Physical Laws

1 Continuity

(Book: 5.1. Note that the book uses weird notations for partial derivatives.)

To solve fluid flow problems, we need to write the basic equations of physics for arbitrary regions of fluid. We start with mass conservation (called the continuity equation in fluid mechanics.)

The continuity equation in words: *material (ie. Lagrangian) regions* conserve mass.

Lagrangian formulation:

$$\boxed{M_{MR} \equiv \int_{MR} \rho \, \mathrm{dV} \qquad \frac{\mathrm{d}M_{MR}}{\mathrm{d}t} = 0} \tag{1}$$

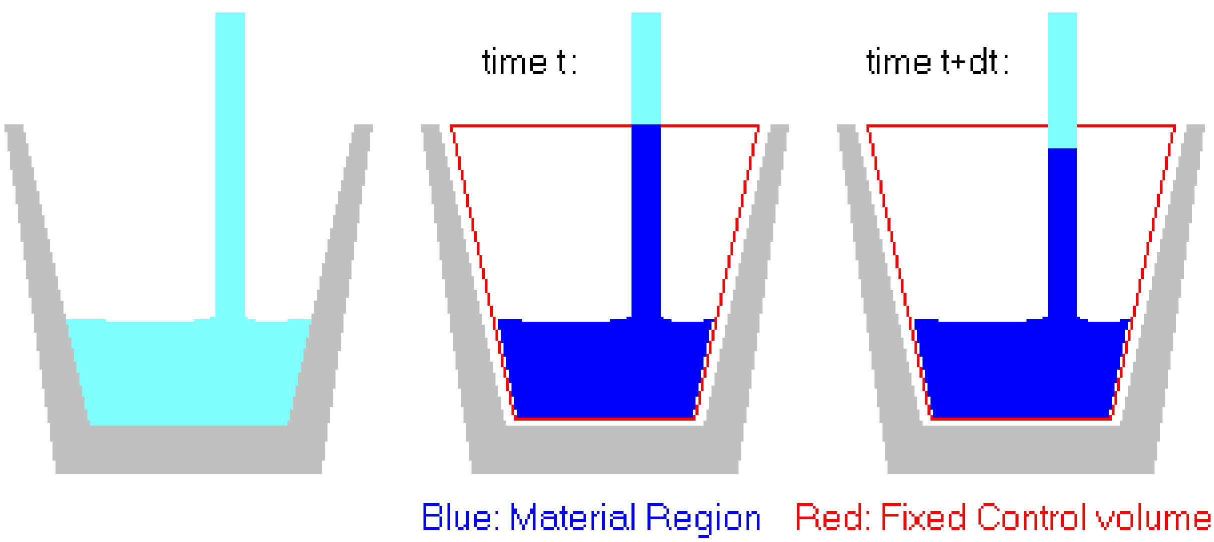
1.1 FCV conversion

Most of the time in fluids, we want the time changes for *fixed* control volumes (FCV):

$$M_{FCV} \equiv \int_{FCV} \rho \, \mathrm{dV} \qquad \frac{\mathrm{d}M_{FCV}}{\mathrm{d}t} = ?$$

The mass of a fixed control volume is not necessarily constant. For example, the fixed control volume might be a glass you are filling. Or it may be a rocket, a balloon, a pressurized aircraft cabin, or many others. We will use the glass being filled as the typical example here.

Trick: find out what happens to M_{FCV} from what happens to the mass of the *material region that coincides with the control volume at whatever time t you are interested in.* You know what happens to the mass of that material region (it remains constant).



Now use mathematics to rewrite the time derivative of the material region in terms of the time derivative of the fixed control volume. They are *not* the same. In the picture above, the material region keeps the same mass, while the control volume mass grows. If you do the math right, (see below for details), what you will find is that

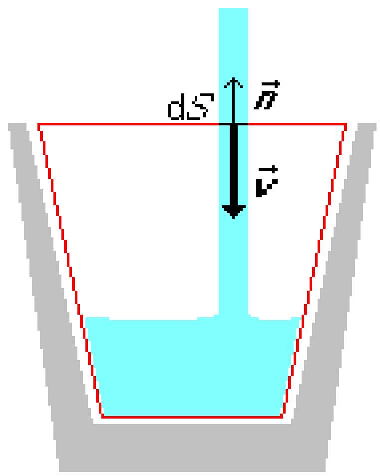
$$\frac{\mathrm{d}M_{MR}}{\mathrm{d}t} = 0$$

rewrites to the following:

$$\boxed{\frac{\mathrm{d}M_{FCV}}{\mathrm{d}t} + \int_{FCV} \rho \vec{v} \cdot \vec{n} \, \mathrm{d}S = 0} \tag{2}$$

In words, the boxed formula says: The rate of increase in mass of the control volume *plus the rate of mass flowing out of the control volume* add to zero. The mass flowing out is the surface integral. It is the *outflow correction* to the physical law for a fixed region.

