EML 5709 **Fall 2007**

Homework Problems

Do not print out this page. Keep checking for changes.

1 9/5

1. 1.4
2. If the density of air at sea level is 1.225 kg/m^3, what is the average spacing of the molecules? Molecular mass of air is 28. For what size of bodies would you expect major problems in trying to use a continuum approximation? If the free path length is 6.6 10^{-8} m, at what size would you expect that normal equations of motion (like the Euler and Navier-Stokes equations) become unusuable? When the body size becomes comparable to the mean molecular spacing or to the mean free path length?
3. Repeat for 200 km height, where the number of molecules is 8 10^{15}/m^3 and the free path length 200 m.
4. Derive the net pressure force per unit volume in the y-direction on a little cube of fluid.

2 9/12

1. 4.1
2. A two-dimensional flow field is given in Eulerian (and Cartesian) coordinates by:

$$u = -y \qquad v = x$$

Integrate the Cartesian particle path of a typical particle in this flow, assuming that the particle is initially at the point $x = \xi$ and $y = \eta$. Sketch the particle path. Write down the Lagrangian description of this flow. (Hint: If you do not remember how to solve systems of ordinary differential equation, here is another way to do it. The equation for x is

$$\frac{Dx}{Dt} = -y.$$

If you differentiate this once with respect to time, the right hand side becomes Dy/Dt and you can then get rid of y using the other equation. That produces a second order equation for x that you can solve. Then go back to the original ODE above to figure out what y is. Then apply the initial conditions.)

3. What is the acceleration vector of the fluid particles for the flow above? So what do you think about the pressure field? How do isobars look? (Hint: remember that the isobars are normal to the gradient of the pressure. The gradient of the pressure is the force per unit area, which is density times acceleration. So you can find the direction of the gradient of the pressure.)
4. Find the streamlines for the flow above from solving

$$d\vec{r}//\vec{v}, \quad dt = 0$$

Do you think this flow would be easier to solve in polar coordinates r, θ?

3 9/19

1. 4.4
2. 4.2 (Note that you can find expressions for the vorticity in cylindrical coordinates in the appendices at the back of the book.)

3. 4.5 ($v_3 = 0$. Note that this is our old stagnation point flow.) Neatly sketch the distortion of a typical small square fluid particle during a very short time interval in this flow.

4. 4.8 ($v_3 = 0$. Note that this is our old stagnation point flow.)

5. 4.9 (Note that you can find expressions for the strain rate tensor in cylindrical coordinates in the appendices at the back of the book.)

4 9/26

1. Find the circulation around a circle around the origin for the difused vortex flow of question 4.2c by directly integrating the line integral. Also find it by integrating the vorticity in accordance with Stokes' theorem. Do you get the same result? If not, explain why not.

2. Find the circulation around a square around the origin for the line vortex flow of question 4.2b by directly integrating the line integral. Also find it by integrating the vorticity in accordance with Stokes' theorem. Do you get the same result? If not, explain why not. (Hint: Note that the flow of the previous question is the same as the one here if seen from a large distance. So, if you do the integrals of the previous question over a large circle and you look at them from far away, things would look the same.)

3. 5.1 (b). Also do and explain $\int_{\mathrm{FR}} \rho \, dV$. Note that this is incompressible *inviscid* flow. The viscous flow is much more complex.

4. If the jet leaves a rocket through an area of 0.5 m^2 at a velocity of 500 m/s relative to the rocket, and the exit density is 0.5 kg/m^3, what can you say about what happens to the total mass of the rocket?

5. 5.1 (a).

6. 5.1 (d).

5 10/03

1. 5.1 (c).

2. 5.1 (e).

3. 5.1 (f).

4. 5.13

5. 5.12. You must first apply mass conservation.

6. 5.3

6 10/10

1. 5.2

2. Rederive the results of question 5.2 by writing integral continuity for the region between two cylinders, respectively two spheres, of radius r_0 and r.

3. 5.6. Hint: take the curl of the equation and simplify.

4. 5.7. Multiply the momentum equation by v_i and simplify.

