Boundary Layers

1 Viscous Flows

(Book 10.2, 3, 5)

Equations for viscous incompressible flows with constant density (these equations also apply to the flow of air at low Mach numbers, eg $M < 0.3$:)

Continuity:

$$\boxed{\frac{\partial u_i}{\partial x_i} = 0}$$

Momentum:

$$\boxed{\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -g\frac{\partial h}{\partial x_i} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}}$$

with $\nu = \mu/\rho$.

The gravity term can be eliminated by defining an artificial (kinetic) pressure

$$p_{\text{kin}} = p + \rho g h$$

However, this does not work if there are free surfaces, since the boundary condition at the free surface would involve the true pressure, not the artificial one.

Differences in flows due to differences in scale can be taken into account by selecting a reference length L and a reference velocity V and then nondimensionalizing all variables with respect to these variables:

$$\bar{t} = \frac{V}{L}t \quad \bar{x}_i = \frac{1}{L}x_i \quad \bar{u}_i = \frac{1}{V}u_i \quad \bar{p} = \frac{1}{\rho V^2}p$$

For example, for flow past a sphere, people typically normalize all lengths with the diameter of the sphere, $L = D$, and all velocities with the incoming stream $V = U$. For an airfoil, the chord is used. For a duct, the height and average velocity, etcetera.

Scaled equations:

Continuity:

$$\boxed{\frac{\partial \bar{u}_i}{\partial \bar{x}_i} = 0}$$

Momentum:

$$\boxed{\frac{\partial \bar{u}_i}{\partial \bar{t}} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial \bar{x}_j} = -\frac{\partial \bar{p}}{\partial \bar{x}_i} + \frac{1}{Re}\frac{\partial^2 \bar{u}_i}{\partial \bar{x}_j \partial \bar{x}_j}}$$

where Re is the nondimensional Reynolds number:

$$\boxed{Re \equiv \frac{UL}{\nu}}$$

A necessary condition for two viscous flows around similar bodies to really be similar is that the Reynolds number is the same. Often there are other nondimensional numbers that also need to be the same due to boundary conditions (free surfaces), variable density effects, additional terms in the governing equations, etcetera.

Since ν is numerically quite small for fluids like water and air, we are typically interested in flows at high Reynolds numbers. Note that as far as the dimensional equations are concerned, a small viscosity ν has the same effect as a large Reynolds number.

Exercise:

Compute the Reynolds number of your car (based on driving velocity and height, and of a plane of characteristic size 20 m flying at Mach 0.8 in standard conditions.

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Exercise:

What is the flow velocity of water in a pipe of diameter 0.1 m if the Reynolds number based on diameter and average velocity is 1.

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Exercise:

List other nondimensional numbers than the Reynolds number that need to be the same for flows that look similar to be really similar. Think of high-speed planes, ships, etcetera.

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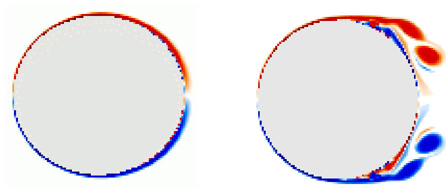
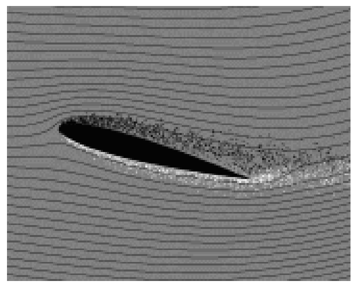
2 Boundary Layers

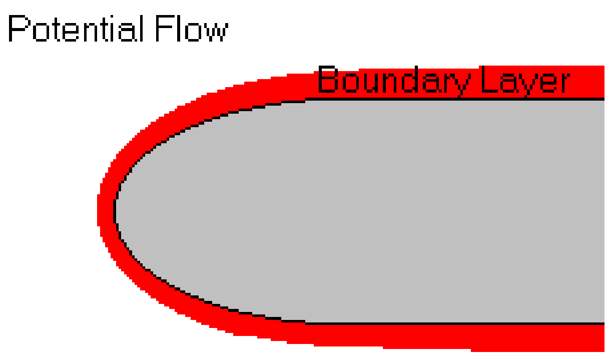
(Book 16.4)

At high Reynold numbers, vorticity is generated by solid surfaces in thin boundary layers. As long as these boundary layers do not separate from the walls, most of the flow field is not much affected by this vorticity.