Look a bit closer at the integral. Outflow corrections are always an integral over the outside surface area S of the volume (which would include the top plane through the rim of the glass, if that is our control volume.) In the integrand, $\vec{v}$ is of course the flow velocity, while $\vec{n}$ is the unit vector normal to the boundary. (For the the top plane through the rim of the glass, that would be a vector of unit length sticking upwards.) So $\vec{v} \cdot \vec{n}$ is *the component of the velocity normal to the boundary* (and positive outward.) It gives the speed of outflow through the boundary. Speed, area and density all increase the mass flow through the boundary.

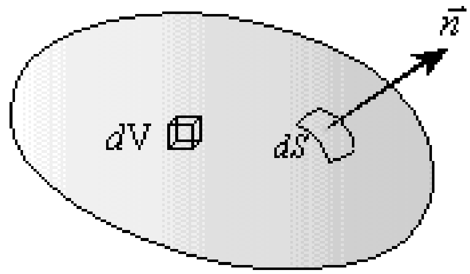


In the picture above, the surface integral is over the entire red outside area of the fixed control volume. But almost all of it is zero. Either the density is zero (no water), or the velocity $\vec{v} \cdot \vec{n}$ normal to the boundary is zero. The only part of the integral that is nonzero is where the water stream cuts through the red control volume surface. If the cross-sectional area of the stream is $\vec{S}_s$ and the velocity has magnitude V_s downward, then $\int \rho \vec{v} \cdot \vec{n} \, \mathrm{d}S = \rho(-V_s)S_s$. So in this example, mass conservation for the glass says

$$\frac{\mathrm{d}M}{\mathrm{d}t} - \rho V_s S_s = 0$$

Bottom line: if you start applying physical laws to fixed spatial regions instead of to material regions, you better be adding outflow corrections. These outflow corrections are always integrals over the outside surface of the control volume.

Note that the mass itself is an integral over the interior volume; $M = \int \rho \, \mathrm{dV}$. Conserved properties like mass are found from integrals over the interior, while outflow corrections are integrals over the boundary.



We did not yet actually derive the above formula. To do so, we must recognize that the time derivative that we know is the time derivative of the mass of the material region. That means we must differentiate an integral $\int \rho \, \mathrm{dV}$ over a region with moving boundaries. Such derivatives can be found using the Leibnitz rule.

- The Leibnitz rule in one dimension:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b f(x,t)\,\mathrm{d}x \equiv \int_a^b \frac{\partial f}{\partial t}\,\mathrm{d}x + f(b,t)\frac{\mathrm{d}b}{\mathrm{d}t} - f(a,t)\frac{\mathrm{d}a}{\mathrm{d}t}$$

- The Leibnitz rule in three dimensions: The interval $[a, b]$ becomes a volume V; the endpoints a and b become the surface S of that volume, and the sum over the two end points becomes an integral over the surface:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_R f(\vec{r},t)\,\mathrm{dV} \equiv \int_V \frac{\partial f}{\partial t}\,\mathrm{dV} + \int_V f\vec{v}_{\text{boundary}}\cdot\vec{n}\,\mathrm{d}S$$

If we apply this to the integral of the mass of the material region $\int_{MR} \rho \, \mathrm{d}v$, we get:

$$0 = \int_{MR} \frac{\partial \rho}{\partial t}\,\mathrm{dV} + \int_{MR} \rho\vec{v}\cdot\vec{n}\,\mathrm{d}S$$

Since the material region was chosen to coincide with the fixed control volume at the given time, the integrals are also integral over the FCV. And since the first integral has fixed boundaries, we can take the time derivative out of it:

$$\int_{FCV} \frac{\partial \rho}{\partial t}\,\mathrm{dV} = \frac{\mathrm{d}M_{FCV}}{\mathrm{d}t}$$

Exercise:

> Evaluate $\int_V \rho\vec{v}\cdot\vec{n}\,\mathrm{d}S$ if $\vec{v} = U\hat{\imath}$, $\rho = 1$, and V is the unit cube $0 \leq x, y, z \leq 1$.

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Instead of using the Leibnitz rule, we can derive the mass equation for a fixed control volume from more physical arguments. This is done on the class web page under Reynolds Transport Theorem.

You should now be able to do questions 5.5 and 5.1 (minus some physical interpretations.)

1.2 PDE conversion

Integral forms of laws like the one just derived for mass conservation are very useful if you want to know the overall properties of a system (its mass, the total force produced by an engine, etcetera.) But if you want to find the detailed properties of a flow at all points, you need Partial Differential Equations (PDEs) that apply at every point. You can find these from the integral form by looking at very small regions.

Here, we want to convert the integral mass conservation equation for a fixed control volume,

$$\frac{\mathrm{d}M_{FCV}}{\mathrm{d}t} + \int_{FCV} \rho\vec{v}\cdot\vec{n}\,\mathrm{d}S = 0,$$

in other words

$$\int_{FCV} \frac{\partial\rho}{\partial t}\,\mathrm{dV} + \int_{FCV} \rho\vec{v}\cdot\vec{n}\,\mathrm{d}S = 0$$

into a PDE that is pointwise valid. We can do this by choosing the FCV as a very small region around whatever point we are looking at. But to work that out, we need to get rid of the surface integral.

