Analysis II — Spring 2011

Homework Problems

Do not print out this page. Keep checking for changes. Complete assignment will normally be available the day after the last lecture whose material is included in the assignment (Monday, normally).

1 01/14 F

Use vector analysis wherever possible.

1. 1st Ed: p13, q31a-f,h-j, 2nd Ed: p17, q31a-i. *if they can be vectors, count them as such.*
2. 1st Ed: p13, q32, 2nd Ed: p17, q32. Do it both graphically and analytically.
3. 1st Ed: p14, q48, 2nd Ed: p19, q46. Use vector calculus only, no trig.
4. 1st Ed: p32, q66, 2nd Ed: p38, q66.
5. 1st Ed: p32, q82, 2nd Ed: p40, q82a, where **B** should be corrected to $(1, -3, 4)$. Do it without finding the actual sides of the parallelogram. Vector calculus only, no trig.
6. 1st Ed: p33, q87, 2nd Ed: p41, q87.
7. 1st Ed: p33, q90, 2nd Ed: p41, q90a.

2 01/21 F

1. 1st Ed: p53, q32, 2nd Ed: p64, q32. Draw the curve neatly.
2. 1st Ed: p54, q47, 2nd Ed: p65, q47. (30 points)
3. 1st Ed: p78, q46, 2nd Ed: p91, q46. $r = \sqrt{x^2 + y^2 + z^2}$
4. 1st Ed: p78, q54, 2nd Ed: p92, q54. You may want to refresh your memory on total derivatives.

3 01/28 F

1. 1st Ed: p78, q60, 2nd Ed: p92, q60. (20 points) Also find two scalar equations that describe the line through P that crosses the surface normally at P.

   Find the unit normal $\vec{n}$ to the surface at P. Now assume that the surface is reflective, satisfying Snell's law. An incoming light beam parallel to the x-axis hits the surface at P. Find a vector equation that describes the path of the *reflected* beam.

   Hint: let $\vec{v}$ be a vector along the light ray. The *component* of $\vec{v}$ in the direction of $\vec{n}$ is $\vec{n} \cdot \vec{v}$. The *component vector* in the direction of $\vec{n}$ is defined as $\vec{v}_1 = \vec{n}(\vec{n} \cdot \vec{v})$. Sketch this vector along with vector $\vec{n}$. In which direction is the remainder $\vec{v}_2 = \vec{v} - \vec{v}_1$? Now figure out what happens to $\vec{v}_1$ and $\vec{v}_2$ during the reflection. Take it from there.

2. 1st Ed: p80, q87, 2nd Ed: p93, q87. (20 points) Compare with a point sink in which

   $$\vec{v} = -\frac{x\hat{\imath} + y\hat{\imath}}{x^2 + y^2}$$

   For each flow, compute the divergence, draw streamlines, and figure out how much fluid passes through a circle of arbitrary radius r. (Since the velocity is radial, the fluid flow through a circle is the magnitude of the velocity times the circumference of the circle.) Based on the results, explain where all the fluid that enters the unit circle disappears. In particular, at an arbitrary point r, θ, what is the amount of fluid disappearing per unit area? So, what do you think of the value of the divergence of the point sink at the origin?

3. 1st Ed: p80, q102, 2nd Ed: p94, q102.
4. 1st Ed: p81, q107, 2nd Ed: p94, q107. (20 points). You need to show that any solution $\vec{E}, \vec{H}$ of Maxwell's equations is given by scalar and vector potentials $\phi, \vec{A}$ as shown. Hints:

   Recall that if the divergence of a vector is zero, the vector is the curl of some other vector $\vec{A}_0$. (Actually, I forgot to tell you that, but you know it now.)

   Also, you can certainly *define* $\vec{E}_\phi$ by setting

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}_0}{\partial t} + \vec{E}_\phi$$

   but it is not automatic that $\vec{E}_\phi$ is minus the gradient of some scalar ϕ_0. That is for you to show.

   Unfortunately, $\vec{A}_0$ and ϕ_0 are not unique and do not normally satisfy (1) in the book. The potentials you need are of the form

$$\vec{A} = \vec{A}_0 + \nabla\psi \qquad \phi = \phi_0 - \frac{1}{c}\frac{\partial \psi}{\partial t}$$

   Show that in those terms,

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi \qquad \vec{H} = \nabla \times \vec{A}$$

   regardless of what you take for ψ. (That is the famous gauge property of the electromagnetic field.) The way that you want to take ψ is so that equation (1) in the book is satisfied. Show that this leads to a partial differential equation for ψ.

   Now substitute into the four Maxwell equations and so find the requirements that $\vec{A}$ and ϕ must satisfy.