We can describe the flow *outside* the regions containing vorticity using potential flow theory. We can describe the flow *in* the thin vorticity layers next to the surface using *boundary-layer theory.*



Boundary layer theory requires high Reynolds numbers. You need very different approximations for low Reynolds numbers.

Exercise:

> Graphically compare the vorticity fields of the Stokes flow around a sphere (problem 6.1 in the book), valid for *small* Reynolds with that around an impulsively started circular for *large* Reynolds number. To get the velocity in the boundary layers of the impulsively started cylinder, assume that if we zoom in on a small piece of the surface of the cylinder, the flow looks like a Stokes' 2nd problem layer, (as seen in a coordinate system in which the plate is at rest and under an angle.)

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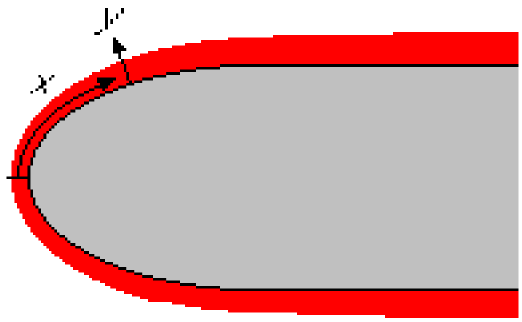
3 Equations

Boundary layer coordinates:



x is the arclength along the wall;
y is the (small) distance from the wall.

Note that despite their names, x and y are *not* Cartesian coordinates. Neither are $u = Dx/Dt$ and $v = Dy/Dt$ Cartesian velocity components.

Exercise:

Identify the boundary layer coordinates and velocities for flow around a circular cylinder in terms of the usual polar variables.

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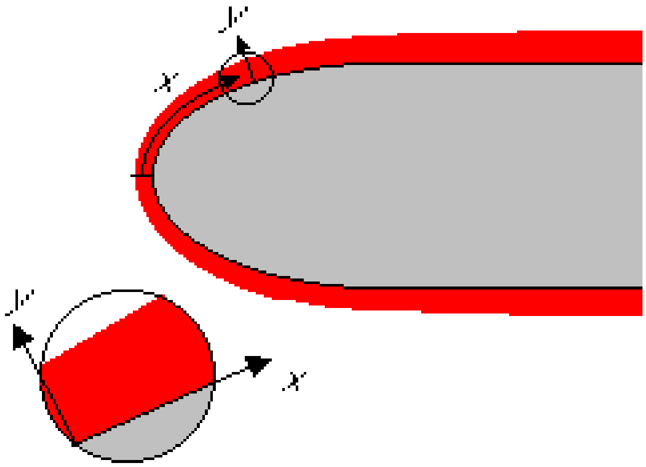
Laminar boundary layer equations:

Seen on the small transverse boundary layer length scale, the boundary layer coordinate system looks approximately Cartesian:



As a result, to first approximation, there are no terms related to the curvature of the coordinate system introduced in the governing equations. In addition, some terms can be neglected since the boundary layer is thin and ν is small. What is left (in 2D) is:

Continuity:

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

which is unchanged compared to the 2D incompressible equations in Cartesian coordinates.

x-Momentum:

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y \partial y}}$$

since the νu_{xx} viscous term is negligible for small ν. Note that the νu_{yy} viscous term is *not* negligible since the smallness of ν is balanced by the fact that u varies rapidly in the direction across the boundary layer, wich has a typical thickness $\Delta y = O(\sqrt{\nu})$.

y-Momentum:

$$\boxed{0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}}$$

Almost all terms can be ignored since the transverse velocity v is small. It follows that the variation of the pressure across the thin boundary layer is negligible; $p = p(x,t)$.

An example of a boundary layer solution is Stokes' second problem, the impulsively started flat plate, in which $v = 0$ and $u = U\mathrm{erfc}\left(y/\sqrt{4\nu t}\right)$. This is an exact solution of the Navier-Stokes, as well as of the boundary layer equations. Note that the thickness of this boundary layer is indeed proportional to $\sqrt{\nu}$.

Exercise:

Write the unsteady boundary layer equations for the boundary layer around a circular cylinder and compare with the full Navier Stokes equations in polar cordinates for the flow. Identify the terms in the full Navier Stakes equations that are ignored in the boundary layer approximation, and explain why these are indeed small.