The *divergence* theorem takes surface integrals to volume integrals and vice-versa.

- Scalar variable f:

$$\int_R f n_i\,\mathrm{d}S = \int_R \frac{\partial f}{\partial x_i}\,\mathrm{dV} \qquad \int_R f\vec{n}\,\mathrm{d}S = \int_R \nabla f\,\mathrm{dV}$$

- Vector variable $\vec{a}$:

$$\int_R a_i n_i\,\mathrm{d}S = \int_R \frac{\partial a_i}{\partial x_i}\,\mathrm{dV} \qquad \int_R \vec{a}\cdot\vec{n}\,\mathrm{d}S = \int_R \nabla\cdot\vec{a}\,\mathrm{dV} \equiv \int_R \mathrm{div}(\vec{a})\,\mathrm{dV}$$

Apply this to mass conservation (with $\vec{a} = \rho\vec{v}$),

$$\int_{FCV} \frac{\partial\rho}{\partial t}\,\mathrm{dV} + \int_{FCV} \rho\vec{v}\cdot\vec{n}\,\mathrm{d}S = 0$$

to get

$$\int_{FCV} \frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\vec{v})\,\mathrm{dV} = 0$$

So the PDE form of the continuity equation is:

$$\boxed{\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\vec{v}) = 0} \tag{3}$$

Index notation (with Einstein summation convention):

$$\boxed{\frac{\partial\rho}{\partial t} + \frac{\partial\rho v_i}{\partial x_i} = 0} \tag{4}$$

Note that in the above equation, each term is simply the derivative of something. An equation that looks like that is called an equation in *conservation form.* It comes directly from a conservation law (here mass conservation) and can directly be turned back into a conservation law by integrating it over a volume.

But if we start differentiating out the individual factors, we get a nonconservative form:

$$\boxed{\frac{\partial\rho}{\partial t} + v_i\frac{\partial\rho}{\partial x_i} + \rho\frac{\partial v_i}{\partial x_i} = 0} \tag{5}$$

In this nonconservative form, we have coefficients in front of the derivatives. Conservative forms are often better in numerical work, since they tend to ensure conservation even if the accuracy is not very good.

But nonconservative forms can be more accurate under other circumstances. Also, they often show important aspects of the physics. For example, the terms in the nonconservative continuity equation can be rewritten as (remember the Lagrangian total derivative):

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i} \qquad \frac{\partial v_i}{\partial x_i} = \mathrm{div}\ \vec{v}$$

so the continuity equation can be written in terms of the material fluxion of density:

$$\boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \mathrm{div}\ \vec{v} = 0} \tag{6}$$

As an important special case, for *incompressible fluids*:

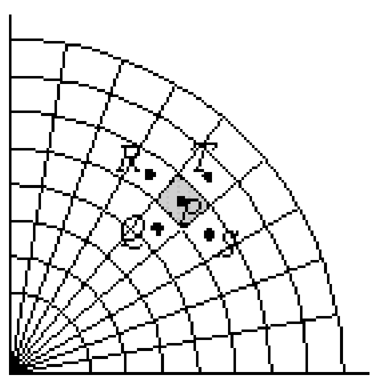
$$\boxed{\mathrm{div}\ \vec{v} = 0} \tag{7}$$

The density does not have to be constant from one fluid element to the next (eg, oil and water, varying salinity, ...) for this to be the case.

You should now be able to do question 5.2

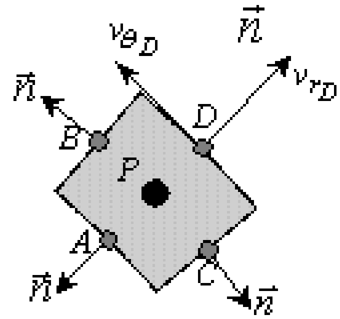
1.3 Example

Problem: We want to do a computation of a fluid flow in two dimensions. We will use polar coordinates. We chop the flow region up into small volumes $(\Delta r, \Delta\theta)$ and now we must derive equations for each of those small volumes. Write an approximate continuity equation for the shaded volume $(\Delta r, \Delta\theta)$ in terms of the densities and polar velocity components at points P, Q, R, S, and T.



Solution: Let us choose the flow quantities at the centers of all the small volumes as the unknowns. We want to write equations for these unknowns. We start with the derived integral mass conservation equation for a fixed region:

$$0 = \int_{FCV} \frac{\partial \rho}{\partial t}\,\mathrm{dV} + \int_{FCV} \rho\vec{v}\cdot\vec{n}\,\mathrm{d}S$$