   Show directly from Maxwell's first and last equation that the charge density must not vary in time. (That is because the current density in the last equation was left out. There should be a $4\pi\vec{j}/c$ in the last equation, as well as in (3).)
5. 1st Ed: p102, q32, 2nd Ed: p122, q32.
6. 1st Ed: p103, q44, 2nd Ed: p123, q44. Do it without using Stokes.

4 02/04 F

1. 1st Ed: p103, q44, 2nd Ed: p123, q44. Do it with Stokes.
2. 1st Ed: p104, q62, 2nd Ed: p124, q62. (20 points) Do both directly and using the divergence theorem. Make sure to include the base of the cone. Use the Cartesian expression for $\vec{n}\,\mathrm{d}S$ to formulate the surface integral, then switch to polar to do it.
3. 1st Ed: p132, q50, 2nd Ed: p154, q50 MODIFIED. Given

$$\vec{v} = \frac{(-y, x)}{x^2 + y^2}$$

   evaluate $\nabla \times \vec{v}$. Also evaluate, presumably using polar coordinates,

$$\oint_{\mathrm{I}} \vec{v} \cdot \mathrm{d}\vec{r} \qquad \oint_{\mathrm{II}} \vec{v} \cdot \mathrm{d}\vec{r}$$

   where path I is the semi circle of radius r going clockwise from $(r, 0)$ to $(-r, 0)$, and path II is the semi circle of radius r going counter-clockwise from $(r, 0)$ to $(-r, 0)$. Explain why the integral over II minus the integral over I is the integral over the closed circle. Explain why Stokes implies that the closed contour integral should be the integral of the z-compont of $\nabla \times \vec{v}$ over the inside of the circle. Then explain why you would then normally expect the contour integral to be zero. That means that the two integrals I and II should be equal, but they are not. Explain what the problem is.

Do you expect integrals over closed contours of different radii to be equal? Why? Are they equal?

Now assume that you allow singular functions to be OK, like Heaviside step functions and Dirac delta functions say. Then figure out in what part of the interior of the circle, $\int\int \nabla \times \vec{v} \cdot \hat{k}\,\mathrm{d}x\mathrm{d}y$ is not zero. So how would you describe $\nabla \times \vec{v}$ for this vector field in terms of singular functions?

4. 1st Ed: p133, q56, 2nd Ed: p155, q56.

5. Read through subsection 10.4 of QMFE[1] and write a half-page summary.

6. Derive $\vec{n}\,\mathrm{d}S$ in terms of $\mathrm{d}\theta$ and $\mathrm{d}\phi$, where (r, θ, ϕ) are spherical coordinates, assuming that the surface is given as $r = f(\theta, \phi)$. Use that:

$$\vec{r} = r\,\hat{\imath}_r \qquad \frac{\partial\,\hat{\imath}_r}{\partial\theta} = \hat{\imath}_\theta \qquad \frac{\partial\,\hat{\imath}_r}{\partial\phi} = \sin\theta\,\hat{\imath}_\phi$$

as will be derived in class. Next generalize the result to the case that the surface is given by the implicit expression $F(r, \theta, \phi) = 0$. One way to do so is to find $\partial r/\partial\theta$ and $\partial r/\partial\phi$ from the total differential

$$\frac{\partial F}{\partial r}\mathrm{d}r + \frac{\partial F}{\partial\theta}\mathrm{d}\theta + \frac{\partial F}{\partial\phi}\mathrm{d}\phi = 0$$

Now clean up your result to something similar to the vector expression that you have in Cartesian coordinates,

$$\vec{n}\,\mathrm{d}S = \frac{\nabla F}{F_z}\,\mathrm{d}x\mathrm{d}y$$

as derived in class. You may want to look up the gradient in spherical coordinates, and the infinitesimal volume element.

7. 1st Ed: p160, q38, 2nd Ed: p183, q38. Simplify as much as possible.

5 02/11 F

1. Finish finding the derivatives of the unit vectors of the spherical coordinate system using the class formulae. Then finish 1st Ed p160 q47, 2nd Ed p183 q47, as started in class, by finding the acceleration. Note that the metric indices h_i for spherical coordinates are in mathematical handbooks. Also,

$$\frac{\partial\hat{\imath}_i}{\partial u_i} = \frac{1}{h_i}\frac{\partial h_i}{\partial u_i}\hat{\imath}_i - \sum_{j=1}^{3}\frac{1}{h_j}\frac{\partial h_i}{\partial u_j}\hat{\imath}_j \qquad \frac{\partial\hat{\imath}_i}{\partial u_j} = \frac{1}{h_i}\frac{\partial h_j}{\partial u_i}\hat{\imath}_j$$

2. Express the acceleration in terms of the spherical velocity components v_r, v_θ, v_ϕ and their first time derivatives, instead of time derivatives of position coordinates. Like $a_r = \dot{v}_r + \ldots$, etc. This is how you do it in fluid mechanics, where time-derivatives of particle position coordinates are normally not used. (So, get rid of the position coordinates with dots on them in favor of the velocity components.)

