EML 5709 Fall 2006

Homework Problems

Do not print out this page. Keep checking for changes.

1 9/6

1. 1.4
2. If the density of air at sea level is 1.225 kg/m^3, what is the average spacing of the molecules? Molecular mass of air is 28. For what size of bodies would you expect major problems in trying to use a continuum approximation? If the free path length is 6.6 10^{-8} m, at what size would you expect that normal equations of motion (like the Euler and Navier-Stokes equations) become unusuable? When the body size becomes comparable to the mean molecular spacing or to the mean free path length?

   Repeat for 200 km height, where the number of molecules is 8 10^{15}/m^3 and the free path length 200 m.

2 9/13

1. 4.1
2. A two-dimensional flow field is given in Eulerian (and Cartesian) coordinates by:

   $$u = -y \qquad v = x$$

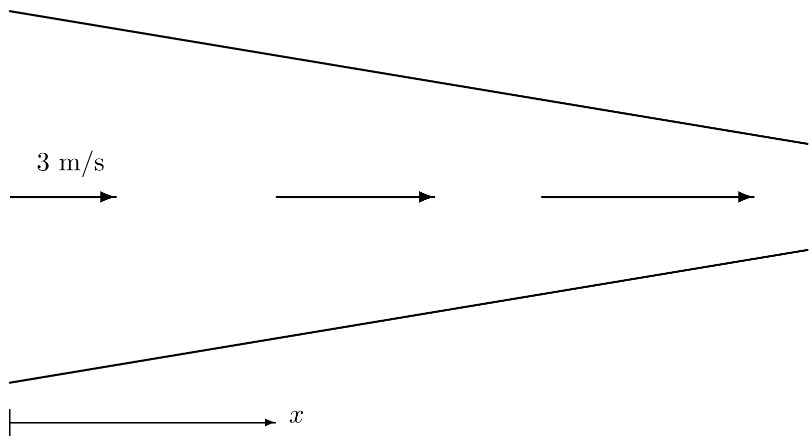
   Integrate the Cartesian particle path of a typical particle in this flow, assuming that the particle is initially at the point $x = \xi$ and $y = \eta$. Sketch the particle path. Write down the Lagrangian description of this flow. (Hint: If you do not remember how to solve systems of ordinary differential equation, differentiate the ODE for x once and then get rid of y in the equation using the other equation. Solve that 2nd order ODE for x. Then go back to the original ODE to figure out what y is. Then apply the initial conditions.)
3. What is the acceleration vector of the fluid particles for the flow above? So what do you think about the pressure field? How do isobars look?
4. Find the streamlines for the flow above from solving

   $$d\vec{r} // \vec{v}, \quad dt = 0$$

   Do you think this flow would be easier to solve in polar coordinates r, θ?

3 9/20

1. Consider the following flow of water through a two dimensional duct:



If the length of the duct is 10 m and the total vertical height of the duct is

$$h(x) = h(0) - 0.05x \qquad (0 \le x \le 10)$$

with $h(0) = 1$ m and the fluid enters at $x = 0$ with a velocity $u_0 = 3$ m/s, show that the centerline velocity at arbitrary x equals

$$u = \frac{3}{1 - 0.05x} \text{ m/s} \qquad v = 0 \text{ m/s}$$

and that the pressure is

$$p = p(0) + 4500 - \frac{4500}{(1 - 0.05x)^2} \text{ Pa}$$

where $p(0)$ is the pressure at $x = 0$, which you can take to be zero. Use mass conservation and Bernoulli. What are the exit velocity and pressure at $x = 10$ compared to the ones at the entrance $x = 0$?

2. See whether or not Euler's differential momentum equations are satisfied on the centerline:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y}$$

(By symmetry, v and $\partial p/\partial y$ are zero on the symmetry line.)

3. Use differential mass conservation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

to determine the sign of $\partial v/\partial y$ on the centerline. So, will v be positive or negative above the centerline? And is that what you would expect?

