C_2 does not exist, show why not.

3. Continuing 5.34, in two very neat xt-planes on raster paper, draw the given two solutions v and w up to time $t = 3$. In particular, draw the two shocks (jump discontinuities) $x = \frac{1}{2}t$ and $x = 1 + \frac{1}{2}t$ in the xt plane for v as fat lines. Similarly draw the single shock in the xt-plane for w; this shock is the straight line $x = 1 + \frac{1}{2}t$ below $t = 2$, but starts curving upward a bit above $t = 2$ as the expansion fan hits it. Draw the characteristic lines as thinner lines. Make sure you draw enough characteristics, especially inside the expansion fan of w.

4. Show that if the Burgers' equation as written gives the physically correct conservation law, then the conserved quantity is $\int u \, \mathrm{d}x$ and its flux is $f = \frac{1}{2}u^2$. Check that the characteristic speed is indeed f'. Then check (for $t \le 2$) that all three shocks satisfy the jump relations:
$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \frac{f_2 - f_1}{u_2 - u_1}$$
where x_s is the location of the shock and 2 stands for the point immediately behind the shock at a given time and 1 to the point immediately before it. Check (for $t \le 2$) that two shocks satisfy the entropy condition
$$f_1' > \frac{f_2 - f_1}{u_2 - u_1} > f_2'$$
but that the first shock of v does not. Check from your graph that indeed the characteristics come out of this shock, rather than run into it. Conclude that solution v has an unphysical expansion shock and that the correct solution is w.

5. In 7.27, acoustics in a pipe with closed ends, assume $\ell = 1$, $a = 1$, $f(x) = x$, and $g(x) = 1$. Graphically identify the extensions $F(x)$ and $G(x)$ of the given $f(x)$ and $g(x)$ to all x that allow the solution u to be written in terms of the infinite pipe D'Alembert solution.

6. Continuing the previous problem, in three separate graphs, draw $u(x, 0)$, $u(x, 0.25)$, and $u(x, 0.5)$. For the latter two graphs, also include the separate terms $\frac{1}{2}F(x-at)$, $\frac{1}{2}F(x+at)$, and $\int_{x-at}^{x+at} G(\xi)\,\mathrm{d}\xi$. Use raster paper or a plotting package. Comment on the boundary conditions. At which times are they satisfied? At which times are they not meaningful? Consider all times $0 \le t < \infty$ and do not approximate.

7. Using the solution of the previous problems, find $u(0.1, 3)$.

12 04/10 F

1. Write the *complete* (Sturm-Liouville) eigenvalue problem for the eigenfunctions of 7.27.
2. Find the eigenfunctions of that problem. Make very sure you do not miss one. Write a symbolic expression for the eigenfunctions in terms of an index, and identify all the values that index takes.
3. Write $f = x$ and $g = 1$ in terms of these eigenfunctions for the case $\ell = 1$. Be very careful with one particular eigenfunction. Note that sometimes you need to write a term of a sum or sequence out separately from the others.
4. Substitute $u(x,t) = \sum_n u_n(t)X_n(x)$ into the PDE to convert it into an ordinary differential for each separate coefficient $u_n(t)$. Solve the ODE. Be very careful with one particular case.
5. By writing the initial conditions in terms of the eigenfunctions, identify the integration constants. Write out a complete summary of the solution. Make sure to identify the values of the symbolic index in each expression.
6. Using some programming language, evaluate the found solution at 101 equally spaced x-values from 0 to ℓ at time $t = 0.25$ and so plot u versus x at that time. Repeat for $t = 0.5$. Include at least 50 nonzero terms in the summations. Take $\ell = 1$ and $a = 1$. Compare with your (or the instructor's) D'Alembert solution. It should show good agreement. What happens if you only include 10 term in the summations?

   To help you get started, a Matlab program that plots the solution to problem 7.28 is provided as an example. You need both p7_28.m[4] and p7_28u.m[5]. This program is valid for the PDE and BC solved in class, with the additional data

   $$a = \tfrac{1}{2}, \quad \ell = \tfrac{1}{2}\pi, \qquad f(x) = \tfrac{1}{2}\pi - x \Rightarrow f_n = \frac{1}{(2n-1)^2}, \qquad g(x) = 0 \Rightarrow g_n = 0.$$

   These may of course not apply for your problem.

   To run the program, enter matlab and type in `p7_28`. If you do not have matlab, a free replacement is octave. Or you can use some other programming and plotting facilities.

13 04/17 F

1. Refer to problem 7.19. Find a function $u_0(x,t)$ that satisfies the inhomogeneous boundary conditions. Define $v = u - u_0$. Find the PDE, BC and IC satisfied by v.
2. Find suitable eigenfunctions in terms of which v may be written, and that satisfy the homogeneous boundary conditions. Work out the Fourier coefficients of the relevant functions in the problem for v as far as possible.
3. Solve for v using separation of variables in terms of integrals of the known functions $f(x)$, $g_0(t)$, and $g_1(t)$. Write the solution for u completely.
4. Assume that $f = 0$, $k = \ell = 1$, and that $u_x = t$ at both $x = 0$ and $x = \ell$. Work out the solution completely.
5. Plot the solution numerically at some relevant times. I suspect that for large times the solution is approximately

   $$u = (x - \tfrac{1}{2})t + \tfrac{1}{6}(x - \tfrac{1}{2})^3 - \tfrac{1}{8}(x - \tfrac{1}{2})$$

   Do your results agree?

[4] `p7_28.m`
[5] `p7_28u.m`

6. Solve the problem of the unidirectional flow of a viscous fluid if a plate at $x = 0$ exerts a given shear stress on the fluid. The PDE is:

$$v_t = \nu v_{xx} \qquad x > 0 \quad t > 0$$

and the initial and boundary conditions are

$$v(x,0) = 0 \qquad \rho\nu v_x(0,t) = f(t) \quad v \sim 0 \text{ for } x \to \infty$$

where ν and ρ are constants and the force per unit area $f(t)$ is a given function.

14 04/24 F

1. Solve 7.26, by Laplace transforming the problem as given in time. This is a good way to practice back transform methods. Note that one factor in $\widehat{u}$ is a simpler function at a shifted value of coordinate s.

2. Solve 7.35 by Laplace transform in time. Clean up completely; only the given function may be in your answer, no Heaviside functions or other weird stuff. There is a minor error in the book's answer.

3. Derive the (real) Fourier series for the 2π-periodic function $f(\theta)$ that satisfies

$$f = 0 \text{ for } |\theta| < \tfrac{1}{2}\pi \qquad f = \tfrac{1}{2} \text{ for } |\theta| = \tfrac{1}{2}\pi \qquad f = 1 \text{ for } \tfrac{1}{2}\pi < |\theta| \le \pi$$

   If possible, check your answer versus 24.17 in the math handbook.

4. Plot the Fourier series for increasing number of terms, $n_{\max} = 1, 3, 7, 15$ using 360 equally spaced plot points in the range from 0 to 2π.

5. Based on these plots, discuss whether the Fourier series really converges to the given function for high enough number of terms:

   (a) When does an actual jump first show up?

   (b) Are there points where the Fourier series does not converge? If so, give the θ values for which it does not converge.

   (c) If it converges everywhere, the maximum difference between the Fourier series and the given function should obviously become zero. You might want to plot the function for $n_{\max} = 100$ using 1800 equally spaced plot points before you answer that. Explain.