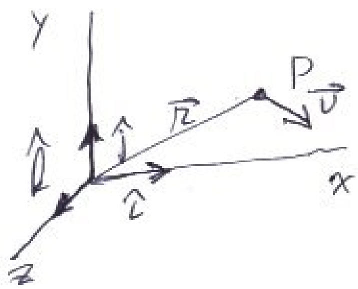
<u>Vector</u> : "Things you can add and multiply by scalars" mentor → Books map

~~Often described as a "column of num~~

<u>Cartesian vectors</u>, <u>1st order tensors</u> } Vectors described in a Cartesian coordinate system that transform in certain ways when you rotate the coordinate system ~~"column of numbers"~~ "column of number"

Example: position vectors, velocity vectors



$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \longrightarrow \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \{r_i\}$$

$i = 1, 2, 3$ index notation

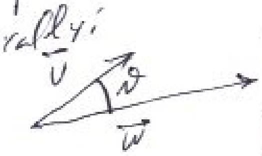
$\downarrow$ $\downarrow$ $\downarrow$
x y z

$$\vec{v} = u\hat{i} + v\hat{j} + w\hat{k} \rightarrow \vec{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \{u_i\}$$

Inner product: $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = \sum_i v_i w_i \equiv v_i w_i$

"Einstein summation convention" leave away the $\sum_i$ if i appears twice

Geometrically:



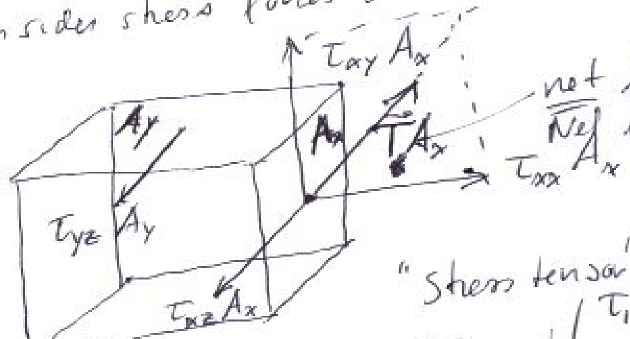
$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos\vartheta$

$|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}} = v_1^2 + v_2^2 + v_3^2$

"index notation"

<u>Matrices</u>, <u>2nd order tensors</u> } "table of numbers"

Consider stress forces on little block of fluid



net force on surface: $\mathbf{R}$

Net force/unit area

$\tau_{xx} \rightarrow \tau_{11}$

$\tau_{xy} \rightarrow \tau_{12}$

etc.

"Stress tensor"

$$\overline{\overline{\tau}} = \begin{array}{cc} & j \rightarrow \\ i \downarrow & \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \end{array} = \{\tau_{ij}\}$$

$i = 1, 2, 3$
$j = 1, 2, 3$

Net force (per unit area) on a small area element ~~...~~ if $\vec{n}$ is the vector of length one (unit vector) normal to the area element

multiplication is row-column dot products

$$\vec{\mathbf{R}} = \begin{pmatrix} \tau_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} \tau_{11} n_1 + \tau_{21} n_2 + \tau_{31} n_3 \\ \tau_{12} n_1 + \tau_{22} n_2 + \tau_{32} n_3 \\ \tau_{13} n_1 + \tau_{22} n_2 + \tau_{33} n_3 \end{pmatrix}$$

e.g. for A_x, $\vec{n} = \hat{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \vec{\mathbf{R}} = \begin{pmatrix} \tau_{11} \\ \tau_{12} \\ \tau_{13} \end{pmatrix} = \begin{pmatrix} \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{pmatrix}$ as before

~~Index notation $\sum_i$~~

Vector Matrix notation:

$$\vec{R} = \overline{\overline{\tau}}^T \vec{n}$$

where superscript T means "transpose";
i.e. ~~swap~~ turn the rows into columns or vice versa.

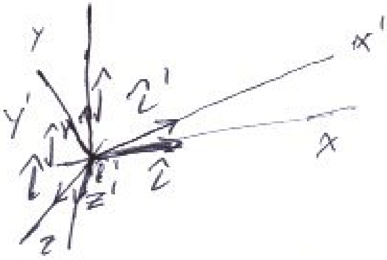
Note: Normally $\overline{\overline{\tau}}$ is "symmetric", meaning that $\overline{\overline{\tau}}^T = \overline{\overline{\tau}}$.
(Exception if there are external moments on the fluid elements that do not disappear in the limit of zero size.)
If symmetric, $\tau_{12} = \tau_{21}$, $\tau_{31} = \tau_{13}$ $\tau_{23} = \tau_{32}$ → in 2D, often written as simply τ
$\tau_{xy} = \tau_{yx}$

Index notation: $R_i = \sum \tau^T_{ij} n_j$: "sum over neighboring indices" $= \tau_{ij}^T n_j$

end

Changes under rotation of coordinate system

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$ old representation



$$\vec{r}\,' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} r_1' \\ r_2' \\ r_3 \end{pmatrix}$$ new representation

Transformation in vector matrix notation

$$\vec{r} = C \vec{r}\,' \qquad \vec{r}\,' = C^{-1}\vec{r} = C^T \vec{r}$$

for coordinate rotations

$C = (\hat{\imath}' \ \hat{\jmath}' \ \hat{k}')$: new basis vectors written in terms of old ones

$$= \begin{pmatrix} \hat{\imath}' \cdot \hat{\imath} & \hat{\jmath}' \cdot \hat{\imath} & \hat{k}' \cdot \hat{\imath} \\ \hat{\imath}' \cdot \hat{\jmath} & \hat{\jmath}' \cdot \hat{\jmath} & \hat{k}' \cdot \hat{\jmath} \\ \hat{\imath}' \cdot \hat{k} & \hat{\jmath}' \cdot \hat{k} & \hat{k}' \cdot \hat{k} \end{pmatrix}$$

where $\hat{\imath}' \cdot \hat{\imath} = \cos\theta_{\hat{\imath}'\hat{\imath}}$
$\hat{\imath}' \cdot \hat{\jmath} = \cos\theta_{\hat{\imath}'\hat{\jmath}}$ etc.

~~For tensors~~ 2nd: For matrices $\overline{\overline{\tau}}' = C^T \overline{\overline{\tau}} C$ (compute row column)

$$\overline{\overline{\tau}} = C \overline{\overline{\tau}}' C^T$$

(Σ_k, Σ_l understood)

Index notation $\tau_{ij} = c_{ik} \tau_{kl} c_{jl}$

Note Cartesian tensors: form of transformation the same for index l as for k.

Diagonalization theorem

~~Th~~ You can turn any symmetric real matrix into a diagonal one if you rotate the axis system just right.