EML 5709 **Fall 2010**

Homework Problems

Do not print out this page. Keep checking for changes. Homeworks should normally be online three school days before they are due. Deviations may occur due to holidays or the end of the semester.

Explain all reasoning.

1 9/1 W

1. If the density of air at sea level is 1.225 kg/m^3, and the molecular mass 28 g/mol, then what is the number of molecules per unit volume? What is the average spacing ℓ of the molecules?

   Consider a molecule of diameter d that moves over one free path length λ. During that motion it will hit another molecule if the center of the other molecule is within a radius d from the path of the molecule. In other words, the center of the other molecule must be inside a cylinder of radius d around the path λ of the first molecule. There should be about one collision in a free path, so there should be about one other molecule within the cylinder. So the free path can be ballparked from setting the volume of the cylinder equal to the average volume per particle:

   $$\pi d^2 \lambda = \text{average volume occupied per particle}$$

   Take the average diameter of the molecules to be 0.3 nm and compute λ. A more careful analysis says you still need to divide this by $\sqrt{2}$, so do so.

   Suppose you have a body of typical size L. Which length, ℓ or λ, relative to L, determines whether you can define a continuum density and velocity? Which length determines whether you can define a continuum density and velocity that you can use to compute the flow development?

2. A carbon nanotude has a diameter of 1 nm. There is a steady flow of standard air around it. Can you define a continuum density and velocity? Do not answer to quickly. Note that in a steady flow you can average over both space and time.

   Now suppose the flow about the nanotube is truly unsteady. Can you define a continuum density and velocity in that case? Comment in particular about the "ensemble average."

   Will you be able to use the Euler or Navier-Stokes to find these continuum fields? If not, will you be able to write modified equations for the continuum quantities that you can use instead?

3. Is a rain droplet in saturated air a Lagrangian region / material region / control mass? How about a droplet in dry air? Explain.

4. For ideal stagnation point as discussed in class, compute the pressure field (Eulerian) from the Bernoulli law. Then verify Newton's second law $\rho\vec{a} = -\nabla p$.

5. Convert the velocities of ideal stagnation point flow to polar coordinates using

   $$v_r = u\cos\theta + v\sin\theta \qquad v_\theta = -u\sin\theta + v\cos\theta \qquad x = r\cos\theta \qquad y = r\sin\theta$$

   Then find the streamlines in polar coordinates. Show that you get the same answer.

6. The velocity field of water waves near the surface is given by

   $$u = \epsilon\sin(kx + \omega t) \qquad v = -\epsilon\cos(kx + \omega t)$$

   where ϵ, k, and ω are all positive constants. Find and draw the streamlines of the flow. You can assume that the water surface is (approximately) at $y = 0$ and the water is below that surface.

7. The pathlines for water waves are more difficult to find. Therefor, assume that ϵ is small. In that case the particle displacements are small, and that allows you to approximate x in the sine and cosine by the x-value x_0 of the initial particle position which is constant:

   $$u = \epsilon\sin(kx_0 + \omega t) \qquad v = -\epsilon\cos(kx_0 + \omega t)$$

   Find and draw the particle paths under that assumption. Compare with the streamlines. Why are they not the same?

2 9/8 W

1. A steady stream of air enter a pipe with a diameter of 1" at a velocity of 20 m/s at a pressure of 1 bar. The pipe has a gradual contraction in diameter to $\frac{1}{2}$". What are the velocity and pressure after the contraction? Ignore viscous effects. (This involves the Bernoulli law and mass conservation of undergraduate fluid mechanics.) This is a steady flow, $\partial\vec{v}/\partial t = 0$, so explain how it is possible for the fluid to change velocity in a steady flow. In particular, write the material derivative of the velocity in Eulerian coordinates and discuss the various terms. Write in particular *also* the expression for the acceleration on the axis of the pipe, taking it as your x-axis.

2. You are driving your Miata at 80 mph on a standard day. At a point on the nose just above the boundary layer the flow velocity of the air relative to the car is 120 mph. What is the pressure on the car surface at that point?