4. 4.5 ($v_3 = 0$. Note that this is our old stagnation point flow.)

5. 4.8 ($v_3 = 0$. Note that this is our old stagnation point flow.)

6. 4.2 (Note that you can find expressions for the vorticity in cylindrical coordinates in the appendices at the back of the book.)

7. 4.9 (Note that you can find expressions for the strain rate tensor in cylindrical coordinates in the appendices at the back of the book.)

4 9/29

1. Find the circulation around a circle around the origin for the difused vortex flow of question 4.2c by directly integrating the line integral. Also find it by integrating the vorticity in accordance with Stokes' theorem. Do you get the same result? If not, explain why not.
2. Find the circulation around a square around the origin for the line vortex flow of question 4.2b by directly integrating the line integral. Also find it by integrating the vorticity in accordance with Stokes' theorem. Do you get the same result? If not, explain why not. (Hint: Note that the flow of the previous question is the same as the one here if seen from a large distance. So, if you do the integrals of the previous question over a large circle and you look at them from far away, things would look the same.)
3. 5.1 (b). Also do and explain $\int_{\mathrm{FR}} \rho \, dV$. Note that this is incompressible *inviscid* flow. The viscous flow is much more complex.
4. If the jet leaves a rocket through an area of 0.5 m^2 at a velocity of 500 m/s relative to the rocket, and the exit density is 0.5 $\mathrm{kg/m}^3$, what can you say about the total mass of the rocket?
5. Find the average exit velocity in the pipe flow of question 5.13. Do the same for the flow of 5.14.
6. If in question 5.12, the liquid comes out of the tube of radius R with a Poiseuille axial velocity

$$v_z = V_{\max}\left(1 - \frac{r^2}{R^2}\right),$$

   and unknown radial and swirl velocity components, and with density ρ, then derive the mass flowing out of the tube per unit time. If at the short distance below the pipe, the stream has contracted to a radius R_c and the velocity has become uniform and equal to $\vec{v} = U\hat{\imath}_z$, then what is the value of the constant U in terms of the other parameters?
7. Suppose you want to compute the flow in a square region. Show how you can reduce the number of unknowns to a finite set by restricting the computation to a finite set of points. Formulate an equation for each (non boundary) point based on the law of mass conservation (continuity).

5 10/4

1. 5.1 (a).
2. 5.1 (d).
3. 5.1 (c) and (e).
4. 5.1 (f).
5. 5.3.

6 10/13

1. 5.11
2. 5.12
3. 5.22

7 10/20

1. In the last problem of set 9/29, you wrote and approximate continuity equation for a small Cartesian volume element in 2D. Show that if the size of the volume element becomes zero, the equation becomes the conservative differential continuity equation.
2. 5.2
3. Rederive the results of question 5.2 by writing integral continuity for the region between two cylinders, respectively two spheres, of radius r_0 and r.

8 10/27

1. 6.3
2. 6.5

9 11/6

All problems are for incompressible flow (constant density and coefficient of viscosity.)

1. 7.9. You can assume that the film thickness is so small that the curvature of the pipe wall can be ignored. In that case, it becomes 2D steady flow along a flat wall of spanwise length $2\pi r_0$ in the z-direction. Steady 2D flow means in this case that velocity and pressure are independent of z and time t and that the velocity in the z-direction $w = 0$. Do not ignore gravity. Assume "developed flow" in which the *velocity components* have become independent of x (taken to be downwards, with y the distance from the cylinder surface). For the boundary conditions at the free surface, assume that the liquid meets air of zero density and constant pressure p_a there. Also write appropriate boundary conditions where the fluid meets the cylinder surface. Do not make any other assumptions than listed above; they should be all you need.
2. What is the wall shear for the previous flow? Explain this value physically.
3. 7.5 Ignore gravity. To be original, assume that, in cylindrical coordinates, the streamlines are circles around the axis (which tells you that some velocity components are zero) and that the velocity and pressure are steady and independent of the axial coordinate z. Do *not* assume that the velocity components are independent of θ. Skip the r-momentum equation for now, you can do without by showing that the pressure gradient in the θ direction must be zero. To show that the pressure gradient in the θ direction is zero, first show that it must be independent of θ, (like we showed in class for duct flow). Next integrate to find the form of the pressure and use the fact that the pressure at a given point must have a unique value, so that if θ increases by 2π, the pressure must return to the same value. (p 785 has the NS equations you need and 781 the continuity equation.)
4. Show that the skipped r-momentum equation in the previous problem can also be satisfied, and give the most general pressure if all equations are satisfied.
5. (noncredit question) Consider the below graph for the minor head losses due to sudden changes in pipe diameter:

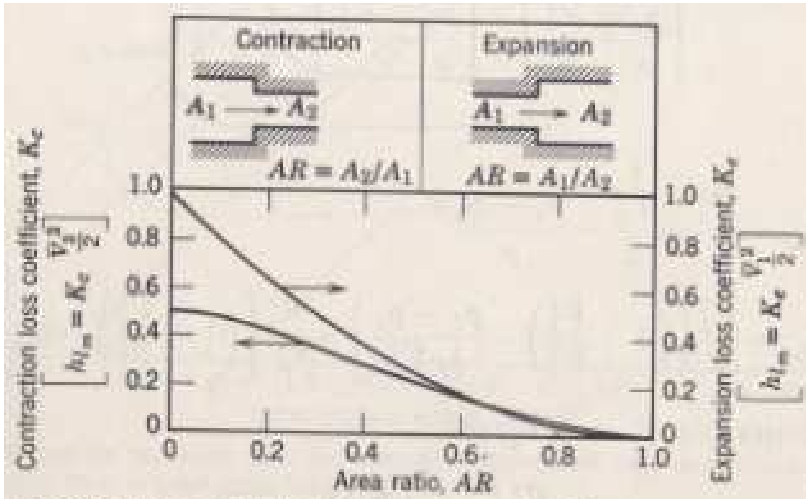


Fig. 8.16 Loss coefficients for flow through sudden area changes (Ref. 5).

Discuss the following issues as well as possible from the sort of flow you would expect.

(a) How come the head loss become zero for an area ratio equal to 1?

(b) Why would the head loss be exactly one for a large expansion? Coincidence?

(c) Why would the head loss be less than one if the expansion is less? If the expansion is less, is not the pipe wall in the expanded pipe closer to the flow, so should the friction with the wall not be more??

(d) Why is there a head loss for a sudden contraction? The mechanism cannot be the same as for the sudden expansion, surely? Or can it?

10 11/15

1. 7.17a corrected. Repeat the Stokes problem analysis, but now assume that instead of the plate velocity for $t > 0$ constant being a constant V_0, the plate velocity is $V_0(t) = At^n$. (So that Stokes' 2nd problem corresponds to the special case $n = 0$.) Assume the velocity profiles become similar in terms of $f_n = u/V_0(t)$ and $\eta = y/\delta(t)$ where $\delta(t)$ is a typical boundary layer thickness to be determined and $f_n(\eta)$ the scaled velocity profile. Plug the similarity assumption $u = V_0(t)f_n(\eta)$ into the PDE, and note that it does not separate if n is not zero. But show that δ is still the same as for Stokes 2nd by writing the PDE at $\eta = 0$ and noting that $f''(0)/nf(0)$ is just a constant that you can take to be 4 (any other value only changes the definition of δ.) With δ found, clean up the PDE to a simple ODE for $f_n(eta)$. Verify that for $n = 0$ it is Stokes' one ($f_0'' + 2\eta f_0' = 0$.)

2. 7.17b Differentiate the Stokes ODE and verify that it produces the ODE for $f_{-1/2}$ if $f_0' = f_{-1/2}$. Since the general solution for the Stokes problem was
$$f_0 = C_1 + C_2 \mathrm{erfc}(\eta),$$
where $\mathrm{erfc}(\infty)$ was zero, its derivative provides a solution for $f_{-1/2}$. Show that it can satisfy both boundary conditions for $f_{-1/2}$, at $\eta = 0$ and $\eta = \infty$. What does C_2 have to be?

