Vorticity Dynamics

1 Geometry

(Book 13.9 III. Read 13.1 for a refresher on circulation, vorticity, and Stokes' theorem.)
Streamlines are lines that are everywhere in the direction of the velocity. Vorticity lines are lines that are everywhere in the direction of the vorticity.
Nope, vortex lines do not have to end on the boundary or be a closed curve.
You should now be able to make 13.8

2 Kelvin's Theorem

(Book 13.10. Read and understand the descriptions of the starting vortex and the bathtub vortex at the end of 13.13.)
The circulation along any closed contour C inside the fluid is defined as

$$\Gamma = \oint \vec{v} \cdot \mathrm{d}\vec{r}$$

Stokes's theorem:

$$\Gamma = \int_S \vec{\omega} \cdot \vec{n} \, \mathrm{d}S$$

where S is any surface that has the contour C as its edge. (Of course, it is also necessary that the velocity field is defined everywhere on S.)
Kelvin's theorem: if

- the closed contour C is a material contour (always made up of the same fluid particles);
- the flow is inviscid;
- the flow is barotropic, where the density depends at most on the pressure (not on both pressure and temperature, say);

then Γ is constant:

$$\boxed{\frac{D\Gamma}{Dt} = 0}$$

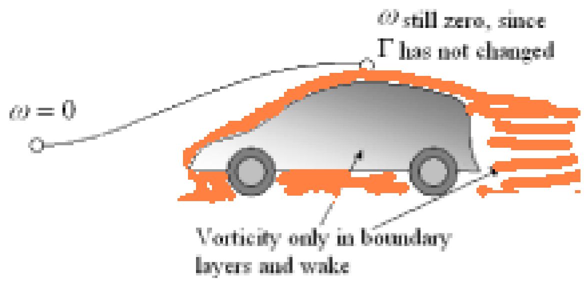
This includes incompressible inviscid flows and isentropic inviscid compressible flows.
Proof of the theorem:

$$\frac{D}{Dt} \oint_{MC} \vec{v} \cdot \mathrm{d}\vec{r} = \oint \frac{D\vec{v}}{Dt} \cdot \mathrm{d}\vec{r} + \oint \vec{v} \cdot \mathrm{d}\frac{D\vec{r}}{Dt} =$$

$$\oint -\frac{1}{\rho} \nabla p \cdot \mathrm{d}\vec{r} + \oint \vec{v} \cdot \mathrm{d}\vec{v} = -\oint \frac{1}{\rho} \mathrm{d}p + \oint \mathrm{d}\tfrac{1}{2} v^2 = 0$$

Applications:

- Flows that start irrotational remain so outside the boundary layers and wake:



- Bathtub vortex. See the description at the end of 3.13
- Starting vortex. See the description at the end of 3.13
- Trailing vortices.

You should now be able to make 13.7