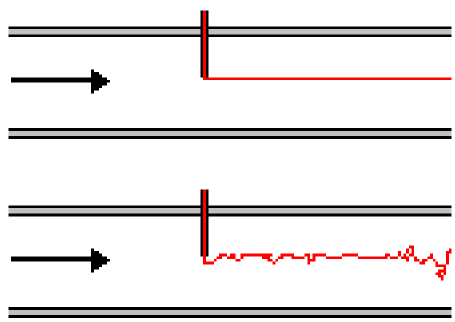
Turbulence

1 Characteristics

(Book 23.2)



Typical characteristics of turbulent flows:

- unsteady, irregular 3D vorticity fluctuations;
- diffusion;
- large Reynolds numbers;
- dissipation.
- local;
- self-sustaining;

2 Reynolds Decomposition

(Book 23.3, 4)

Decompose the flow quantities in average and fluctuating components. Time average:

$$\boxed{\overline{\vec{v}} = \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T \vec{v}\,(\vec{r},t)\ \mathrm{d}t\right)}$$

Fluctuating part:

$$\vec{v} = \overline{\vec{v}} + \vec{v}'$$

Exercise:

Sketch a velocity or pressure trace in a point in a turbulent pipe flow. In the figure, indicate what the average and fluctuating quantities are. What are the average and fluctuating velocities at the surface of the pipe? Sketch the mean (average) velocity profile in turbulent flow and in laminar flow in the same graph, assuming the two flows have the same net mass flow.

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Reynolds averaged equations can be found by averaging the Navier Stokes equations:

Continuity

$$\overline{\frac{\partial \overline{u_i} + u_i'}{\partial x_i}} = 0$$

Now from the definition of time average, the time average of a time average leaves that time average unchanged. Also the time average of any fluctuating component is zero. Further, the time average is a linear operation and commutes with differentiation, so

$$\boxed{\frac{\partial \overline{u_i}}{\partial x_i} = 0}$$

Viscous Stress Tensor: Assuming μ is constant,

$$\overline{\tau_{ij}} = \mu \left(\overline{\frac{\partial \overline{u_i} + u_i'}{\partial x_j}} + \overline{\frac{\partial \overline{u_j} + u_j'}{\partial x_i}} \right)$$

$$\boxed{\overline{\tau_{ij}} = \mu \left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right)}$$

Momentum Equations: Using the conservation form,

$$\overline{\frac{\partial \rho \left(\overline{u_i} + u_i'\right)}{\partial t}} + \overline{\frac{\partial \rho \left(\overline{u_i} + u_i'\right)\left(\overline{u_j} + u_j'\right)}{\partial x_j}} = -\overline{\frac{\partial \overline{p} + p'}{\partial x_i}} + \overline{\frac{\partial \overline{\tau_{ij}} + \tau_{ij}'}{\partial x_j}}$$

$$\frac{\partial \rho \overline{u_i}\,\overline{u_j}}{\partial x_j} + \frac{\partial \rho \overline{u_i' u_j'}}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \overline{\tau_{ij}}}{\partial x_j}$$

It is seen that the turbulent velocity fluctuations act as an additional shear stress called the Reynolds stress:

$$\boxed{\tau_{ij}^R = -\rho \overline{u_i' u_j'}}$$

$$\boxed{\frac{\partial \rho \overline{u_i}\,\overline{u_j}}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \overline{\tau_{ij}} + \tau_{ij}^R}{\partial x_j}}$$

Unfortunately, the Reynolds stress is not known unless you solve the full unsteady Navier-Stokes equations. To avoid this humongous task, guessing (also known as modelling) is needed.

Exercise:

Compare the sizes of the Reynolds stress and the laminar stress in high Reynold number flows. In the turbulent pipe flow, is the Reynolds stress everywhere much larger than the laminar stress? If not, at what locations is the laminar stress larger?

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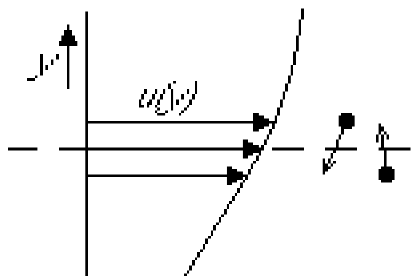
Molecular model of (laminar) viscosity:

(Book 6.2)



Consider the net momentum transfer per unit area through the plane indicated by the broken line. This momentum transfer produces the shear stress. Molecules that cross the plane travelled towards the plane over a distance on the order of a mean free path length λ without colliding. So the molecules reaching the plane from above have an average x-velocity that is roughly $\lambda \partial u/\partial y$ higher than the average u at the plane, decreasing the x momentum above the broken line. The fluid particles that reach the plane from below have an average x-velocity that is roughly $-\lambda \partial u/\partial y$ lower than the average u at the plane, also decreasing the x momentum above the broken line. As a result, the fluid above the plane is slowed down at the plane. The force per unit area follows as the part of $\rho u \vec{v} \cdot \vec{n}$ that not cancels between the downward and upward moving molecules. The crossing velocity corresponding to $\vec{v} \cdot \vec{n}$ is of the order of the mean molecular speed, which is of the order of the speed of sound a. So the stress is of order $\rho \lambda a \partial u/\partial y$, and the kinematic viscosity of order λa.