7 10/17

1. 6.1
2. 6.2 Discuss in view of the fact, (6.1), that the Reynolds number must be small in such flows.
3. 6.3
4. 6.4

8 10/24

1. 7.5. You may assume that the velocity and pressure are independent of the angular and axial positions in cylindrical coordinates. Also assume that the flow is steady with no velocity component in the axial direction. Do not assume the radial velocity is zero.
2. In 7.5, what is the power needed to keep the rod rotating, per unit axial length? What is the pressure difference between the surfaces of the pipe and the rod?
3. 7.6. Do not ignore gravity, but assume the pipe is horizontal. Careful, the gravity vector is *not* constant in polar coordinates. Do not ignore the pressure gradients: assume the pressure can be any function $p = p(r, \theta, z, t)$. Merely assume that the pressure distribution at the end of the pipe and rod combination is the same as the one at the start. For the velocity assume $v_r = v_\theta = 0$ and $v_z = v_z(r, z)$. Anything else must be derived. Give both velocity and pressure field.
4. 7.9. You can assume that the film thickness is so small that the curvature of the pipe wall can be ignored. In that case, it becomes 2D steady flow along a flat wall of spanwise length $2\pi r_0$ in the z-direction. Take the x-axis downwards. Assume $\vec{v} = \vec{v}(y)$ only (developed flow), $w = 0$ (two-dimensional flow), and $p = p(x, y, z)$. Everything else must be derived; give both pressure and velocity field. Do not ignore gravity. For the boundary conditions at the free surface, assume that the liquid meets air of zero density and constant pressure p_a there. Also write appropriate boundary conditions where the fluid meets the cylinder surface.

9 10/31

1. For the case of question 7.6, what is the force required to pull the rod through the axis, per unit length? In 7.9, what is the net downward shear force on the pipe? Does the simple answer surprise you? Why not?
2. 7.4. Argue your answer. In what terms would you ballpark the answer? What is the importance of the pressure level? Of the flow velocity? What are the relevant values involved? What are the most important uncertainties?
3. 7.1a Assume only that the velocity only depends on r, $\vec{v} = \vec{v}(r)$, that $p = p(r, \theta, z, t)$ is arbitrary, and that the pipe is horizontal. Use the effective pressure. Show that two velocity components must be zero. (You should be able to show that the effective pressure is independent of θ from the appropriate momentum equation by noting that p at $\theta = 2\pi$ must be the same as at $\theta = 0$; otherwise just assume it is. Also note that the velocity can obviously not be infinitely large on the pipe centerline.)
4. 7.1b Continuing the previous question, derive the velocity and pressure fields.

10 11/07

1. 7.14. (In terms of class notations, U is now a function V_0 of time, not a constant. Diffusion equation is another name for the heat equation. The integration can be done using integration by parts since erfc has an easy derivative.)

2. 7.16. With "outward velocity," the motion of a typical position in the middle of the boundary layer is meant, not a particle velocity.

3. 7.17a. Deduce the suitable forms for A and δ, and and then write the ODE. State the conditions for such flows to be possible.

4. 7.17b. Solve the case $n = \frac{1}{2}$. To do so, differentiate the ODE for f once and then renotate f' by g. Function g satisfies the same equation as the function in Stokes 2nd problem discussed in class, and must therefor have the same general solution: a multiple of erfc plus a constant. Verify it. The final solution is, of course, essentially the same as in 7.14, but must be derived independently.

11 11/14

1. Do bathtub vortices have opposite spin in the southern hemisphere as they have in the northern one? Derive some ballpark number for the exit speed of a bathtub vortex at the north pole and one at the south pole, assuming the bath water is initially at rest compared to the earth. What do you conclude about the starting question?

2. A Boeng 747 has a maximum take-off weight of about 400,000 kg and take-off speed of about 75 m/s. The wing span is 65 m. Estimated the circulation in the trailing vortices, and from that, ballpark the typical circulatory velocities around the trailing vortices. Compare to the typical take-off speed of a Cessna 52, 50 mph.