3. Newton's second law per unit volume reads

$$\rho\frac{D\vec{v}}{Dt} = \vec{f}$$

where f is the net force on the fluid per unit volume. Assume inviscid flow, in which case the force per unit volume is minus the pressure gradient plus the force of gravity, so

$$\rho\frac{D\vec{v}}{Dt} = -\nabla p + \rho\vec{g}$$

Write these equations out fully in Eulerian coordinates for each of the three Cartesian velocity components u, v, and w. Use the Eulerian form of the material derivative as given in class. The equations you should get are known as the Euler equations.

4. Substitute the Eulerian velocity field of stagnation point flow into the Euler equations that you obtained above. You get three equations for the pressure, one giving its x-derivative, one its y derivative, and the third its z-derivative. More than one equation for a single unknown p is usually too much, but show that in this case, there is indeed a solution p that satisfies all three equations. Find out what it is. Take gravity to be in the negative y-direction, so that $g_x = g_z = 0$ and $g_y = -g$.

5. The velocity field in Couette flow is given by

$$u = cy \qquad v = 0 \qquad w = 0$$

Draw the streamlines of this flow and a couple of velocity profiles. Find the velocity derivative tensor A, the strain rate tensor S, and the matrix W.

6. In Poiseuille flow (laminar flow through a pipe), the velocity field is in cylindrical coordinates given by

$$\vec{v} = \widehat{\imath}_z v_{\max}\left(1 - \frac{r^2}{R^2}\right)$$

where $v_{\max}$ is the velocity on the centerline of the pipe and R the pipe radius. Use Appendices B and C to find the velocity derivative and strain rate tensors of this flow. *Do not guess.* Evaluate the strain rate tensor at $r = 0$, $\frac{1}{2}R$ and R. What can you say about the straining of small fluid particles on the axis?

3 9/15 W

1. If the surface temperature of a river is given by $T = 2x + 3y + ct$ and the surface water flows with a speed $\vec{v} = \widehat{\imath} - \widehat{\jmath}$, then what is c assuming that the water particles stay at the same temperature? (Hint: $\mathrm{D}T/\mathrm{D}t = 0$ if the water particles stay at the same temperature. Write this out mathematically.)

2. A boat is cornering through this river such that its position is given by $x_b = f_1(t)$, $y_b = f_2(t)$. What is the rate of change $\mathrm{d}T/\mathrm{d}t$ of the water temperature experienced by the boat in terms of the functions f_1 and f_2?

3. Is the Poisseuille flow of the previous homework an incompressible flow? Derive the principal strain rates and the principal strain directions as a function of the radial position. Also find the vorticity.

4. For the Poisseuille flow of the previous question, describe what motions small particles perform. Neatly sketch a particle, that was spherical at time t, at time $t+\mathrm{d}t$. Show both the individual motions and the combined motion. Repeat for an initially cubical particle.

5. Write the velocity derivative tensor for two-dimensional ideal stagnation point flow. (With the velocity field as discussed in class.) From this tensor, decide whether or not ideal stagnation point flow is an incompressible flow. Find the strain rate tensor. Diagonalize it. What are the principal strain rates? What are the principal strain axes (i.e. the directions of $\widehat{\imath}'$, $\widehat{\jmath}'$, and $\widehat{k}'$)? Neatly sketch the deformation of an initially cubical particle (aligned with the principal strain axes), during a small time interval. Also sketch the deformation of an initially spherical particle. Also show the complete particle changes when you include the solid body rotation.

4 9/24 F

1. If you put a cup of coffee at the center of a rotating turn table and wait, eventually, the coffee will be executing a "solid body rotation" in which the velocity field is, in cylindrical coordinates:

$$\vec{v} = \widehat{\imath}_\theta \Omega r$$

where Ω is the angular velocity of the turn table. Draw samples of the streamlines of this flow. Find the vorticity and the strain rate tensor for this flow, using the expressions in appendices B and C. Show that indeed the coffee moves as a solid body; i.e. the fluid particles do not deform, and that for a solid body motion like this, indeed the vorticity is twice the angular velocity.