3. 7.17c. Now go the other way, to get the requested solution at $n = 1/2$. Differentiate the equation for $f_{1/2}$, and you will get an equation for $f_{1/2}'$ for which you know the solution. Integrate to find $f_{1/2}$ itself, and make sure that the boundary condition at $\eta = \infty$ is satisfied. Don't worry too much about the boundary condition at $\eta = 0$. (This process can be repeated to find solutions for any half-integer value of n.)

4. 7.17d. Is there a value of n for which the *shear stress* that the plate applies to the fluid is constant? If so, sketch the plate velocity for that case as a function of time.

5. It is sometimes claimed that bathtub vortices rotate counterclockwise in the northern hemispere and clockwise in the southern one. Assume you are on the north pole and fill a cylindrically symmetric bathtub of radius 1 m with water. When a circular contour of water particles of initial radius 1 m goes out the drain of radius 1 cm, the tangential rotating velocity of the cicular contour increases according to Kelvin's theorem. Find out how much the tangential velocity was when the water was at rest compared to the tub with the drain closed, and from that, the tangential velocity when it is going in the drain. Express in terms of the revolutions per second the contour makes.

6. Solve the incompressible irrotational flow around an expanding cylinder of radius $r_0(t)$. Write the partial differential equation and boundary conditions. Solve after assuming that the tangential velocity component v_θ is zero by symmetry. (Actually, you might notice that the tangential velocity component does *not* have to be zero, but anyway.)

11 11/22

1. The potential of irrotational transverse flow around a circular cylinder of radius r_0 is

$$\phi = U\left(r + \frac{r_0^2}{r}\right)\cos(\theta) - \frac{\bar{\Gamma}}{2\pi}\theta$$

   where U is the magnitude of the velocity at large distances, and $\bar{\Gamma}$ the circulation around the cylinder. Verify that the correct boundary conditions at the surface of the cylinder and at large distances are satisfied.

2. Verify that the circulation around the cylinder is indeed $\bar{\Gamma}$ by integrating around a suitable contour.

3. For the flow of the previous question, find the pressure on the cylinder surface as a function of angular location.

4. For the flow of the previous question, find the force on the cylinder assuming the viscosity is zero, so there are no viscous stresses on the cylinder surface. Is d'Alembert satisfied? Is Kutta-Joukowski satisfied?

5. Derive the streamfunction of ideal stagnation point flow. The steps are similar to the ones used to derive the corresponding potential flow in class. From the streamfunction, determine the mathematical form of the streamlines.

12 12/01

1. Derive the streamfunction of irrotational incompressible flow around a cylinder from solution of the PDE. The steps are similar to the ones used in class to derive the potential.

2. Compute approximate values of the Reynolds number of the following flows:

   (a) your car, assuming it drives;

   (b) a passenger plane flying somewhat below the speed of sound (assume an aerodynamic chord of 30 ft);

   (c) flow in a 1 cm water pipe if it comes out of the faucet at .5 m/s,

   In the last example, how fast would it come out if the Reynolds number is 1? How fast at the transition from laminar to turbulent flow?

3. If the complex potential flow of a source and a line vortex equal

$$F = \frac{Q}{2\pi}\ln(z) \qquad F = \mathrm{i}\frac{\bar{\Gamma}}{2\pi}\ln(z)$$

then what would be the real velocity potentials ϕ? (use polar coordinates.) Differentiate to find the velocities and compare to questions 4.2 and 4.9.

4. According to potential flow theory, what would be the lift per unit span of a flat-plate airfoil of chord 2 m moving at 100 m/s at sea level at an angle of attack of 10 degrees? What would be the drag?