3. Solve the following partial differential equation around a circle:

$$\nabla^2 \psi = 0 \text{ for all } r_0 < r < \infty,\, \theta$$

with boundary conditions:

$$\psi(r_0, \theta) = 0 \qquad \psi(\infty, \theta) \sim Ur \sin(\theta)$$

Hint: this is similar, but not the same as the problem solved in class. The physical meaning of the problem will become clear in next week's homework.

12 11/21

1. Use the software of your choice (mathlab, mathcad, mathematica, ...) to plot lines of constant streamfunction ψ for irrotational, incompressible flow around a cylinder. Take $U = r_0 = 1$.

2. Also plot the isobars. Take $p_\infty = 0$. (Check that you have the right pressure values at the stagnation points and top and bottom points of the cylinder.) Are the pressures on front and rear of the cylinder the same? Are they the same on top and bottom?

3. Now add $\ln(r)$ to the streamfunction. (This does not change the boundary conditions since $\ln 1 = 0$ and $v_r = 1/r = 0$ at large r.) Replot the streamlines.

4. Replot the isobars. Comment on the pressures. Will the force still be zero? If not, in what direction will it be?

5. Add another $3\ln(r)$ to the streamfunction. Replot the streamlines.

6. Replot the pressure.

13 11/28

This homework considers the flow around a flat plate airfoil and a symmetric Joukowski airfoil using conformal mapping from the flow around a cylinder.

The flow around the cylinder is described in a z-plane by the complex velocity potential:

$$F = U\left(z + \frac{r_0^2}{z}\right) - iV\left(z - \frac{r_0^2}{z}\right) + \frac{\mathrm{i}\Gamma}{2\pi}\ln(z/r_0)$$

where

$$z = x + \mathrm{i}y$$

Take the radius of the cylinder to be

$$r_0 = 1 + \epsilon$$

with value for ϵ as specified below. Take $U = 1$ and $V = \tan(20^\circ)$ for an incoming flow at an angle of attack of 20 degrees. The values of Γ to use are specified below.

The airfoil plane is described in terms of a coordinate

$$\zeta = x' + \mathrm{i}y'$$

which can be computed from z using the shifted Joukowski transform

$$\zeta = z - \epsilon + \frac{1}{z - \epsilon}$$

Use a software package like matlab that can handle complex numbers, and create a polar coordinates mesh around the cylinder in the z-plane. (Which will correspond to a nonpolar mesh in the ζ-plane.) In matlab, `real()` will evaluate the real part of a complex number and `imag()` the imaginary part.

1. Taking $\epsilon = 0$ and $\Gamma = 0$, evaluate the imaginary part of F, giving the streamfunction ψ, and then use a contour plotting package to plot the streamlines in the z-plane. You may need to first evaluate x and y as the real and imaginary parts of z. Verify that the streamlines describe the circulationless ideal flow around a cylinder.

2. Evaluate the pressure from Bernoulli, using the fact the the derivative of F gives the complex conjugate velocity, so $|\vec{v}|^2 = |\mathrm{d}F/\mathrm{d}z|^2$, (make sure to use the absolute value function here, not just the square, since F is complex). Take the pressure zero at infinity. Plot isobars.

3. Now evaluate the ζ values, and then plot the streamlines in the x', y'-plane. Verify that you get the flow around a flat plate at an angle of attack, but that the Kutta condition is *not* satisfied.

4. Evaluate the circulation from the Kutta condition that $\mathrm{d}F/\mathrm{d}z$ must be zero at the trailing edge $z = r_0$.

5. Replot the streamlines in the z-plane.

6. Replot the streamlines in the ζ-plane.

7. Set $\epsilon = 0.1$. Replot the streamlines in the z-plane.

8. Replot the streamlines in the ζ-plane.

14 12/05

1. Identify the boundary layer variables x, y, u, and v for the case of a circular cylinder of radius r_0 in terms of the cylindrical variables r, θ, v_r, and v_θ. Be careful of the difference between r and the radius of the cylinder r_0.