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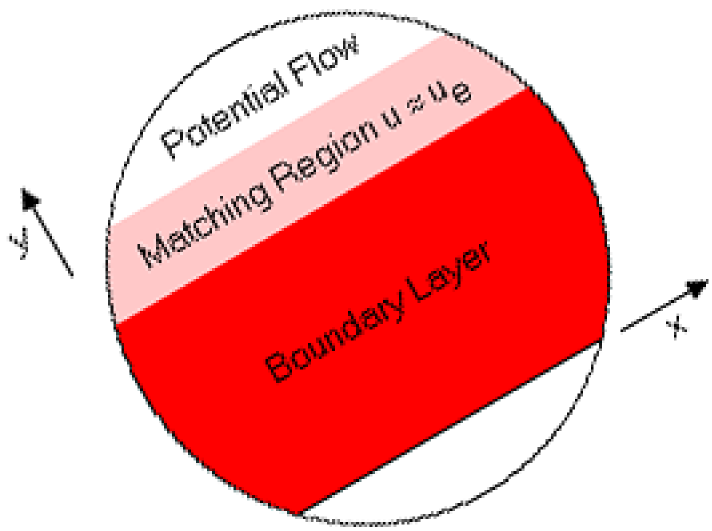
Boundary conditions at a solid, stationary, impermeable wall:

$$u = v = 0 \text{ at } y = 0$$

Boundary conditions above the boundary layer follow from the fact that directly above the boundary layer, there is a "matching" region in which *both* the boundary layer solution and the potential flow solution are valid approximations:



In the matching region, as far as the potential flow is concerned y is small, but as far as the boundary layer solution is concerned, $y >> \sqrt{\nu}$. Since y is small, the velocity $u = u_e(x,t)$ is approximately the wall slip velocity found from the potential flow solution (in which the thin boundary layer is ignored.) But the velocity u_e must also be the velocity in the boundary layer solution for $y >> \sqrt{\nu}$. The same holds for the pressure:

$$u \approx u_e \quad p \approx p_e \text{ for } \frac{y}{\sqrt{\nu}} \to \infty$$

Note that $p_e + \frac{1}{2}\rho u_e^2$ is constant, since the Bernoulli law applies in the matching region.

Exercise:

> Identify the velocity u_e at the outer edge of the boundary layer around the impulsively started circular cylinder (i.e, in the matching region). Also identify the pressure $p(x, y, t)$ at any arbitrary point inside this boundary layer.

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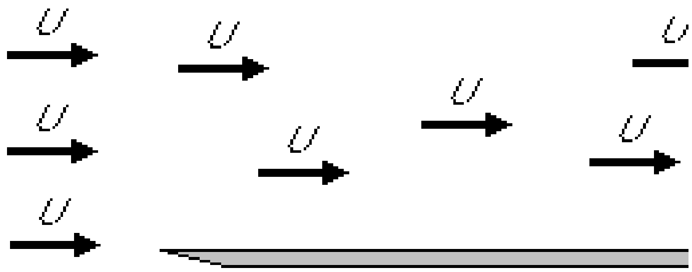
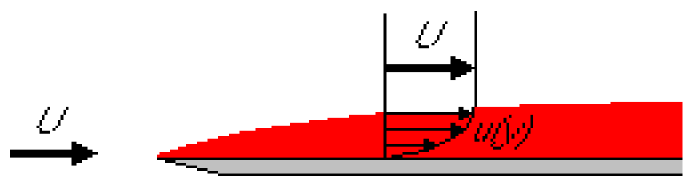
4 Flat Plate

(Book 20.1)

We want to derive the Blasius steady boundary layer on a semi-infinite flat plate:





Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Longrightarrow \quad u = \frac{\partial \psi}{\partial y},\ v = -\frac{\partial \psi}{\partial x}$$

x-Momentum:

$$\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\cancel{\frac{\partial p}{\partial x}} + \nu\frac{\partial^2 u}{\partial y \partial y}$$

Note that since the velocity is constant, U, in the matching region, the pressure must be constant.