2. An "ideal vortex flow" is described in cylindrical coordinates by

$$\vec{v} = \frac{C}{2\pi r}\widehat{\imath}_\theta$$

where C is some constant. Draw samples of the streamlines of this flow. In cylindrical coordinates, nabla is given by

$$\nabla = \widehat{\imath}_r \frac{\partial}{\partial r} + \widehat{\imath}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \widehat{\imath}_z \frac{\partial}{\partial z}$$

Evaluate

$$\begin{vmatrix} \widehat{\imath}_r & \widehat{\imath}_\theta & \widehat{\imath}_z \\ \dfrac{\partial}{\partial r} & \dfrac{1}{r}\dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ v_r & v_\theta & v_z \end{vmatrix}$$

for this flow. Compare the answer with the vorticity $\omega = \nabla \times \vec{v}$, for which you can find the correct expressions in appendix B. Explain why the determinant does not give the correct result for the vorticity. In particular note that in

$$\nabla \times \vec{v} = \left(\widehat{\imath}_r \frac{\partial}{\partial r} + \widehat{\imath}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \widehat{\imath}_z \frac{\partial}{\partial z}\right) \times \left(\widehat{\imath}_r v_r + \widehat{\imath}_\theta v_\theta + \widehat{\imath}_z v_z\right)$$

the unit vectors $\widehat{\imath}_r$ and $\widehat{\imath}_\theta$ are *not* constants but depend on θ. ($\mathrm{d}\widehat{\imath}_r/\mathrm{d}\theta = \widehat{\imath}_\theta$ and $\mathrm{d}\widehat{\imath}_\theta/\mathrm{d}\theta = -\widehat{\imath}_r$.)

3. The "circulation" Γ along a closed contour is defined as

$$\Gamma \equiv \oint \vec{v} \cdot \mathrm{d}\vec{r}$$

For both the solid body rotation of question 1, and the vortex flow of question 2, find the circulation along the unit circle in the x, y-plane. Next, *only for the vortex flow*, find the circulation along the closed curve consisting of the following segments:

(a) The part of the curve $y = \cosh(x)$, $z = 0$ from $x = 2$ to $x = -3$;

(b) vertically downward to the curve $y = -\cosh(\cosh(x))$, $z = 0$ at $x = -3$;

(c) following the curve $y = -\cosh(\cosh(x))$, $z = 0$ from $x = -3$ to the point $x = 0$, hence $y = -\cosh(1)$, $z = 0$;

(d) in a straight line along the z direction to the point $x = 0$, $y = -\cosh(1)$, $z = 3.5$;

(e) in a straight line from $x = 0$, $y = -\cosh(1)$, $z = 3.5$ to the starting point $x = 2$, $y = \cosh(2)$, $z = 0$.

Note that in cylindrical coordinates

$$\mathrm{d}\vec{r} = \widehat{\imath}_r \mathrm{d}r + \widehat{\imath}_\theta r \mathrm{d}\theta + \widehat{\imath}_z \mathrm{d}z \qquad \vec{v} = \widehat{\imath}_r v_r + \widehat{\imath}_\theta v_\theta + \widehat{\imath}_z v_z$$

4. According to the Stokes theorem of Calculus III, you should have

$$\oint \vec{v} \cdot \mathrm{d}\vec{r} = \int \nabla \times \vec{v} \cdot \vec{n} \, \mathrm{d}S$$

where the second integral is over the inside of the contour. So instead of integrating the circulation Γ as you did in question 3, you could have integrated the component of vorticity normal to the circle over the inside of the circle. Show that if you do that integral using the vorticity that you found for solid body rotation in question 1, you do indeed get the same answer as you got in question 3. Fine. But now show that if you do the integral of the vorticity over the inside of the circle for the vortex flow of question 2, you do *not* get the same answer for the circulation as in question 3. Explain which value is correct. And why the other value is wrong.

5. Write down the worked-out mathematical expressions for the integrals requested in question 5.1. Explain their physical meaning. Don't worry about actually doing the integrations. However, show integrands and limits completely worked out.

   Take the surfaces S_I, S_{II}, S_{III}, and S_{IV} to be one unit length in the z-direction. (To figure out the correct direction of the normal vector $\vec{n}$ at a given surface point, note that the control volume in this case is the right half of the region in between two cylinders of radii r_0 and R_0 and of unit length in the z-direction. The vector $\vec{n}$ is a unit normal vector sticking *out* of this control volume.)