We apply this to the shaded region. We will make approximations to express the two terms in terms of the density at only a finite number of points:

$$\int_{FV} \frac{\partial \rho}{\partial t} \,\mathrm{dV} \equiv \frac{\mathrm{d}}{\mathrm{d}t} \int_{FV} \rho \,\mathrm{dV} \qquad \int_{FV} \rho \,\mathrm{dV} \approx \rho_P r_P \Delta r \Delta\theta \qquad \Longrightarrow \qquad \int_{FCV} \frac{\partial \rho}{\partial t} \,\mathrm{dV} \approx \frac{\partial \rho_P}{\partial t} r_P \Delta r \Delta\theta$$

$$\int_{FV} \rho \vec{v} \cdot \vec{n} \,\mathrm{d}S \approx \rho_D v_{rD} r_D \Delta\theta + \rho_B v_{\theta B} \Delta r - \rho_A v_{rA} r_A \Delta\theta - \rho_C v_{\theta C} \Delta r$$

Finally we can get rid of the side points A, B, C, and D by approximating further, eg, $v_{rD} \approx \frac{1}{2}(v_{rP} + v_{rT})$. There may be further considerations, see Intro to Computational Mechanics.

Exercise:

> Derive the continuity partial differential equation in polar coordinates by taking the limit $\Delta r, \Delta\theta \to 0$. (See the appendices in the book for this equation in various coordinate systems.)

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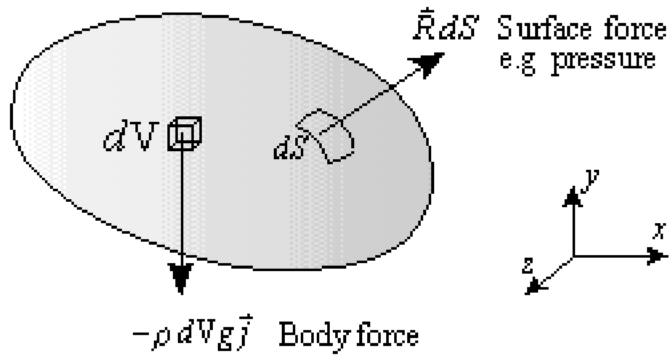
In the homework, you will do all this in spherical coordinates, i.e. derive both the equation for a finite volume $\Delta r, \Delta\theta, \Delta\phi$ and the PDE. The mathematical handbook has a couple of different pictures of spherical coordinates; many calculus books have a picture of a volume $\Delta r, \Delta\theta, \Delta\phi$.

2 Forces

(Book: 5.3, 4, 5)

To figure out fluid motion, we need the forces on the fluid. The two types of forces we will deal with in this class are:

- body forces acting in the interior:
  - gravity: $\vec{g}$ per unit mass;
- surface forces:
  - pressure only (*inviscid* flow): $-p\vec{n}$ per unit outside surface;
  - pressure plus viscous stresses (viscous flow): $-pn_i + \tau_{ji} n_j$ in the i-direction, per unit outside surface.



So, including all these, the total external force on a volume of fluid will be:

$$\boxed{\vec{F} = \int \rho\vec{g}\,\mathrm{dV} - \int p\vec{n}\,\mathrm{d}S + \int \bar{\bar{\tau}}\vec{n}\,\mathrm{d}S \qquad F_i = \int \rho g_i\,\mathrm{dV} - \int pn_i\,\mathrm{d}S + \int \tau_{ji}n_j\,\mathrm{d}S} \tag{8}$$

Exercise:

> Derive the Archimedes law for a submarine of arbitrary shape if the pressure is given by $p = -\rho g y$, where the y-axis is vertically upward. Hint: use the divergence theorem to convert the surface integral into a volume integral.

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To shorten the writing, we define the total stress tensor as:

$$\boxed{T_{ji} = -p\delta_{ji} + \tau_{ji}} \tag{9}$$

We alse rewrite $\vec{g}$ often in terms of the gravitational potential:

$$\vec{g} = -g\nabla h \qquad g_i = -g\frac{\partial h}{\partial x_i}$$

where h is the height above the surface of the earth.

You should now be able to do question 5.3

3 Momentum Conservation

(Book: 5.2)

As you may guess, Newton's second law is the most important equation for fluid flow, besides mass conservation. We will write Newton's law in the form of conservation of linear momentum.

The momentum equation in words: the rate of change of linear momentum of a *material region* equals the net force on the region:

The Lagrangian formulation is:

$$\vec{I}_{MR} \equiv \int_{MR} \rho\vec{v}\,\mathrm{dV} \qquad \frac{\mathrm{d}\vec{I}_{MR}}{\mathrm{d}t} = \vec{F}$$

Note: by the definition of the center of gravity (mass, really), $\int_{MR} \vec{v}\,\mathrm{d}m \equiv M_{MR}\vec{v}_{cg}$, where $\vec{v}_{cg}$ is the velocity of the center of gravity, so we can also write the momentum equation as $M_{MR}\vec{a}_{cg} = \vec{F}$, which is Newton's law in the more conventional form.

Note: the book says the momentum equation above cannot be derived from Newton's equation for a point mass. However, setting $\mathrm{d}\vec{F} = \rho\,\mathrm{dV}\vec{a}$ for an infinitesimal element of fluid with volume dV and integrating using action = -reaction does produce the momentum equation above.

3.1 FCV conversion

For practical reasons, we would again like to convert the momentum equation into one for a fixed control volume.