Exercise:

Compare the value of λa of standard air with its dynamic viscosity.

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Mixing length idea:

Assume that the turbulent transport of momentum is similar to the molecular one. Fluid at a given plane originates from some transverse distance ℓ with much of its velocity difference left intact during the trip. Then the kinematic eddy viscosity would be $\ell v'$, where v' is the typical vertical velocity fluctuation. From continuity, $u'_x + v'_y = 0$, so assuming that there is no strong directionality in eddy length scales, v' is of the order u', which was estimated as $\ell \partial u/\partial y$, giving an eddy viscosity

$$\boxed{\nu^R = \ell^2 \left| \frac{\partial \overline{u}}{\partial y} \right|}$$

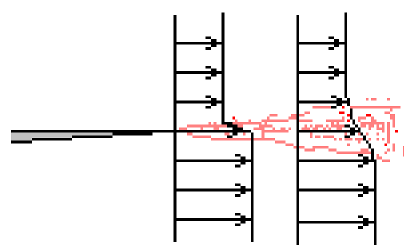
Assuming ℓ is a known quantity, this gives governing equations that are partial differential equations. The estimate of ℓ could be the typical transverse length scale in free turbulence or the distance from the wall in a surface layer.

Many things are wrong in the story: For one, turbulent motion is not small compared to the transverse scales of the flows, so $u' \neq \ell \partial u/\partial y$ and is in fact related to the velocity at finite distances, making the entire idea of having universal partial differential equations (involving local quantities only) suspect. Also, the turbulent fluctuations are not independent of the velocity field like a, the turbulent shear stress would always be predicted to be exactly zero at points of $\partial \overline{u}/\partial y = 0$ even if there is no symmetry around that point, the larger turbulence scales are directional, etc.

3 Free Turbulence

(Book 23.5, 6)

Mixing layers:



$u = U_{\text{ave}} + \Delta U f(y/\delta) \qquad \delta \propto x$

Asuming that the layer is dominated by a balance of inviscid instability (stronger for thinner layers) creating turbulence, and the diffusive effects of turbulence (which thicken the layer), viscosity would not play a big part. Then there is no obvious nondimensional quantity that the nondimensional ratio δ/x could depend on, giving δ linear in x. For the same reason, the velocity profile will be similar.

Exercise:

Discuss how well that seems to agree with flow visualizations.

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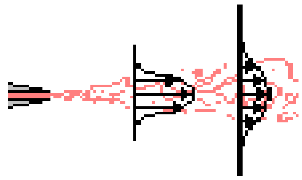
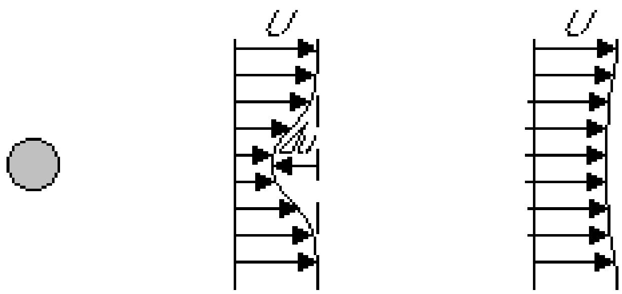
Jets:



$$u = U(x) f(y/\delta) \qquad \delta \propto x \qquad U \propto x^{-1/2} \text{ plane, } U \propto x^{-1} \text{ axisymmetric}$$

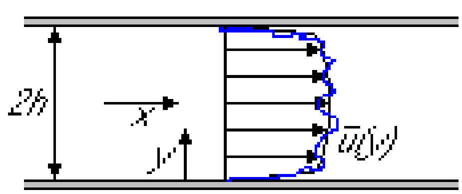
Because of integral momentum conservation, the integral $\int_{y=-\infty}^{\infty} \rho u^2 \, \mathrm{d}y$ must be constant, so, assuming similarity, we get the decay rates for the velocity.