6. 5.14

5 9/29 W

1. 5.11. Cearly define what control volume you are using.

2. 5.12. This question explains why the water stream coming out of a faucet contracts in area shortly below the faucet exit. As always, both mass and momentum conservation are needed.

   The faucet exit velocity is assumed to be of the form of Poisseuille flow:

$$v_z = v_{\max}\left(1 - \frac{r^2}{R^2}\right)$$

   You can assume that the stress tensor at the faucet exit is of the form (in cylindrical coordinates)

$$\bar{\bar{\tau}} = \begin{pmatrix} 0 & 0 & \tau_0 r/R \\ 0 & 0 & 0 \\ \tau_0 r/R & 0 & 0 \end{pmatrix}$$

in other words, much like the strain rate tensor that you derived earlier for Poisseuille flow.

Take the faucet exit as the entrance of your control volume. Take as exit to your control volume a slighly lower plane at which the radius of the jet has stabilized to R_2 and the flow velocity has become uniform (independent of r). For a uniform flow velocity there are no viscous stresses. Gravity and pressure forces can be ignored compared to the high viscous forces in this very viscous fluid.

3. Write a finite volume discretization for the x-momentum equation for the little finite volume in polar coordinates. Just like the continuity equation done in class, your final equation should *only* involve pressures, densities, and velocities at the center points of the finite volumes. Ignore the viscous stresses for now.

   The unknown velocities used in the computation should be taken to be the polar components v_r and v_θ. But momentum conservation for x-momentum is asked. (Conservation of r-momentum or θ-momentum would be complete nonsense.) So you will need to write the x-component of velocity in terms of the polar unknowns. Note that

$$\widehat{\imath}_r = \cos(\theta)\widehat{\imath} + \sin(\theta)\widehat{\jmath} \qquad \widehat{\imath}_\theta = -\sin(\theta)\widehat{\imath} + \cos(\theta)\widehat{\jmath}$$

4. Assuming that there are known viscous stresses at the centers of the sides of the finite element, what additional terms do you get in the obtained equation due to viscous forces? Assume the stress tensor is given in polar form. (So τ_{rr}, $\tau_{r\theta}$, etcetera.)

6 10/6 W

1. Unlike the ideal point vortex you analyzed in a previous homework, a true vortex diffuses out with time, and its velocity field is given by

$$\vec{v} = \widehat{\imath}_\theta \frac{C}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right]$$

   Find the vorticity of this flow field. Also find the circulation along a circle of an arbitrary radius r. Then show that Stokes theorem *does* work for this flow. (The velocity is zero at $r = 0$; just apply l'Hopital.) Finally show that if the coefficient of viscosity is very small, the vorticity is only nonzero in some narrow spike near the origin, so that it looks almost like a an ideal vortex. (But the vorticity still integrates to Γ, despite the small radius of the region with appreciable vorticity.)

2. 5.2 Make sure to first write the full equations for the special case that the density is constant before assuming a radial flow. Make that a neat graph, and include the streamlines.

3. 5.3. This is two-dimensional Poiseuille flow (in a duct instead of a pipe). T_{ij} is the book's notation for the complete surface stress including the pressure,

$$T_{ij} = -p\delta_{ij} + \tau_{ij}$$

   where δ_{ij} is called the Kronecker delta or unit matrix. So the book is really saying the pressure is -5 and there is an additional viscous stress $\tau_{xy} = -2\mu v_0 y/h^2$. Watch it, the expression $n_j\tau_{ji}$ gives the stress components in the x, y, z-axis system.

4. 5.6. Z is the height h. The final sentence is to be shown by you based on the obtained result. Hints: take the curl of the equation and simplify. Formulae for nabla are in the vector analysis section of math handbooks. If there is a density gradient, then the density is not constant. And neither is the pressure. T_{ij} is the book's notation for the complete surface stress, so the book is saying there is no viscous stress. (That is self-evident anyway, since a still fluid cannot have a strain rate to create viscous forces.)

7 10/15 F

1. 6.1. Use the appendices. Based on the results, discuss whether this is incompressible flow, and in what direction the viscous stresses on the surface of the sphere are. Also state in which direction the inviscid stress on the surface is.