To do this, we again use the Leibnitz rule on the components of the momentum equation for the material region that instantaneously coincides with the fixed control volume:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{MR} \rho v_i\,\mathrm{dV} = \int_{FCV} \frac{\partial \rho v_i}{\partial t}\,\mathrm{dV} + \int_{FCV} \rho v_i \vec{v}\cdot\vec{n}\,\mathrm{d}S = F_i$$

Hence, for the same reasons as for the continuity equation,

$$\boxed{\vec{I}_{FCV} \equiv \int_{FCV} \rho\vec{v}\,\mathrm{dV} \qquad \frac{\mathrm{d}\vec{I}_{FCV}}{\mathrm{d}t} + \int_{FCV} \rho\vec{v}\,(\vec{v}\cdot\vec{n})\,\mathrm{d}S = \vec{F}} \tag{10}$$

Again, we get a surface integral as an outflow correction in addition to what we have in physics. Note the additional $\vec{v}$ in this integral compared to the continuity integral: we are now integrating *momentum* flowing out, not mass flowing out.

You should now be able to do questions 5.1, 5.11, 5.12, 5.13, 5.14

3.2 PDE conversion

(Book: 5.7)

Next, we will convert the momentum equation into a PDE as a tool to eventually find detailed solutions of flow fields. The momentum equations for inviscid flow in PDE form are called the *Euler* equations. The momentum equations for viscous flow in PDE form are called the *Navier-Stokes* equations.

We start with the derived integral momentum equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{FCV} \rho\vec{v}\,\mathrm{dV} + \int_{FCV} \rho\vec{v}\,(\vec{v}\cdot\vec{n})\,\mathrm{d}S = \vec{F}$$

Write this out in index notation, using the expression for F_i from the previous section:

$$\int_{FCV} \frac{\partial \rho v_i}{\partial t}\,\mathrm{dV} + \int_{FCV} \rho v_i v_j n_j\,\mathrm{d}S = \int_{FCV} \rho g_i\,\mathrm{dV} + \int_{FCV} T_{ji} n_j\,\mathrm{d}S$$

Use the divergence theorem twice:

$$\int_{FCV} \left(\frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} - \rho g_i - \frac{\partial T_{ji}}{\partial x_j} \right) \mathrm{dV} = 0$$

So, the conservation PDE form of the momentum equations is:

$$\boxed{\frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} = \rho g_i + \frac{\partial T_{ji}}{\partial x_j}} \tag{11}$$

To get the also useful nonconservative form, we can again differentiate out:

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} = v_i \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} \right) + \rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right)$$

The terms between the first set of parentheses are zero according to the continuity equation derived earlier. So we get the nonconservative Navier-Stokes equations:

$$\boxed{\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = \rho g_i + \frac{\partial T_{ji}}{\partial x_j}} \tag{12}$$

We can again make this more physical by recognizing the Lagrangian derivative of the velocity in it:

$$\boxed{\rho \frac{D v_i}{D t} = \rho g_i + \frac{\partial T_{ji}}{\partial x_j}} \tag{13}$$

Hence $\rho \vec{g}$ is the gravity force per unit volume and the *gradient* of the stress tensor is the *net* surface force per unit volume.

Exercise:

> Derive the corresponding Euler equation by restricting the total stress T_{ij}. Write the equation in vector form. Explain by a simple example that the pressure does *not* give you the net force per unit volume on the fluid, but that you need the pressure gradient.

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You should now be able to do questions 5.4, 5.6

4 Heat Conduction

(Book: 5.9)

To figure out most flows, (some simple incompressible flows do not need it), we will need to use energy conservation in addition to mass and linear momentum conservation. That requires that heat conduction is considered.

The heat flux per unit area is called the heat flux vector $\vec{q}$. The total heat flowing out of a volume V per unit time is:

$$\boxed{\dot{Q}^{\nearrow} = \int \vec{q} \cdot \vec{n} \, \mathrm{d}S \qquad \dot{Q}^{\nearrow} = \int q_i n_i \, \mathrm{d}S} \tag{14}$$

5 Energy Conservation

(Book: 5.9)

The energy equation in words: the change of energy of a *material region* is the net work done on the region and heat added to the region.

Lagrangian formulation:

$$\boxed{E_{tMR} \equiv \int_{MR} \rho\left(e + \tfrac{1}{2}\vec{v}\cdot\vec{v}\right)\,\mathrm{dV} \qquad \frac{\mathrm{d}E_{tMR}}{\mathrm{d}t} = \dot{W}^{\swarrow} + \dot{Q}^{\swarrow}} \tag{15}$$

Note: we will account for gravity as an external force doing work, rather than including gravitational potential energy in the energy.

An expression for the heat flowing in was given in the previous section (with a minus sign, since the previous section gave outflow instead of inflow.)

