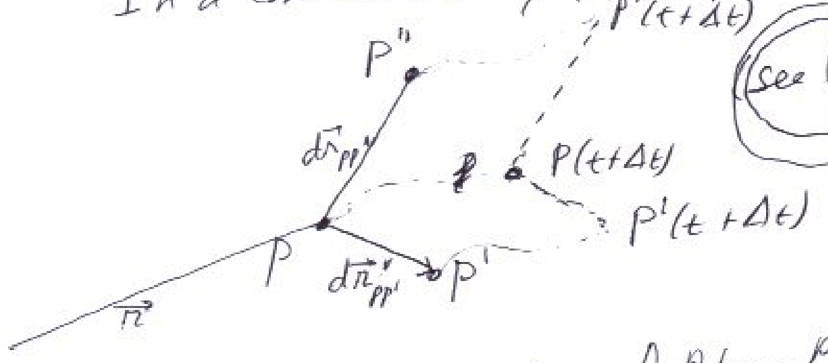
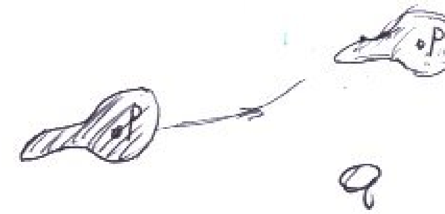
(1) Deformation and rotation of small fluid particles 3.4

In a constant velocity field, fluid particles do not deform or rotate (5)



(see back)



It is the relative motion of P' or P'' relative to P that cause the deformations and rotations.

$$\boxed{d\vec{v} = \frac{\partial \vec{v}}{\partial \vec{r}}\, d\vec{r} \text{ causes deformations and rotations}}$$

So $\frac{\partial \vec{v}}{\partial \vec{r}} = \left\{\frac{\partial u_i}{\partial x_j}\right\}$ is to blame

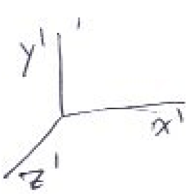
Decomposition:

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{symmetric, "strain rate tensor" } S} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{antisymmetric "rotation tensor" } R} \rightarrow R^T = -R$$

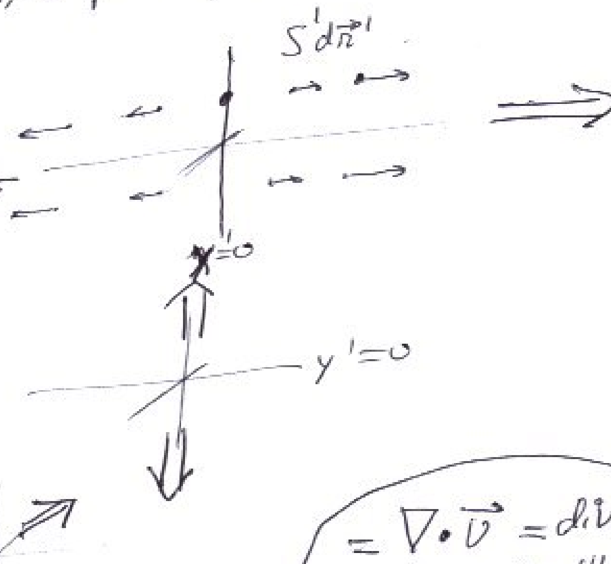
Effect of S: If you rotate the axis system just right, S becomes diagonal

$$S' = \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix}$$

s_1, s_2, s_3 principal strain rates.

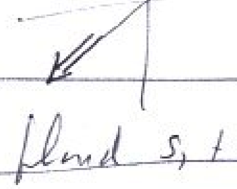


$\begin{pmatrix} s_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$: uniform stretching in x



$\begin{pmatrix} 0 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ uniform stretching in y

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & s_3 \end{pmatrix}$ z



For incompressible fluid $s_1 + s_2 + s_3 = 0$

$$= \nabla \cdot \vec{v} = \operatorname{div} \vec{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Effect of R

Define
$$\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$$
$$\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$
$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\boxed{\vec{\omega} \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} \equiv \nabla \times \vec{v}}$$

Then
$$R = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

$$\frac{1}{2} R\, d\vec{r} = \frac{1}{2}\begin{pmatrix} \omega_y dz - \omega_z dy \\ \omega_z dx - \omega_x dz \\ \omega_x dy - \omega_y dx \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \times d\vec{r}$$

= Solid body rotation with angular velocity $\frac{1}{2}\vec{\omega}$

$\boxed{\text{vorticity is twice the angular velocity}}$

as far as it is reasonably defined