2. 6.2 Discuss your result in view of the fact, as stated in (6.1), that the Reynolds number must be small for Stokes flow to be valid. What does it physically mean?

3. 7.5. Use the appendices. You may only assume that $v_r = v_r(r)$, $v_\theta = v_\theta(r)$, $v_z = 0$, and $p = p(r, \theta)$ in cylindrical coordinates. Do not assume that the radial velocity is zero, derive it. Do not assume that he pressure is independent of θ, derive it. Ignore gravity as the question says. Note that p must have the same value at $\theta = 0$ and 2π because physically it is the same point. Answer for v_θ:

$$\frac{\Omega r_0^2 r_1}{r_1^2 - r_0^2}\left(\frac{r_1}{r} - \frac{r}{r_1}\right)$$

4. In 7.5, what is the power needed to keep the rod rotating, per unit axial length? What is the pressure difference between the surfaces of the pipe and the rod?

8 10/20 W

1. 7.9. You can assume that the film thickness is so small that the curvature of the pipe wall can be ignored. In that case, it becomes 2D steady flow in the x-direction along a flat wall of spanwise length $2\pi r_0$ in the z-direction. Take the x-axis downwards. Assume $v = 0$ (vertical streamlines), $u = u(x, y)$ and $w = 0$ (two-dimensional flow), and that $p = p(x, y, z)$. Everything else must be derived; derive both pressure and velocity field. Do not ignore gravity. For the boundary conditions at the free surface, assume that the liquid meets air of zero density and constant pressure p_a there. Air of zero density cannot exert nonzero tangential forces. Also write appropriate boundary conditions where the fluid meets the cylinder surface.

2. 7.6. Do not ignore gravity, but assume the pipe is horizontal. Do not use the effective pressure. Careful, the gravity vector is *not* constant in polar coordinates. Do not ignore the pressure gradients: assume the pressure can be any function $p = p(r, \theta, z, t)$ and derive anything else. Merely assume that the pressure distribution at the end of the pipe and rod combination is the same as the one at the start. For the velocity assume $v_r = v_\theta = 0$ and $v_z = v_z(r, z)$. Anything else must be derived. Give both velocity and pressure field.

3. For the case of question 7.6, what is the force required to pull the rod through the axis, per unit length? In 7.9, (previous homework), what is the net downward shear force on the pipe? Does the simple answer surprise you? Why not?

4. 7.1a Assume only that the cylindrical velocity components only depend on r, $v_r, v_\theta, v_z = v_r, v_\theta, v_z(r)$, that $p = p(r, \theta, z, t)$ is arbitrary, and that the pipe is horizontal. Use the effective pressure. Show that two velocity components must be zero. (You should be able to show that the effective pressure is independent of θ from the appropriate momentum equation by noting that p at $\theta = 2\pi$ must be the same as at $\theta = 0$; otherwise just assume it is. Also note that the velocity can obviously not be infinitely large on the pipe centerline.)

5. 7.1b Continuing the previous question, derive the velocity and pressure fields.

9 10/27 W

1. (A small part of 7.17 with $n = 1$.) Assume that an infinite flat plate normal to $\widehat{\jmath}$ accelerates from rest, so that its velocity is given by $u_p \widehat{\imath} = \dot{U} t \widehat{\imath}$ where $\dot{U}$ is a constant. There is a viscous Newtonian fluid

above the plate. Assuming only that $\vec{v} = \vec{v}(y,t)$, $w = 0$, and that the effective pressure far above the plate is constant, derive a partial differential equation and boundary conditions for the flow velocity of the viscous fluid. List them in the plane of the independent variables.

2. (A small part of 7.17 with $n = 1$.) Assuming that the velocity profile is similar, derive that

$$f - \frac{\dot{\delta} t}{\delta} \eta f' = \frac{\nu t}{\delta^2} f''$$

where $f(\eta)$ is the similar velocity profile and $\delta(t)$ is the boundary layer thickness used to get similarity. By examining the above equation at the plate, where $\eta = 0$, show that within a constant, δ must be the same as in Stokes' second problem. Take it the same, then write the final equation for the similar profile f.