Work done per unit time equals force times velocity, or to be precise, the scalar product of the velocity and force vectors. The total work on the fluid region is therefor:

$$\boxed{\dot{W}^{\swarrow} = \int_{FCV} \rho v_i g_i\,\mathrm{dV} + \int_{FCV} v_i T_{ji} n_j\,\mathrm{d}S} \tag{16}$$

For an inviscid fluid, and in vector notation, this simplifies to:

$$\boxed{\dot{W}^{\swarrow} = \int_{FCV} \rho\vec{g}\cdot\vec{v}\,\mathrm{dV} - \int_{FCV} p\vec{v}\cdot\vec{n}\,\mathrm{d}S} \tag{17}$$

5.1 FCV conversion

To get an equation for the energy in a fixed control volume, we proceed as before for the continuity and momentum equations. The result is:

$$\boxed{\begin{array}{l} E_{tFCV} \equiv \displaystyle\int_{FCV} \rho\left(e + \tfrac{1}{2}\vec{v}\cdot\vec{v}\right)\,\mathrm{dV} \\ \dfrac{\mathrm{d}E_{tFCV}}{\mathrm{d}t} + \displaystyle\int_{FCV} \rho\left(e + \tfrac{1}{2}\vec{v}\cdot\vec{v}\right)(\vec{v}\cdot\vec{n})\,\mathrm{d}S = \dot{W}^{\swarrow} + \dot{Q}^{\swarrow} \end{array}} \tag{18}$$

Again we have a surface outflow correction term.

5.2 PDE conversion

To convert to a PDE, we again proceed as we did for the continuity and momentum equations.

First write out the energy equation in index notation, including the expressions for $\dot{W}^{\swarrow}$ and $\dot{Q}^{\swarrow}$ in the left hand side:

$$\int_{FCV} \frac{\partial\rho\left(e + \frac{1}{2}\vec{v}\cdot\vec{v}\right)}{\partial t}\,\mathrm{dV} + \int_{FCV} \rho\left(e + \tfrac{1}{2}\vec{v}\cdot\vec{v}\right) v_j n_j\,\mathrm{d}S = \\ \int_{FCV} \rho g_i v_i\,\mathrm{dV} + \int_{FCV} v_i T_{ji} n_j\,\mathrm{d}S - \int_{FCV} q_i n_i\,\mathrm{d}S$$

Using the divergence theorem:

$$\int_{FCV} \left(\frac{\partial\rho\left(e + \frac{1}{2}\vec{v}\cdot\vec{v}\right)}{\partial t} + \frac{\partial\rho\left(e + \frac{1}{2}\vec{v}\cdot\vec{v}\right) v_j}{\partial x_j} - \rho g_i v_i - \frac{\partial v_i T_{ji}}{\partial x_j} + \frac{\partial q_i}{\partial x_i} \right)\mathrm{dV} = 0$$

The conservation form of the energy equation is therefor:

$$\boxed{\frac{\partial\rho\left(e+\frac{1}{2}\vec{v}\cdot\vec{v}\right)}{\partial t}+\frac{\partial\rho\left(e+\frac{1}{2}\vec{v}\cdot\vec{v}\right)v_j}{\partial x_j}=\rho g_i v_i+\frac{\partial v_i T_{ji}}{\partial x_j}-\frac{\partial q_i}{\partial x_i}} \tag{19}$$

The usual nonconservative form can be derived as before, and is:

$$\boxed{\rho\frac{\partial\left(e+\frac{1}{2}\vec{v}\cdot\vec{v}\right)}{\partial t}+\rho v_j\frac{\partial\left(e+\frac{1}{2}\vec{v}\cdot\vec{v}\right)}{\partial x_j}=\rho g_i v_i+\frac{\partial T_{ji}v_i}{\partial x_j}-\frac{\partial q_i}{\partial x_i}} \tag{20}$$

In terms of a Lagrangian derivative,

$$\boxed{\rho\frac{D\left(e+\frac{1}{2}\vec{v}\cdot\vec{v}\right)}{Dt}=\rho g_i v_i+\frac{\partial T_{ji}v_i}{\partial x_j}-\frac{\partial q_i}{\partial x_i}} \tag{21}$$

You should now be able to do questions 5.8

6 Kinetic Energy

(Book: 5.10)

The energy equation derived above tells you the combined changes in internal energy and kinetic energy of the fluid. Often, you are interested in the changes in those quantities separately instead of combined. After all, if you are interested in the motion, the kinetic energy is important, while if you are interested in the state of the fluid, the internal energy is.

Fortunately, we can get an equation for kinetic energy by itself in terms of the forces by multiplying the momentum equation,

$$\rho\frac{Dv_i}{Dt}=\rho g_i+\frac{\partial T_{ji}}{\partial x_j}$$

by v_i and summing over i:

$$\boxed{\rho\frac{D\frac{1}{2}v_i v_i}{Dt}=\rho g_i v_i+\frac{\partial T_{ji}}{\partial x_j}v_i} \tag{22}$$

It is seen that kinetic energy of the fluid elements changes due to work done by the gravity and stress gradient forces.