5. What would be the circulation around the airfoil of the previous question?

6. Identify the boundary layer variables x, y, u, and v for the case of a circular cylinder of radius r_0 in terms of the cylindrical variables r, θ, v_r, and v_θ.

7. Using the result of the previous question, write the continuity equation in cylindrical coordinates from table C.3 in terms of the boundary layer coordinates and comment on the differences from the boundary layer continuity equation. Is the difference small?

8. Similarly, write the r and θ momentum equations of table C.5 in 2D and cross out the terms the boundary layer approximation ignores. Ignore gravity.

13 12/06

1. Write the boundary layer equations for the unsteady boundary layer flow around a cylinder that is impulsively started from rest at time $t = 0$ into a velocity U compared to the ambient air. (In other words, relative to the cylinder, the flow velocity far away equals $U\hat{\imath}$.) Give the pressure inside the boundary layer and the boundary conditions at the wall and at the outside edge of the boundary layer.

2. The potential flow towards a sink of fluid at the origin has polar velocity components

$$v_r = -\frac{Q}{2\pi r} \qquad v_\theta = 0$$

An infinitely thin, semi-infinite flat plate is placed in this flow field along the positive x-axis. Write the equations for the boundary layer problem along the top of the plate.

3. Fully specify the boundary conditions for the boundary layer problem at the surface at the plate and just above the boundary layer. Identify the pressure at all points in the boundary layer. What happens to the pressure when the sink is approached?

4. Formulate an appropriate similarity assumption $u = u_e(x)f'(\eta)$, $\eta = y/\delta(x)$. Sketch unscaled and scaled velocity profiles.

5. Satisfy continuity by defining a streamfunction for the boundary layer flow. Write the momentum equation and the boundary conditions in terms of this streamfunction. (Actually, the v-boundary condition would need to replaced by the condition that the wall is the $\psi = 0$ streamline, if you are picky.) Answer:

$$f'^2 - ff'' - \frac{u_e\delta'}{u_e'\delta}ff'' = 1 + \frac{\nu}{u_e'\delta^2}f'''$$

$$f(0) = f'(0) = 0 \qquad f'(\infty) = 1$$

6. Evaluate the momentum equation at the wall, noting the wall boundary conditions and the fact that $f'''(0)$ is just some constant that you can take equal to minus one. (Any other value just changes the definition of δ, not the physical flow.) Determine what the representative boundary layer thickness $\delta(x)$ is. Answer the question whether the boundary layer gets thicker when more and more fluid is being retarded by the viscous forces. If it does not, explain why not.

7. Plug the expression for the boundary layer thickness into the momentum equation and note that f drops out. So our *third* order equation for f' is really a *second* order equation for a variable $g = f'$. Reduce this second order equation to a *first* order equation for g' considered as a function of g. Use the chain rule of differentiation. Answer:
$$g^2 = 1 + \frac{\mathrm{d}g'}{\mathrm{d}g} g'$$

8. Solve this equation for g' as a function of g. The integration constant can be found from the boundary condition at infinite η: $f'(\infty) = 1$, so $f''(\infty) = 0$. Answer (with the right sign of the square root:)
$$\sqrt{\frac{3}{2}} g' = \sqrt{g^3 - 3g + 2}$$

9. Show that the cubic in the above first order ordinary differential for g can be factored as $(1-g)^2(2+g)$, and then integrate to find η as a function of g. Use a wall boundary condition to find the integration constant. Then invert the formula for η to find g, hence the velocity profile f' as a function of η. Congratulations. You have just solved the boundary layer equations analytically for this flow. Answer:
$$f'(\eta) = 3\tanh^2\left(\frac{\eta}{\sqrt{2}} + \tanh^{-1}\left(\sqrt{\frac{2}{3}}\right)\right) - 2$$

   with
$$u = -\frac{Q}{2\pi x} f'(\eta) \quad v = \frac{y}{x} u \quad p = p_\infty - \frac{\rho Q^2}{8\pi^2 x^2} \qquad \eta = \frac{y}{\sqrt{2\pi\nu/Qx}}$$