3. (A small part of 7.17 with $n = 1$.) Differentiate the equation for f twice with respect to η, and so show that $g = f''$ satisfies the equation

$$g'' + 2\eta g' = 0$$

This equation is the same as the one for f in Stokes' second problem, and was solved in class. The general solution was

$$g(\eta) = C_1 \int_{\bar{\eta}=\eta}^{\infty} e^{-\bar{\eta}^2} \, \mathrm{d}\bar{\eta} + C_2$$

Explain why C_2 must be zero. Explain why then f' can be found as

$$f'(\eta) = -\int_{\bar{\eta}=\eta}^{\infty} g(\bar{\eta}) \, \mathrm{d}\bar{\eta} = -C_1 \int_{\bar{\eta}=\eta}^{\infty} \int_{\bar{\bar{\eta}}=\bar{\eta}}^{\infty} e^{-\bar{\bar{\eta}}^2} \, \mathrm{d}\bar{\bar{\eta}} \, \mathrm{d}\bar{\eta}$$

Draw the region of integration in the $\bar{\eta}, \bar{\bar{\eta}}$-plane. Use the picture to change the order of integration in the multiple integral and integrate $\bar{\eta}$ out. Show that

$$f'(\eta) = C_1 \left[\eta \int_{\bar{\bar{\eta}}=\eta}^{\infty} e^{-\bar{\bar{\eta}}^2} \, \mathrm{d}\bar{\bar{\eta}} - \frac{1}{2} e^{-\eta^2} \right]$$

Integrate once more to find $f(\eta)$. Apply the boundary condition to find C_1.

10 11/5 F

1. Do bathtub vortices have opposite spin in the southern hemisphere as they have in the northern one? Derive some ballpark number for the exit speed of a bathtub vortex at the north pole and one at the south pole, assuming the bath water is initially at rest compared to the rotating earth. Use Kelvin's theorem. Note that the theorem applies to an inertial frame, not that of the rotating earth. What do you conclude about the starting question?

2. A Boeng 747 has a maximum take-off weight of about 400,000 kg and take-off speed of about 75 m/s. The wing span is 65 m. Estimate the circulation around the wing from the Kutta-Joukowski relation. This same circulation is around the trailing wingtip vortices. From that, ballpark the typical circulatory velocities around the trailing vortices, assuming that they have maybe a thickness of a quarter of the span. Compare to the typical take-off speed of a Cessna 52, 50 mph.

3. Find boundary conditions for the streamfunction for transverse ideal flow around a circular cylinder. The velocity far away from the cylinder is $U\hat{\imath}$ and the radius of the cylinder is r_0. Note that appendix D.2 has an error. The correct equation is

$$v_\theta = -\frac{\partial \psi}{\partial r}$$

4. Following similar lines as in class, but watching the new boundary conditions, solve the equation for the streamfunction around the circular cylinder. Before continuing, check your results for the radial and tangential velocity components at the surface of the cylinder against the one from the velocity potential solution obtained in class. Is the velocity at the top and bottom points $2U$? Are the stagnation points correct?

5. Find the pressure on the surface of the cylinder.

11 11/12

1. Integrate the pressure forces over the surface of the irrotational flow about a cylinder to get the net force on the cylinder.

2. Now add to the velocity field the velocity field of an ideal vortex,

$$\vec{v} = \frac{\Gamma}{2\pi r}\widehat{\imath}_\theta$$

Check whether the correct flow boundary conditions are still satisfied at the surface of the cylinder and far from the cylinder. Integrate the pressure again, and compare the forces to D'Alembert and Kutta-Joukowski.

3. Write the complete velocity potential and streamfunction of cylinder plus circulation.

12 11/19 F

1. Sketch streamlines for the potential flow

$$F = z^{2/3}$$

Explain why such a flow might be relevant to flow about a corner. What is the (effective) pressure on the positive x-axis? Comment on what happens to the pressure when $x = 0$.

2. Find the polar velocity components and pressure of the source flow

$$F = \frac{Q}{2\pi}\ln z$$

where Q is a constant. Show that this is a valid solution of the Navier Stokes equations for the flow outside a cylindrical balloon whose radius R is expanding according to the relationship

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{Q}{2\pi R}$$

Then show that this means that the cross sectional area of the balloon is linearly increasing with time.