2. Using the result of the previous question, write the true continuity equation in cylindrical coordinates from table C.3 in terms of the boundary layer coordinates and comment on the differences from the boundary layer continuity equation. Is the difference small if the boundary layer is thin? If it is thick?

3. According to what was said in class, if a circular cylinder is impulsively set into motion, the velocity profile in the boundary layer at small times is given by the Stokes layer profile $u = u_e \operatorname{erf}(y/\sqrt{4\nu t})$ where u_e is the potential flow velocity just above the boundary layer, equal to $2U \sin\theta$, (θ measured clockwise from the front stagnation point.) And y is, of course, the *boundary layer* coordinate. The corresponding velocity component in the potential flow, which hopefully in question 1 you found to be v_θ, is given by $\partial\phi/r\partial\theta$ or $U(1 + r_0^2/r^2)\sin\theta$ or $\frac{1}{2}u_e(1 + r_0^2/r^2)$. A *composite* profile can be formed by adding the two expressions and substracting the part u_e that they have in common:

$$u = u_e \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right) + \tfrac{1}{2}u_e\left(1 + \frac{r_0^2}{(r_0+y)^2}\right) - u_e$$

Plot this composite velocity profile (1) near the front of the cylinder, $\theta = 30°$, (2) on top of the cylinder, $\theta = 90°$, and (3) near the rear stagnation point, $\theta = 150°$. Take $U = r_0 = 1$, $\nu = 0.001$, and $t = 1$. Plot first over the y-range from 0 to $3r_0$ and then from 0 to $0.15r_0$ to see the details. Use a spacing of no more than 0.003 in your y-values. You do not have to plot the first and third profiles slanted over 30 degrees, just plot them in the normal way (i.e. y vertical and u horizontal.) Note that the composite profile, like the true one, is not exactly constant and actually somewhat less than u_e immediately above the boundary layer. The higher the Reynolds number, the smaller these errors, but the harder to see the profile.

4. According to Blasius, (the same one as from the flat plate, he was a student of Prandtl,) to better approximation, the velocity in the boundary layer is given by two terms,

$$u_e \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right) + t u_e \frac{\mathrm{d}u_e}{\mathrm{d}x}\zeta_1'\left(\frac{y}{\sqrt{4\nu t}}\right)$$

The second term is new. Add this second term to the composite solution above and replot. You should now see that flow reversal has occured at the rear of the cylinder, a first step in creating separation and the wake.

BTW, the function ζ_1' as found by Blasius can be evaluated by saving the following code as a file `zeta1p.m`:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% Blasius second function for impulsively started boundary layer flow. %%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% This really returns function zeta_1', which is the scaled second order
% contribution to the velocity profile.  Argument eta may be an array.

% start of the function definition
function z1p = zeta1p(eta)

% constant
c1=1/sqrt(pi);

% store the values of the exponential
ex=exp(-eta.^2);

% store the values of the complementary error function
erc=erfc(eta);

% compute function zeta_1' according to the result of Blasius
```

```
z1p=-3*c1*eta.*ex.*erc+.5*(2*eta.^2-1).*erc.^2+2/pi*ex.^2+c1*eta.*ex...
+2*erc-4/(3*pi)*ex+(3+4/(3*pi))*c1*(eta.*ex-.5/c1*(2*eta.^2+1).*erc);
```

Also, $\mathrm{d}u_e/\mathrm{d}x$ is u_e with the sine replaced by a cosine (and an r_0, but it is one). Only replot over the range from 0 to 0.15.

5. Finishing the class notes, derive the equations for the Blasius solution for steady laminar boundary layer flow along a flat plate. Do so by making the similarity assumption that $u = Uf'(\eta)$ where $\eta = y/\delta$, with $\delta = \delta(x)$ a typical boundary layer thickness. Use a separation of variables argument similar to the one we used in the Stokes second problem to show that $\delta = \sqrt{\nu x/U}$, except for a chosen constant that is not important, and that function f satisfies the ODE:

$$f''' + \frac{1}{2}ff'' = 0$$

with boundary conditions:

$$f(0) = f'(0) = 0 \qquad f'(\infty) = 1$$

6. Integrate the Blasius equation numerically to $\eta = 10$. To do so, define the following computational variables:

$$x_1 = f \qquad x_2 = f' \qquad x_3 = f''$$

Verify that these variables satisfy the system of first order ordinary differential equations:

$$x_1' = x_2 \qquad x_2' = x_3 \qquad x_3' = -\frac{1}{2}x_1x_3$$

To integrate this system numerically in Matlab or similar, you must define it numerically by creating a file `blaseq.m` containing:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Definition of the first order system of ODE that represents the Blasius %%
%%%% equation for the boundary layer along a semi-infinite flat plate. %%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Notations:
%                   x(1) = f,   x(2) = f',   x(3) = f''
% The initial condition for f'' at eta = 0 should be about 0.33206.  The
% other two initial conditions are zero.

% Return the derivatives of the x-values
function xp = blaseq(eta,x)
xp=[x(2); x(3); -.5*x(1)*x(3)];
```