Converted back to conservation form it becomes:

$$\boxed{\frac{\partial\rho\frac{1}{2}v_i v_i}{\partial t}+\frac{\partial\rho\frac{1}{2}v_i v_i v_j}{\partial x_j}=\rho g_i v_i+\frac{\partial T_{ji}}{\partial x_j}v_i} \tag{23}$$

Converted back to integral form:

$$\boxed{K_{MR}=\int_{MR}\rho\tfrac{1}{2}v_i v_i\,\mathrm{dV}\qquad\frac{\mathrm{d}K_{MR}}{\mathrm{dt}}=\int_{MR}\rho g_i v_i\,\mathrm{dV}+\int_{MR}\frac{\partial T_{ji}}{\partial x_j}v_i\,\mathrm{dV}} \tag{24}$$

In words, the kinetic energy changes due to work done by gravity and the work done by stress-induced forces on the fluid.

The kinetic energy equation is not a new, additional, equation, but a consequence of momentum.

You should now be able to do questions 5.7

7 Thermodynamics

(Book: 5.10, 12)

Now we want to get an equation for the thermodynamic properties only (no kinetic energy.)

This can be done by substracting the kinetic energy equation

$$\rho \frac{D\frac{1}{2}v_i v_i}{Dt} = \rho g_i v_i + \frac{\partial T_{ji}}{\partial x_j} v_i$$

from the energy equation

$$\rho \frac{D\left(e + \frac{1}{2}\vec{v}\cdot\vec{v}\right)}{Dt} = \rho g_i v_i + \frac{\partial T_{ji} v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}$$

to give:

$$\rho \frac{De}{Dt} = T_{ji} \frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}$$

To clean up even more, split the stress tensor T_{ji} into the thermodynamical pressure and the viscous stress, $T_{ji} = -p\delta_{ji} + \tau_{ji}$:

$$\rho \frac{De}{Dt} = -p \frac{\partial v_i}{\partial x_i} + \tau_{ji} \frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}$$

Then use the continuity equation,

$$\frac{1}{\rho} \frac{D\rho}{Dt} + \operatorname{div} \vec{v} = 0$$

to make the pressure term also fully thermodynamical. Taking it to the left hand side, we get

$$\rho \left(\frac{De}{Dt} - \frac{p}{\rho^2} \frac{D\rho}{Dt} \right) = \tau_{ji} \frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}$$

Finally, compare the left hand side to the second law to get a very simple and important final result:

$$\boxed{\rho T \frac{Ds}{Dt} = \tau_{ji} \frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}} \qquad (25)$$

Assuming that the stress tensor is symmetric, the first term is also equal to $\tau_{ij} s_{ij}$ and is called the dissipation.

Exercise:

> Comment on the properties of reversible processes in fluid mechanics. Also comment on the sign of the viscous stress term.

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The entropy equation is not an additional equation, but can be used as an alternative to the energy equation. Obviously, it is a lot simpler.

8 Temperature Equation

(Book: 5.11)

In many cases, it is more convenient to have an equation for the temperature instead of either the entropy equation or the original energy equation.

To derive such an equation, note that according to thermodynamics (Maxwell):

$$T\,Ds = c_p\,DT - T\left(\frac{\partial \mathrm{v}}{\partial T}\right)_p Dp$$

Multiply by $\rho = 1/\mathrm{v}$:

$$\rho T\,Ds = \rho c_p\,DT - T\beta\,Dp \qquad \beta = \frac{1}{\mathrm{v}}\left(\frac{\partial \mathrm{v}}{\partial T}\right)_p$$

The variable β is called bulk expansion coefficient. Note that for an ideal gas, $\beta T = 1$.

The equation for temperature is therefor:

$$\boxed{\rho c_p \frac{DT}{Dt} - \beta T \frac{Dp}{Dt} = \tau_{ji}\frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}} \tag{26}$$

The temperature equation is not an additional equation, but can be used as an alternative to the entropy or original energy equation.

9 Arbitrary Regions

Sometimes, writing the conservation laws for fixed volumes is not enough. For example, an emptying balloon is not a fixed volume. Nor is the human heart, an accelerating rocket, etcetera. But they are not material regions either. They are arbitrary regions (or control volumes).

Fortunately it is not difficult to go from the equation for a fixed volume to one for a moving volume. Simply use the Leibnitz also on the moving region, the same way we used it on the material region. (Do not forget that the velocity of the boundary $\vec{v}_b$ of the moving region is not the same as the fluid velocity $\vec{v}$ on that boundary.) We will skip the derivation.

The only point to remember is: if the boundary of the region you are considering moves with speed $\vec{v}_b$, you need to replace the $\vec{v}\cdot\vec{n}$ in the outflow correction integrals with $(\vec{v}-\vec{v}_b)\cdot\vec{n}$. And it is not difficult to see why: if the boundary moves with the same speed as the fluid at that boundary, no fluid goes in or out!