3. Show that the potential flow

$$F = \frac{Q}{2\pi}\ln z$$

where Q is not a constant but equal to $2\pi t$ is an *exact* solution of the viscous Navier-Stokes equation for flow around a balloon whose radius expands as $R = t$. Then find the pressure, using the correct Bernoulli equation for an *unsteady* potential flow. Comment on the pressure far from the balloon.

4. In the familiar potential flow around a cylinder,

$$F = U\left(z + \frac{r_0^2}{z}\right)$$

the Uz term produces the incoming uniform flow and the Ur_0^2/z term produces the flow induced by the cylinder. That means that if the *fluid* is at rest at infinity and it is the cylinder that moves, the potential is given by

$$F = -\dot{x}_0 \frac{r_0^2}{z - x_0}$$

where $x_0(t)$ is the position of the center of the cylinder on the x-axis. Find the time derivative $\partial F/\partial t$ and the spatial derivative $W = \partial F/\partial z$. Watch it: both x_0 and $\dot{x}_0$ in F depend on time. Now evaluate these derivatives on the surface of the cylinder where $z - x_0 = r_0 e^{i\theta}$. (Here the angle θ is measured from the center of the cylinder, not from the origin.) Then find $\partial\phi/\partial t$ as the real part of $\partial F/\partial t$. Also find the square magnitude of the velocity as $W\bar{W}$, where $\bar{W}$ is the complex conjugate of W. Use this to find the pressure on the surface of the cylinder. Answer:

$$p_{\text{eff}} = p_\infty - \frac{1}{2}\rho\dot{x}_0^2 + \rho r_0 \ddot{x}_0 \cos\theta + \rho\dot{x}_0^2 \cos 2\theta$$

5. From the pressure of the previous question, find the force on the cylinder. Show that it implies that to accelerate the cylinder, in addition to the force required to accelerate the cylinder itself, there will be an additional force as if an additional mass equal to an amount of fluid with the volume of the cylinder also must be accelerated. Explain why an apparent mass effect must be there on behalf of the second law of thermodynamics.

6. Write the complex velocity potential F for nose flow. From its derivative, identify the location of the stagnation point. Identify the streamfunction from F. From it, find the value of the streamfunction at the stagnation point. From that, find the equation for the wall streamline, which passes through the stagnation point. From that, find the thickness of the nose far downstream. From the derivative of F, find the square magnitude of the velocity in terms of polar coordinates. Write the pressure on the wall as a function of the polar angle θ by writting the equation for the wall streamline in polar coordinates and plugging it in the square velocity.

13 11/24 W

1. Draw the streamlines of the uniform flow $F = Uz$, with U a positive real constant. Now define a new coordinate $\zeta = \sqrt{z}$. Plot the streamlines of the same flow F in the ζ-plane.

2. The flow around a circular cylinder is

$$F = U\left(z + \frac{1}{z}\right)$$

Apply the Joukowski transformationw

$$\zeta = z + \frac{1}{z}$$

to find a new coordinate ζ. How does the cylinder flow in the z-plane look in the ζ plane? How does the cylinder look in the ζ-plane? So, are you surprised by the flow field?

That was too boring. Take the flow around a circle of radius 2,

$$F = U\left(z + \frac{4}{z}\right)$$

and apply again the Joukowski transformation:

$$\zeta = z + \frac{1}{z}$$

Show that the circle transforms to an ellipse in the ζ plane by setting $z = 2e^{\mathrm{i}\theta}$ on the surface of the circle and transforming that. What is the aspect ratio of the ellipse? Find the velocity at the top and bottom points of the ellipse and compare with the value $2U$ that applies for a circle. To find the velocity, use

$$v_\xi - \mathrm{i}v_\eta = \frac{\mathrm{d}F}{\mathrm{d}\zeta} = \frac{\mathrm{d}F}{\mathrm{d}z}\Big/\frac{\mathrm{d}\zeta}{\mathrm{d}z}$$

where v_ξ and v_η are the velocity components of the flow in the ζ plane.