Then you can integrate the system using a statement like

```
[eta_vals, x_vals] = ode45(@blaseq, [0 10], [0; 0; tau_wall]);
```

where `tau_wall` is $f''(0)$, which physically corresponds to the scaled nondimensional wall shear. According to the book, its value is 0.33206. Verify that this is right by also trying the values 0.2 and 0.4 and see what is wrong then. (Hint: examine array `x_vals`.)

7. Plot the scaled nondimensional velocity profile in the normal way (i.e. y/δ vertical and u/U horizontal.) Take the range of y/δ to be from 0 to 10. Hint: to get hold of column number i of array `x_vals`, use `x_vals(:,i)`.

8. The vorticity in a boundary layer can be approximated as

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \approx -\frac{\partial u}{\partial y}$$

since v is small and x is not. Show that the scaled vorticity $-\omega_z\sqrt{\nu x/U^3}$ corresponds to $f''(\eta)$. Plot the scaled vorticity profile. Does the boundary layer flow turn into a potential flow for large y/δ as the matching idea requires?

15 12/07

1. Plot the wall shear in Blasius flow along a flat plate against x. Take the density, potential flow velocity and plate length to be one. Take the Reynolds numbers to be 100, 1,000, and 10,000 and restrict the x-values to plot τ_w on a scale from zero to one. In matlab, you would use the `plot` and maybe `hold` commands. Verify that the highest Reynold number has the lowest wall shear.

2. Plot the streamlines around the impulsively started circular cylinder. According to the Blasius two-term solution mentioned earlier, the streamfunction is given by:

$$\psi = \tfrac{1}{2}u_e\left[r - \frac{r_0^2}{r}\right] + u_e\zeta_0\left(\frac{y}{\sqrt{4\nu t}}\right)\sqrt{4\nu t} + tu_e\frac{\mathrm{d}u_e}{\mathrm{d}x}\zeta_1\left(\frac{y}{\sqrt{4\nu t}}\right)\sqrt{4\nu t} - u_e y$$

with $u_e = 2U\sin\theta$ again the velocity just above the boundary layer. You should be able to recognize the first term as the potential flow streamfunction. The second term represents the approximation of the boundary layer flow as Stokes' second problem; ζ_0 is the integral of the error function. The third term is the second order time term added by Blasius. The final $u_e y$ term is the "matching" part that the boundary layer solution and the potential flow have in common; it must be substracted in order to avoid having it double. Of course, y is again the *boundary layer* coordinate.

Function ζ_1 can in principle be found by integrating function ζ_1' of the previous homework, but you saw what a mess that was. It is neater to simply integrate the system of ordinary differential equations satisfied by ζ_0 and ζ_1:

$$\zeta_0''' + 2\eta\zeta_0'' = 0 \qquad \zeta_1''' + 2\eta\zeta_1'' - 4\zeta_1' = 4({\zeta_0'}^2 - \zeta_0\zeta_0'' - 1)$$

This can be converted into a first order system, a matlab version of which you can find here[1]. This can be integrated and the streamlines then plotted; this is done by the matlab program that you can find here[2].