So mass conservation for an arbitrary control volume is:

$$\boxed{M_{CV} \equiv \int_{CV} \rho\,\mathrm{dV} \qquad \frac{\mathrm{d}M_{CV}}{\mathrm{d}t} + \int_{MR} \rho\,(\vec{v}-\vec{v}_b)\cdot\vec{n}\,\mathrm{d}S = 0} \tag{27}$$

Similarly the momentum equation becomes:

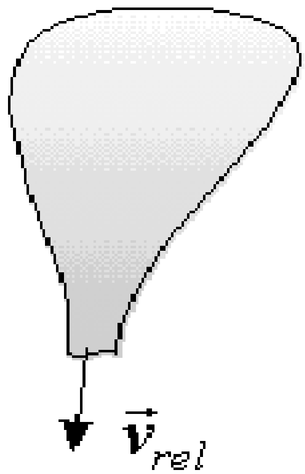
$$\boxed{\vec{I}_{CV} \equiv \int_{CV} \rho\vec{v}\,\mathrm{dV} \qquad \frac{\mathrm{d}\vec{I}_{CV}}{\mathrm{d}t} + \int_{CV} \rho\vec{v}\,(\vec{v}-\vec{v}_b)\cdot\vec{n}\,\mathrm{d}S = \vec{F}} \tag{28}$$

Note that only the velocity in the dot product is affected.

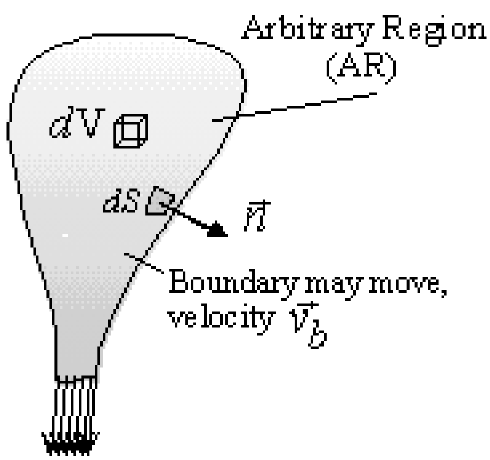
Often, it is more convenient to use a moving coordinate system, say a coordinate system fixed to the accelerating rocket instead of to the ground. Trick two to remember is that you can get a momentum equation involving velocities relative to the moving coordinate system, (instead of absolute velocities), by substracting the continuity equation times the coordinate system velocity from the momentum equation above.

Example:

Problem: Write the equation of motion of a balloon in terms of the exit area and the relative exit velocity of the air from the balloon.



Solution: The control volume is the balloon. This is an arbitrary region.



Mass conservation:

$$\frac{\mathrm{d}M}{\mathrm{d}t} + \int_{\text{exit}} \rho\,(\vec{v} - \vec{v}_b)\cdot\vec{n}\,\mathrm{d}S + \int_{\text{rubber}} \rho\,(\vec{v} - \vec{v}_b)\cdot\vec{n}\,\mathrm{d}S = 0$$

The final integral over the rubber is zero (air velocity $\vec{v}$ and rubber velocity $\vec{v}_b$ are equal at the balloon surface.) The integral over the exit we will call $\dot{m}_e$, so

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\dot{m}_e$$

Assuming that the relative air velocity at the exit is normal to it and constant:

$$\dot{m}_e \equiv \int_{\text{exit}} \rho\,(\vec{v} - \vec{v}_b)\cdot\vec{n}\,\mathrm{d}S \approx \rho v_{\text{rel exit}} S_{\text{exit}}$$

Now the momentum equation. It reads

$$\frac{\mathrm{d}\vec{I}}{\mathrm{d}t} + \int_{\text{exit}} \rho\vec{v}\,(\vec{v}-\vec{v}_b)\cdot\vec{n}\,\mathrm{d}S + \int_{\text{rubber}} \rho\vec{v}\,(\vec{v}-\vec{v}_b)\cdot\vec{n}\,\mathrm{d}S = \vec{F}$$

We write the linear momentum as $\vec{I} = M\vec{v}_{\text{average}}$, in which v_{average} is an average velocity of the balloon and the air inside. Also, the integral over the rubber is again zero, and assuming that at the exit the air velocity $\vec{v}$ is about constant across the exit:

$$\frac{\mathrm{d}M\vec{v}_{\text{average}}}{\mathrm{d}t} + \dot{m}_e\vec{v}_{\text{exit}} = \vec{F}$$

Substract $\vec{v}_{\text{average}}$ times the continuity equation:

$$M\frac{\mathrm{d}\vec{v}_{\text{average}}}{\mathrm{d}t} + \dot{m}_e(\vec{v}_{\text{exit}} - \vec{v}_{\text{average}}) = \vec{F}$$

Ignoring the difference between average velocity and the boundary velocity of the exit, the difference in velocities is just the relative velocity of the air at the exit. Also, we will assume one-dimensional motion and identify the force as being minus the drag force. So we finally get:

$$M\frac{\mathrm{d}v_{\text{average}}}{\mathrm{d}t} = -D + \dot{m}_e v_{\text{rel exit}}$$

where D is the drag force and $\dot{m}_e v_{\text{rel exit}}$ the thrust force due to outflow. This can be solved along with

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\dot{m}_e$$

•

You should now be able to do questions 5.15, 5.16, 5.17.