3. Notes 1.2.1.1

4. Notes 1.2.1.3 and 1.2.1.4

5. Notes 1.2.1.5

6. Notes 1.2.1.6

7. Notes 1.2.1.7

8. Notes 1.2.1.8

9. Notes 1.2.1.9

[1] http://www.eng.fsu.edu/~dommelen/quantum/style_a/maxwell.html

6 02/18 F

1. Notes 1.2.2.1.
2. Notes 1.2.3.1
3. Notes 1.2.3.3
4. Notes 1.2.3.5
5. Notes 1.6.1.1
6. Notes 1.6.1.2
7. Notes 1.6.3.2
8. Notes 1.6.3.3
9. Notes 1.6.3.5

7 02/25 F

1. 2.19b, h. Show a picture of the different regions.
2. Notes 1.3.2.1
3. Notes 1.4.3.1
4. Notes 1.4.4.1
5. 2.22b,g. Draw the characteristics very neatly in the xy-plane,
6. 2.28d. First find a particular solution. Next convert the remaining homogeneous problem to characteristic coordinates. Show that the homogeneous solution satisfies

$$2u_{h,\xi\eta} = u_{h,\eta}$$

   Solve this ODE to find $u_{h,\eta}$, then integrate $u_{h,\eta}$ with respect to η to find u_h and u. Write the total solution in terms of x and y.

7. 2.28f. In this case, leave the inhomogeneous term in there, don't try to find a particular solution for the original PDE. Transform the full problem to characteristic coordinates. Show that the solution satisfies

$$4u_{\xi\eta} - 2u_\xi \pm e^\eta = 0$$

   where $\pm$ indicates the sign of xy, or

$$4\xi\eta u_{\xi\eta} - 2\xi u_\xi + \eta = 0$$

   or

$$4\xi\eta u_{\xi\eta} + 2\xi u_\xi - \frac{1}{\eta} = 0$$

   or equivalent, depending on exactly how you define the characteristic coordinates. Solve this ODE for u_ξ, then integrate with respect to ξ to find u. Write the solution in terms of x and y.

8 03/04 F

1. 2.28c. Use the 2D procedure. Show that the equation may be simplified to

$$u_{\xi\xi} = 0$$

   Solve this equation and write the solution in terms of x and y.

2. Notes 1.5.3.1

3. 2.28k. Reduce the PDE to the form

$$u_\eta = \left(e^{-\xi} + \frac{1}{\eta}\right) u_{\xi\xi}$$

   Now discuss the properly posedness for the initial value problem, recalling from the class notes that the backward heat equation is not properly posed. In particular, given an interval $\xi_1 \leq \xi \leq \xi_2$, with an initial condition at some value of η_0 and boundary conditions at ξ_1 and ξ_2, can the PDE be numerically solved to find u at large η? If η_0 is positive? If η_0 is a small negative number? If η_0 is a large negative number?

4. 3.44. This is mostly the uniqueness proof given in class, which can also be found in the notes and more generally in solved problems 3.14-3.16. However, here you will want to write out the two parts of the surface integral separately since the boundary conditions are a mixture of the two cases 3.14 and 3.15 (with $c = 0$).

5. Notes 1.7.1.1

6. Notes 1.7.1.2

9 03/18 F

1. Notes 2.1.1.1

2. Notes 2.1.1.2

3. Notes 2.2.2.1

10 03/25 F

1. Notes 2.3.3.1

2. Notes 2.3.6.1

3. Notes 2.3.6.2

11 04/01 F

1. 5.25. Also: (c) Assume that

$$f(x) = e^{-|x-2|}$$

   In a single very neat plot, draw $u(x,1)$, $u(x,2)$, and $u(x,3)$ versus x.

2. 5.26b. Ignore the hint. Simplify your answer as much as possible. Draw the characteristic very neatly in the x, y-plane.

3. 5.27(a). Include a sketch of the characteristic lines. Is the solution you get valid everywhere?

4. 5.27(b). Do not try to use an initial condition written in terms of two different, related, variables. Get rid of either x or y in the condition. Then call the argument of your undetermined function γ and rewrite its expression in terms of γ. Include a sketch of the characteristic lines.

5. 5.29 Explain why there is no solution.