3. Read through the cylinder flow program cylinder.m[1]. Save it as type "all files" and run it in Matlab or Octave 3.0 or higher. Make plots to show the effect of changing the angle-of-attack parameter and of the circulation. In particular, plot the flow for the values of the circulation for which the stagnation points are at opposite sides of the cylinder, for which they are apart by a 90° angle, for which they coincide, and for which they are completely off the cylinder. Use screen capture or the *print* command to make hardcopies of the plots. Try *help print* if needed.

4. Read through the cylinder flow program airfoil.m[2]. Save it as type "all files" and run it in Matlab or Octave 3.0 or higher. Do a "clear all" first to get rid of the cylinder stuff. Make plots to show zero thickness airfoil. Change the radius of the cylinder to produce a symmetric Joukowski airfoil with and without lift. Destroy the top/bottom symmetry of the conformal mapping to create camber. Be sure to create a reasonable cambered airfoil shape, thickness ratio about 15%, angle of attack about 15 degrees.

5. Airfoils have a suction peak at the nose. Read through and download pressure.m[3]. Adjust to get a good picture. Mark both the stagnation point and the suction peak in the picture. Comment on the pressure at the trailing edge. How big is it?

6. According to potential flow theory, what would be the lift per unit span of a flat-plate airfoil of chord 2 m moving at 100 m/s at sea level at an angle of attack of 10 degrees? What would be the drag? What would be the circulation around the airfoil?

14 12/3 F

1. Compute approximate values of the Reynolds number of the following flows:
   (a) your car, assuming it drives;
   (b) a passenger plane flying somewhat below the speed of sound (assume an aerodynamic chord of 30 ft);
   (c) flow in a 1 cm water pipe if it comes out of the faucet at .5 m/s,

   In the last example, how fast would it come out if the Reynolds number is 1? How fast at the transition from laminar to turbulent flow?

2. Using suitable neat graphics, show that the boundary layer variables for the boundary layer around a circular cylinder of radius r_0 in a cross flow with velocity at infinity equal to U and pressure at infinity p_∞ are given by:
$$x = r_0\theta \qquad y = r - r_0 \qquad u = v_\theta \qquad v = v_r$$
   Write the appropriate equations for the unsteady boundary layer flow around a circular cylinder in terms of the *boundary layer variables* above.

3. Rewrite the exact Navier-Stokes equations in polar coordinates, (the continuity equation and the r and θ momentum equations) in terms of the boundary layer variables x, y, u, and v and the radius of the cylinder r_0.

4. Compare the equations you got above with the exact equations in polar coordinates, (the continuity equation and the r and θ momentum equations). Explain for each discrepancy why the difference is small if the boundary layer is thin.

5. Reconsider the unsteady boundary layer flow around a circular cylinder in terms of the boundary layer equations. Assuming that the potential flow outside the boundary layer is steady and unseparated, give all boundary conditions to be satisfied. Make sure to write them in terms of boundary layer variables only. Solve the pressure field inside the boundary layer.

[1] `cylinder.m`
[2] `airfoil.m`
[3] `pressure.m`

6. According to potential flow theory, what would be the lift per unit span of a flat-plate airfoil of chord 2 m moving at 30 m/s at sea level at an angle of attack of 10 degrees? What would be the viscous drag if you compute it as if the airfoil is a flat plate aligned with the flow with that chord and the flow is laminar? Only include the shear stress over the last 98% of the chord, since near the leading edge the shear stress will be much different from an aligned flat plate. What is the lift to drag ratio? Comment on the value. Use $\rho = 1.225$ kg/m^3 and $\nu = 14.5\ 10^{-6}$ m^2/s.

7. Assume that a flow enters a two dimensional duct of constant area. If no boundary layers developed along the wall, the centerline velocity of the flow would stay constant. Assuming that a Blasius boundary layer develops along each wall, what is the correct expression for the centerline velocity?

8. Continuing the previous question. Approximate the Blasius velocity profile to be parabolic up to $\eta = 3$, and constant from there on. At what point along the duct would you estimate that developed flow starts based on that approximation? Sketch the velocity profile at this point, as well as at the start of the duct and at the point of the duct where the range $0 \leq \eta \leq 3$ corresponds to $\frac{1}{8}$ of the duct height accurately in a single graph. Remember the previous question while doing this!