2D Wakes:



For 2D wake flow, δ/x will vary with the nondimensional parameter $\Delta U/U$. The profiles will be similar since the shape should tend to a finite limit far downstream. Because of the momentum integral, $\delta \Delta U$ remains constant going downstream. Estimating the change in momentum $\rho Du/Dt \sim uu_x \sim \rho U(\Delta U)_x$ as being caused by a Reynolds shear force $\partial\tau^R/\partial y \sim \tau^R/\delta \sim \rho\overline{u'v'}/\delta \sim \rho\left(\Delta U\right)^2/\delta$, (eg, a mixing length estimate with $\ell = \delta$,) we get that δ/x increases as $\sqrt{x}$ and $\Delta U/U$ decreases as $1/\sqrt{x}$ going downstream.

4 Channel Flow



Assume that the flow is independent of x.

Continuity

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0 \quad \Longrightarrow \quad \overline{v} = 0$$

Stress Tensor:

$$\overline{\tau_{xy}} \equiv \tau^{\ell} = \mu \frac{\partial \overline{u}}{\partial y}$$

$$\tau_{xy}^R \equiv \tau^R = -\rho\overline{u'v'} \qquad \tau_{yy}^R = -\rho\overline{v'v'}$$

x Momentum:

$$0 = -\frac{\partial \overline{p}}{\partial x} + \frac{\partial \tau^\ell + \tau^R}{\partial y}$$

y-Momentum:

$$0 = -\frac{\partial \overline{p}}{\partial y} + \frac{\partial \overline{\tau_{yy}^R}}{\partial y}$$

From y-momentum:

$$\boxed{p - \tau_{yy}^R = p_w(x)}$$

where p_w is the pressure on the wall.

Put in x-momentum (with τ_{yy} independent of x):

$$\boxed{\tau^\ell + \tau_{xy}^R = -\frac{\mathrm{d}p_w}{\mathrm{d}x}(h-y) \quad \tau^\ell = \mu\frac{\partial \overline{u}}{\partial y} \quad \tau^R = -\rho\overline{u'v'}}$$

So, the *total* stress is linear in y, and vanishes (by symmetry) on the center line.

Since the Reynolds number is high (i.e. μ small) the laminar shear stress must be small over most of the cross section. However, the dissipation is not so small, and the loss of energy must come from the pressure gradient. Hence the pressure gradient required is much larger than for laminar flow. To balance, the Reynolds stress must be much larger than the stress in laminar duct flow.

However, at the wall, the Reynolds stress is zero because of no slip. So at the wall and close to it, the laminar shear stress must be large, which is only possible if $\partial\overline{u}/\partial y$ is very large. But, since the profile is monotonuous to the center, this can only be true in a thin sublayer at the wall. Outside that layer, the Reynolds stress must dominate.

Exercise:

> Sketch the velocity profiles for laminar and turbulent flow with the same mass flow through the duct. Indicate the layer near the wall and how it is different from the laminar flow case.

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The magnitude of the shear stress at the wall is written in terms of a "friction velocity" u^*:

$$\boxed{\tau_0 = -h\frac{\mathrm{d}p_w}{\mathrm{d}x} \equiv \rho {u^*}^2}$$

In the surface layer, we have the "law of the wall":

$$\boxed{\frac{\overline{u}}{u^*} = f\left(\frac{yu^*}{\nu}\right)}$$

The bottom part of the surface layer is called viscous sublayer since the effective Reynolds number becomes too low for turbulence to sustain itself.

In the core region:

$$\rho u^{*2}\frac{y-h}{h}+\tau^R=0$$

"Velocity defect law":

$$\boxed{\frac{\overline{u}-u\mathrm{max}}{u^*}=F\left(\frac{y}{h}\right)}$$

The matching region is called inertial sublayer. In this layer, both results must be valid. In particular, if we match gradients,

$$\frac{y}{u^*}\frac{\mathrm{d}u}{\mathrm{d}y}=f'\left(\frac{yu^*}{\nu}\right)\frac{yu^*}{\nu}=F'\left(\frac{y}{h}\right)\frac{y}{h}=\frac{1}{\kappa}$$

which requires:

$$\boxed{\frac{\overline{u}}{u^*}=\frac{1}{\kappa}\ln\left(\frac{yu^*}{\nu}\right)+C_f\qquad\left(\frac{yu^*}{\nu}\to\infty\right)}$$

$$\boxed{\frac{\overline{u}-u\mathrm{max}}{u^*}=\frac{1}{\kappa}\ln\left(\frac{y}{h}\right)+C_F\qquad\left(\frac{y}{h}\to 0\right)}$$

Substracting, we get the "logarithmic friction law":

$$\boxed{\frac{u\mathrm{max}}{u^*}=\frac{1}{\kappa}\ln\left(\frac{hu^*}{\nu}\right)+C}$$

Exercise:

Discuss figure 23.13 and how it verifies and does not verify the above discussion for the case of a boundary layer.

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