Run the program and verify that recirculatory streamlines occur at the rear of the cylinder, as a first step in creating the wake.

Now copy the program to another name and remove the second order term of Blasius. Remove all traces, such as variables that are no longer used and update all comments. Replot, and comment on why the term added by Blasius is important for understanding the flow development.

Now copy the program to another name and remove the first order, Stokes second problem, term too. Comment on the effect that this term has compared to the potential flow streamlines. (You can let this program inherit the r-values from the previous one, but add a comment near the start of the file that it needs to do so.)

3. Now copy the program to another name and modify it to plot the lines of constant vorticity according to the full Blasius expansion. Note that the potential flow term in the streamfunction expression above has no vorticity, and that the vorticity of the boundary layer terms can be approximated as $-\partial u/\partial y = -\partial^2\psi/\partial y^2$. The derivatives of the ζ functions can be found at various locations in the array `zeta_vals` returned by `ode45`. Since the vorticity values at high Reynolds numbers are large, you may want to

[1] `zetasys.m`
[2] `cylpsi2.m`

plot from -110 to 110 in increments of 20. In particular, avoid plotting the zero vorticity line since it is extremely round-off sensitive. Comment on where the vorticity can and cannot be found in high-Reynolds number flow. According to the Stokes second problem approximation, the boundary layer thickness would be the same along the cylinder, take it $\sqrt{4\nu t}$. Is this still true when Blasius second term is included? If not, where is the boundary layer thicker? What would you think about what happens to the boundary thicknesses at the front and rear for large times (where the Blasius solution no longer applies?)

4. Program `blaspsi.m` found here[3] plots the streamlines around a semi-infinite flat plate. Using the matlab `hold` and `plot` commands, add the displacement thickness curve δ^* to the graph in blue, and the rough total boundary layer thickness δ in red (for the latter, you may assume that the boundary layer vorticity is pretty much gone at $\eta = 4$.) You can find the wall x-values $x > 0, y = 0$ somewhere in the two-dimensional mesh array `x`. Are the streamlines parallel to the plate with the displacement thickness added? If not, why not? To better see which streamlines are parallel to the thickened plate, also plot displacement curves shifted upwards by 0.025, 0.05, and 0.1, in green. Comment on in what sense the effect of the boundary layer is to thicken the plate by the displacement thickness.

   You may wonder about the use of "optimal coordinates." The Blasius solution is written in terms of conformally mapped coordinates

$$x' + iy' = \zeta = \sqrt{z} = \sqrt{x + iy}$$

   For a thin boundary layer, y is small and then by approximation

$$x' \approx \sqrt{x} \text{ and } y' \approx \frac{\mathrm{d}\zeta}{\mathrm{d}z} y = \frac{1}{2\sqrt{x}} y$$

   Using these approximations, the Blasius expression for the streamfunction is

$$\psi = \sqrt{\nu U x} f\left(\sqrt{\frac{U}{\nu x}} y\right) \approx \sqrt{\nu U} x' f\left(2\sqrt{\frac{U}{\nu}} y'\right)$$

   Note that the final expression is perfectly regular, free of the artifacts caused by the factors $\sqrt{x}$ in the original Blasius expression. Not only that, it turns out that the final expression already contains the boundary layer displacement effect on the potential flow, without having to add that explicitly!

5. Read the descriptions in the book on how the steady flow field around a sphere and cylinder varies qualitatively with Reynolds number, from the bottom of page 323 on. Summarize it in about hundred words. (No more than 200.)

6. Look at Shankar Subramaniam's results[4] of *unsteady* flow around a circular cylinder. Summarize it in about hundred words. (No more than 200.)

7. Look at Dr. Van Dommelen's results[5] for unsteady flow around an airfoil. (Can you also see the "starting vortex" that comes off the trailing edge before the flow breaks away from the leading edge?) Summarize it in about hundred words. (No more than 200.)

[3] `blaspsi.m`
[4] `http://www.eng.fsu.edu/~dommelen/papers/subram/style_a/node56.html`
[5] `http://www.eng.fsu.edu/~dommelen/research/airfoil/airfoil.html`