12 04/08 F

1. In 7.27, acoustics in a pipe with closed ends, assume $\ell = 1$, $a = 1$, $f(x) = x$, and $g(x) = 1$. Graphically identify the extensions $F(x)$ and $G(x)$ of the given $f(x)$ and $g(x)$ to all x that allow the solution u to be written in terms of the infinite pipe D'Alembert solution. (F and G may have been called $\bar{f}$ and $\bar{g}$ in class.)

2. Continuing the previous problem, in three separate graphs, draw $u(x, 0)$, $u(x, 0.25)$, and $u(x, 0.5)$. For the latter two graphs, also include the separate terms $\frac{1}{2}F(x - at)$, $\frac{1}{2}F(x + at)$, and $\int_{x-at}^{x+at} G(\xi)\,\mathrm{d}\xi$. Use raster paper or a plotting package. *Use the D'Alembert solution only to plot, do not use a separation of variables solution.* Comment on the boundary conditions. At which times are they satisfied? At which times are they not meaningful? Consider all times $0 \le t < \infty$ and do not approximate.

3. Using the D'Alembert solution of the previous problems, find $u(0.1, 3)$.

4. Write the *complete* (Sturm-Liouville) eigenvalue problem for the eigenfunctions of 7.27.

5. Find the eigenfunctions of that problem. Make very sure you do not miss one. Write a symbolic expression for the eigenfunctions in terms of an index, and identify all the values that that index takes.

6. Continuing the previous homework, write $f = x$ and $g = 1$ in terms of the eigenfunctions you found for the case $\ell = 1$. Be very careful with one particular eigenfunction. Note that sometimes you need to write a term in a sum or sequence out separately from the others.

7. Substitute $u(x, t) = \sum_n u_n(t)X_n(x)$ into the PDE to convert it into an ordinary differential for each separate coefficient $u_n(t)$. Solve the ODE. Be very careful with one particular case.

8. By writing the initial conditions in terms of the eigenfunctions, identify the integration constants. Write out a complete summary of the solution. Make sure to identify the values of your numbering index in each expression.

13 04/15 F

1. Reconsider the separation of variables solution you derived in the previous homework. (If you got it wrong, the correct solution is online.) Using some programming language, evaluate the found solution at 101 equally spaced x-values from 0 to ℓ at time $t = 0.25$ and so plot u versus x at that time. Repeat for $t = 0.5$. Include at least 50 nonzero terms in the summations. Take $\ell = 1$ and $a = 1$. Compare with your (or the instructor's) D'Alembert solution. It should show good agreement. What happens if you only include 10 term in the summations?

   To help you get started, a Matlab program that plots the solution to problem 7.28 is provided as an example. You need both p7_28.m[2] and p7_28u.m[3]. This program is valid for the PDE and BC solved in class, with the additional data

   $$a = \tfrac{1}{2}, \quad \ell = \tfrac{1}{2}\pi, \qquad f(x) = \tfrac{1}{2}\pi - x \Rightarrow f_n = \frac{1}{(2n-1)^2}, \qquad g(x) = 0 \Rightarrow g_n = 0.$$

   These may of course not apply for your problem.

   To run the program, enter matlab and type in `p7_28`. If you do not have matlab, a free replacement is octave. Or you can use some other programming and plotting facilities.

2. Solve 7.26, by Laplace transforming the problem as given in time. This is a good way to practice back transform methods. Note that one factor in $\widehat{u}$ is a simpler function at a shifted value of coordinate s.

3. Solve 7.35 by Laplace transform in time. Clean up completely; only the given function may be in your answer, no Heaviside functions or other weird stuff. There is a minor error in the book's answer.

[2] `p7_28.m`
[3] `p7_28u.m`

14 04/22 F

1. Refer to problem 7.19. Find a function $u_0(x,t)$ that satisfies the inhomogeneous boundary conditions. Define $v = u - u_0$. Find the PDE, BC and IC satisfied by v.

2. Find suitable eigenfunctions in terms of which v may be written, and that satisfy the homogeneous boundary conditions. Write the relevant known functions in terms of these eigenfunctions and give the expressions for their Fourier coefficients.

3. Solve for v using separation of variables in terms of integrals of the known functions $f(x)$, $g_0(t)$, and $g_1(t)$. Write the solution for u completely.

4. Assume that $f = 0$, $k = \ell = 1$, and that $u_x = t$ at both $x = 0$ and $x = \ell$. Work out the solution completely.

5. Plot the solution numerically at some relevant times. I suspect that for large times the solution is approximately

$$u = (x - \tfrac{1}{2})t + \tfrac{1}{6}(x - \tfrac{1}{2})^3 - \tfrac{1}{8}(x - \tfrac{1}{2})$$

Do your results agree?