Try a similarity solution. If $\delta(x)$ is a typical boundary layer thickness:

$$u = Uf'\left(\frac{y}{\delta}\right) = \psi_y \quad \psi = U\delta f\left(\frac{y}{\delta}\right) \quad v = -\psi_x = -U\left(\delta' f\left(\frac{y}{\delta}\right) - \delta f'\left(\frac{y}{\delta}\right)\frac{y}{\delta}\frac{\delta'}{\delta}\right)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y \partial y}$$

$$-Uf'\left(\frac{y}{\delta}\right)Uf''\left(\frac{y}{\delta}\right)\frac{y}{\delta}\frac{\delta'}{\delta} - U\left(\delta' f\left(\frac{y}{\delta}\right) - \delta f'\left(\frac{y}{\delta}\right)\frac{y}{\delta}\frac{\delta'}{\delta}\right)Uf''\frac{1}{\delta} = Uf'''\frac{1}{\delta^2}$$

$$-\frac{f'''}{ff''} = \frac{U\delta\delta'}{\nu} = \text{constant} = \tfrac{1}{2}$$

Typical boundary layer thickness:

$$\delta = \sqrt{\frac{\nu x}{U}} = \frac{x}{\sqrt{Ux/\nu}} = \frac{x}{\sqrt{Re_x}}$$

Shape of the velocity profile: $u = Uf'(y/\delta)$ with

$$f''' + \tfrac{1}{2}ff'' = 0 \qquad f(0) = f'(0) = 0 \quad f'(\infty) = 1$$

Numerical solution produces $f''(0) = 0.33206$, which fixes the wall shear.

Exercise:

Find the shear force on a plate of length 0.5 m moving flush through air at a speed of 0.5 m/sec

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Exercise:

Derive the steady boundary layer along a semi-infinite plate if instead of a uniform flow, there is a sink sitting at the nose of the plate. In other words, the potential flow just above the boundary layer is $u_e = -m/(2\pi x)$. The boundary layer solution is again similar, but the boundary layer thickness is now linear in x. In other words, $u(x,y) = u_e(x)f'(y/x)$.

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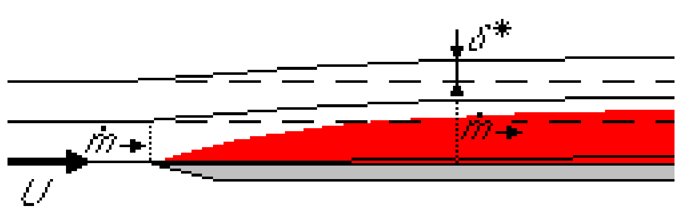
5 Displacement Thickness

It is not exactly true that the boundary layer does not affect the potential flow at all. The potential flow streamlines just above the boundary layer will be slightly displaced away from the wall by the boundary layer:



Suppose a potential flow streamline in the matching region would be at a distance h from the wall if there was no boundary layer, then it has streamfunction value $\psi = u_e h$. The displaced distance $\bar{h}$ follows from:

$$\psi = u_e h = \int_0^{\bar{h}} u \, \mathrm{d}y = \int_0^{\bar{h}} u - u_e \, \mathrm{d}y + u_e \bar{h}$$

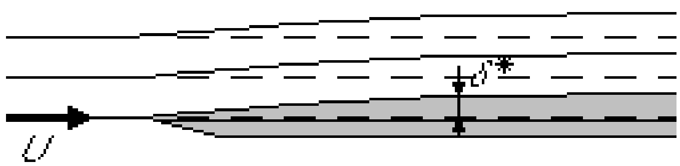
The displacement of the streamline follows:

$$\bar{h} - h = \delta^* = \int_0^{\bar{h}} 1 - \frac{u}{u_e} \, \mathrm{d}y$$

Since the integrand disappears above the boundary layer, this displacement is the same for all streamlines in the matching region. Assuming we use the boundary layer solution for the velocity profile, we may as well write

$$\boxed{\delta^* = \int_0^{\infty} 1 - \frac{u}{u_e} \, \mathrm{d}y}$$

The result of this displacement is that as far as the potential flow is concerned, the body is thicker by an amount δ^*:



From the solution of the Blasius profile, it follows that for a flat plate in a uniform flow,

$$\delta^* = 1.72\sqrt{\frac{\nu x}{U}} = 1.72\frac{x}{\sqrt{Re_x}}$$

Exercise:

Find the displacement thickness at the end of the plate from two exercises back.

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Exercise:

For the inlet of question 7.1, determine how the pressure changes near the start of the inlet.

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